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On the existence of mild solutions for totally nonlinear Caputo-Hadamard fractional differential equations

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Abstract

The existence of mild solutions of a totally nonlinear Caputo-Hadamard fractional differential equation is investigated using the Krasnoselskii-Burton fixed point theorem and some results are presented. Two example are given to illustrate our obtained results.

Keywords: Fractional differential equations Krasnoselskii-Burton's fixed point Large contraction Existence Mild solutions.
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1. Introduction

Fractional differential equations arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of fractional differential equations have received the attention of many authors, see [2]-[14], [17]-[33]. Inspired and motivated by the references [1]-[33], we study the existence of mild solutions for the totally nonlinear fractional differential equation

$$\begin{cases} D_{1^+}^{\alpha} h\left(x\left(t\right)\right) = f\left(t, x\left(t\right)\right), \ t \in \left(1, T\right], \\ x\left(1\right) = 0, \end{cases} \tag{1}$$

where $D_{1^+}^{\alpha}$ is the Caputo-Hadamard fractional derivative of order $\alpha \in (0,1)$, $h : \mathbb{R} \to \mathbb{R}$ is a continuous function with $h(0) = 0$ and $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. That is, there exists a positive constant $L_f > 0$ such that

$$\left|f\left(t, x\right) - f\left(t, y\right)\right| \leq L_f \left|x - y\right|. \tag{2}$$

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By a mild solution of (1) we mean a function $x \in C([1,T],\mathbb{R})$ that satisfies the corresponding integral equation of (1), where $C([1,T],\mathbb{R})$ is the space of continuous real-valued functions defined on $[1,T]$.

Equation (1) is obviously totally nonlinear. This is due to the presence of the terms h and f and the fact that the functions h and f are arbitraries non necessarily linears.

Authors in previous mathematical studies on fractional differential equations did not consider the totally nonlinear equations. Hence, our study is a novel contribution to the fractional differential equations.

To show the existence of mild solutions, we transform (1) into an integral equation and then use the Krasnoselskii-Burton fixed point theorem. The obtained integral equation splits in the sum of two mappings, one is a large contraction and the other is compact.

The organization of this paper is as follows. In Section 2, we give some definitions, lemmas and preliminary results needed in later sections. Also, we present the inversion of (1) and state the Krasnoselskii-Burton fixed point theorem. For details on the Krasnoselskii-Burton theorem we refer the reader to [16]. In Section 3, we present our main results on the existence of mild solutions of (1) and give two example to illustrate our obtained results.

2. Preliminaries

In this section, we introduce some notations, definitions and preliminary facts which are used throughout this paper.

Let $X = C([1,T],\mathbb{R})$ be the Banach space of all real-valued continuous functions, endowed with the maximum norm

$$\|x\| = \sup_{t\in[1,T]} |x(t)|.$$

Definition 2.1 ([22]). *The Hadamard fractional integral of order $\alpha > 0$ of a function $x : [1,\infty) \longrightarrow \mathbb{R}$ is given by*

$$I_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s},$$

where Γ is the gamma function.

Definition 2.2 ([22]). *The Caputo-Hadamard fractional derivative of order $\alpha > 0$ of a function $x : [1,\infty) \longrightarrow \mathbb{R}$ is given by*

$$D_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n(x)(s) \frac{ds}{s},$$

where $\delta^n = \left(t\frac{d}{dt}\right)^n$ and $n = [\alpha] + 1$.

Lemma 2.3 ([22]). *Suppose that $x \in C^{n-1}([1,+\infty),\mathbb{R})$ and $x^{(n)}$ exists almost everywhere on any bounded interval of $[1,\infty)$. Then*

$$I_{1^+}^{\alpha} D_{1^+}^{\alpha} x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(1)}{k!} (\log t)^k.$$

In particular, when $\alpha \in (0,1)$, $I_{1^+}^{\alpha} D_{1^+}^{\alpha} x(t) = x(t) - x(1)$.

From Lemma 2.3, we deduce the following lemma.

Lemma 2.4. *$x \in C([1,T],\mathbb{R})$ is a mild solution of (1) if x satisfies*

$$x(t) = H(x(t)) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s,x(s)) \frac{ds}{s},\ t \in [1,T]. \tag{3}$$

where

$$H(x) = x - h(x) \tag{4}$$

Definition 2.5 (Large contraction [16]). *Let $(\mathbb{M}, d)$ be a metric space and consider $\mathcal{B} : \mathbb{M} \to \mathbb{M}$. Then $\mathcal{B}$ is said to be a large contraction if given $x, y \in \mathbb{M}$ with $x \neq y$ then $d(\mathcal{B}x, \mathcal{B}y) \leq d(x, y)$ and if for all $\varepsilon > 0$ there exists a $\delta < 1$ such that*

$$[x, y \in \mathbb{M},\ d(x, y) \geq \varepsilon] \Rightarrow d(\mathcal{B}x, \mathcal{B}y) \leq \delta d(x, y).$$

Theorem 2.6 (Krasnoselskii-Burton [16]). *Let $\mathbb{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{M}$ such that*

(i) $x, y \in \mathbb{M}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,

(ii) $\mathcal{A}$ is compact and continuous,

(iii) $\mathcal{B}$ is a large contraction mapping,

Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

We will use this theorem to show the existence of mild solutions for (1).

Theorem 2.7 ([1]). *Let $\|.\|$ be the supremum norm, $\mathbb{M} = \{x \in X : \|x\| \leq r\}$, where r is a positive constant. Suppose that h is satisfying the following conditions*

(H1) h is continuous on $[-r, r]$ and differentiable on $(-r, r)$,

(H2) the function h is strictly increasing on $[-r, r]$,

(H3) $\sup_{x\in(-r,r)} h'(x) \leq 1$.

Then, the mapping H define by (4) is a large contraction on $\mathbb{M}$.

3. Main results

By using Theorem 2.6, we prove in this section the existence of mild solutions for (1). To apply Theorem 2.6 we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset $\mathbb{M}$ of $\mathbb{B}$ and construct two mappings, one is a large contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (X, \|.\|)$ and

$$\mathbb{M} = \{x \in X : \|x\| \leq r\}, \tag{5}$$

where the positive constant r is satisfied the following inequality

$$J \frac{(rL_f + \sigma_f)(\log T)^{\alpha}}{\Gamma(\alpha + 1)} \leq r, \tag{6}$$

with $\sigma_f = \sup_{t\in[1,T]} |f(t, 0)|$ and $J \geq 3$ is a constant.

We express (3) as

$$x(t) = (\mathcal{A}x)(t) + (\mathcal{B}x)(t) = (\mathcal{S}x)(t), \tag{7}$$

where the operators $\mathcal{A}, \mathcal{B} : \mathbb{M} \to X$ are defined by

$$(\mathcal{A}x)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s}, \tag{8}$$

and

$$(\mathcal{B}x)(t) = H(x(t)). \tag{9}$$

Lemma 3.1. *Assume that (2) and (6) hold. Then, the operator $\mathcal{A} : \mathbb{M} \longrightarrow \mathbb{M}$ is compact and continuous.*

Proof. Let $\mathcal{A}$ be defined by (8). Clearly, $\mathcal{A}x$ is bounded and continuous. Observe that in view of (2), we arrive at

$$\begin{aligned} |f(t, x)| &\leq |f(t, x) - f(t, 0) + f(t, 0)| \\ &\leq |f(t, x) - f(t, 0)| + |f(t, 0)| \\ &\leq L_f \|x\| + \sigma_f. \end{aligned}$$

We have

$$\begin{aligned}
|(\mathcal{A}x)(t)| &\leq \left|\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |f(s,x(s))|\frac{ds}{s} \\
&\leq \frac{(rL_f+\sigma_f)(\log T)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Thus,

$$\|\mathcal{A}x\| \leq \frac{r}{J} \leq r.$$

Hence, $\mathcal{A}:\mathbb{M}\to\mathbb{M}$ which implies $\mathcal{A}(\mathbb{M})$ is uniformly bounded.

To prove the continuity of $\mathcal{A}$, we consider a sequence (x_n) converging to x. We have

$$\begin{aligned}
&|(\mathcal{A}x_n)(t)-(\mathcal{A}x)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |f(s,x_n(s))-f(s,x(s))|\frac{ds}{s} \\
&\leq \frac{L_f}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |x_n(s)-x(s)|\frac{ds}{s} \\
&\leq \frac{L_f}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{ds}{s}\|x_n-x\| \\
&\leq \frac{L_f(\log T)^{\alpha}}{\Gamma(\alpha+1)}\|x_n-x\|.
\end{aligned}$$

From the above analysis we obtain

$$\|\mathcal{A}x_n-\mathcal{A}x\| \leq \frac{L_f(\log T)^{\alpha}}{\Gamma(\alpha+1)}\|x_n-x\|.$$

Hence whenever $x_n\to x$, $\mathcal{A}x_n\to\mathcal{A}x$. This shows the continuity of $\mathcal{A}$.

To prove $\mathcal{A}$ is compact. We will prove that $\mathcal{A}(\mathbb{M})$ is equicontinuous. Let $x\in\mathbb{M}$, then for any $t_1,t_2\in[1,T]$ with $t_1<t_2$, we have

$$\begin{aligned}
&|(\mathcal{A}x)(t_1)-(\mathcal{A}x)(t_2)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^{t_1} \left|\left(\log\frac{t_1}{s}\right)^{\alpha-1}-\left(\log\frac{t_2}{s}\right)^{\alpha-1}\right| |f(s,x(s))|\frac{ds}{s} \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \left(\log\frac{t_2}{s}\right)^{\alpha-1} |f(s,x(s))|\frac{ds}{s} \\
&\leq \frac{rL_f+\sigma_f}{\Gamma(\alpha)}\left(\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\alpha-1}-\left(\log\frac{t_2}{s}\right)^{\alpha-1}\right)\frac{ds}{s}+\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}\right) \\
&\leq \frac{rL_f+\sigma_f}{\Gamma(\alpha+1)}\left((\log t_1)^{\alpha}-(\log t_2)^{\alpha}+2\left(\log\frac{t_2}{t_1}\right)^{\alpha}\right) \\
&\leq 2\frac{rL_f+\sigma_f}{\Gamma(\alpha+1)}\left(\log\frac{t_2}{t_1}\right)^{\alpha},
\end{aligned}$$

which is independent of x and tends to zero as $t_2\to t_1$. Thus that $\mathcal{A}(\mathbb{M})$ is equicontinuous. So, the compactness of $\mathcal{A}$ follows by the Ascoli-Arzela theorem. □

For the next we suppose that

$$\max\left(\left|H\left(-r\right)\right|,\left|H\left(r\right)\right|\right)\leq\frac{\left(J-1\right)}{J}r. \tag{10}$$

Lemma 3.2. *Let $\mathcal{B}$ be defined by (9). Suppose (10) and all conditions of Theorem 2.7 hold. Then $\mathcal{B}:\mathbb{M}\rightarrow\mathbb{M}$ is a large contraction.*

Proof. Let $\mathcal{B}$ be defined by (9). Obviously, $\mathcal{B}x$ is continuous. For having $\mathcal{B}x\in\mathbb{M}$ we will prove that $\left\|\mathcal{B}x\right\|\leq r$. Let $x\in\mathbb{M}$, by (10) we get

$$\begin{aligned}\left|\left(\mathcal{B}x\right)\left(t\right)\right| &\leq \left|H\left(x\left(t\right)\right)\right| \\ &\leq \max\left\{\left|H\left(-r\right)\right|,\left|H\left(r\right)\right|\right\}\leq\frac{\left(J-1\right)r}{J}\leq r.\end{aligned}$$

Then, for any $x\in\mathbb{M}$, we have

$$\left\|\mathcal{B}x\right\|\leq r.$$

Consequently, we have $\mathcal{B}:\mathbb{M}\rightarrow\mathbb{M}$.

It remains to prove that $\mathcal{B}$ is a large contraction. By Theorem 2.7, H is a large contraction on $\mathbb{M}$, then for any $x,y\in\mathbb{M}$ with $x\neq y$ we have

$$\begin{aligned}\left|\left(\mathcal{B}x\right)\left(t\right)-\left(\mathcal{B}y\right)\left(t\right)\right| &= \left|H\left(x\left(t\right)\right)-H\left(y\left(t\right)\right)\right| \\ &\leq \left\|x-y\right\|.\end{aligned}$$

Then $\left\|\mathcal{B}x-\mathcal{B}y\right\|\leq\left\|x-y\right\|$. Now, let $\varepsilon\in(0,1)$ be given and let $x,y\in\mathbb{M}$, with $\left\|x-y\right\|\geq\varepsilon$, from the proof of Theorem 2.7, we have found a $\delta\in(0,1)$, such that

$$\left|\left(Hx\right)\left(t\right)-\left(Hy\right)\left(t\right)\right|\leq\delta\left\|x-y\right\|.$$

Thus,

$$\left\|\mathcal{B}x-\mathcal{B}y\right\|\leq\delta\left\|x-y\right\|.$$

The proof is complete. □

Theorem 3.3. *Suppose the hypotheses of Lemmas 3.1 and 3.2 hold. Let $\mathbb{M}$ defined by (5), Then (1) has a mild solution in $\mathbb{M}$.*

Proof. By Lemma 3.1, $\mathcal{A}:\mathbb{M}\rightarrow\mathbb{M}$ is continuous and compact. Also, from Lemma 3.2, the mapping $\mathcal{B}:\mathbb{M}\rightarrow\mathbb{M}$ is a large contraction. Next, we prove that if $x,y\in\mathbb{M}$, we have $\left\|\mathcal{A}x+\mathcal{B}y\right\|\leq r$. Let $x,y\in\mathbb{M}$ with $\left\|x\right\|,\left\|y\right\|\leq r$. By (6) and (10), we obtain

$$\begin{aligned}\left\|\mathcal{A}x+\mathcal{B}y\right\|\leq\left\|Ax\right\|+\left\|By\right\| &\leq \frac{\left(rL_f+\sigma_f\right)\left(\log T\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{\left(J-1\right)r}{J} \\ &\leq \frac{r}{J}+\frac{\left(J-1\right)r}{J}=r.\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii-Burton theorem are satisfied. Thus there exists a fixed point $z\in\mathbb{M}$ such that $z=\mathcal{A}z+\mathcal{B}z$. By Lemma 2.3 this fixed point is a mild solution of (1). Hence (1) has a mild solution. □

Example 3.4. *Let us consider the following totally nonlinear fractional differential equation*

$$\left\{\begin{array}{l} D_{1^+}^{\frac{1}{3}}\left(x^3\left(t\right)\right)=10^{-3}\left(1+\cos\left(x\left(t\right)\right)\right),\ t\in\left(1,e\right], \\ x\left(1\right)=0, \end{array}\right. \tag{11}$$

where $\alpha = 1/3$, $T = e$, $h(x) = x^3$, $h(0) = 0$, $f(t, x(t)) = 10^{-3}(1 + \cos(x(t)))$. *Define*

$$\mathbb{M} = \left\{ x \in X : \|x\| \le \sqrt{3}/3 \right\}.$$

Using Theorem 2.7, the mapping

$$H(x) = x - x^3,$$

is a large contraction on the set $\mathbb{M}$ *for* $r = \sqrt{3}/3$. *For* $x, y \in \mathbb{M}$ *there exists a positive constant* $L_f = 10^{-3} > 0$ *such that*

$$|f(t, x) - f(t, y)| \le 10^{-3} |x - y|,$$

and

$$\sigma_f = \sup_{t \in [1,e]} |f(t, 0)| = 2 \times 10^{-3}.$$

By substituting r, α, T, L_f, σ_f, $\Gamma(\alpha + 1)$ *in (6), we get*

$$J \le \frac{\Gamma(\alpha + 1)}{(rL_f + \sigma_f)(\log T)^{\alpha}} r \simeq 200.04,$$

then, we obtain $J \in [3, 200]$. *Hence, by Theorem 3.3, (11) has a mild solution in* $\mathbb{M}$.

Example 3.5. *Let us consider the following totally nonlinear fractional differential equation*

$$\begin{cases} D_{1^+}^{\frac{1}{2}} \left(x^5(t) \right) = 10^{-2} \left(\frac{3}{1+t} + \frac{|x(t)|}{1+|x(t)|} \right), \; t \in (1, e], \\ x(1) = 0, \end{cases} \tag{12}$$

where $\alpha = 1/2$, $T = e$, $h(x) = x^5$, $h(0) = 0$, $f(t, x(t)) = 10^{-2}\left(\frac{3}{1+t} + \frac{|x(t)|}{1+|x(t)|} \right)$. *Define*

$$\mathbb{M} = \left\{ x \in X : \|x\| \le 5^{-1/4} \right\}.$$

Using Theorem 2.7, the mapping

$$H(x) = x - x^5,$$

is a large contraction on the set $\mathbb{M}$ *for* $r = 5^{-1/4}$. *For* $x, y \in \mathbb{M}$ *there exists a positive constant* $L_f = 10^{-2} > 0$ *such that*

$$|f(t, x) - f(t, y)| \le 10^{-2} |x - y|,$$

and

$$\sigma_f = \sup_{t \in [1,e]} |f(t, 0)| = \frac{3}{2} \times 10^{-2}.$$

By substituting r, α, T, L_f, σ_f, $\Gamma(\alpha + 1)$ *in (6), we get*

$$J \le \frac{\Gamma(\alpha + 1)}{(rL_f + \sigma_f)(\log T)^{\alpha}} r \simeq 27.33,$$

then, we obtain $J \in [3, 27]$. *Hence, by Theorem 3.3, (12) has a mild solution in* $\mathbb{M}$.

4. Conclusion

In this paper, by using the Krasnoselskii-Burton fixed point theorem, we obtained several sufficient conditions which guarantee the existence of mild solutions for a totally nonlinear Caputo-Hadamard fractional differential equation. Finally, two examples have been given to illustrate our obtained results. In the future, we can study the existence and stability of nonnegative mild solutions of the problem (1).

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