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Research Article

# Hybrid proximal point algorithm for solving split equilibrium problems and its applications

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**Abstract**

This paper deals with split equilibrium problems in Banach spaces. The presented algorithm is based on the hybrid algorithm and the proximal point algorithm and has been used for finding the solution of split equilibrium problems. Under some standard assumptions on equilibrium bifunctions, it is proven that the generated sequences by the presented scheme are strongly convergent. Finally, the efficiency of the proposed method is demonstrated through some examples. Also, comparative results verify that the proposed method is more effective than the other existing methods in the literature. Furthermore, an application of the presented algorithm in Hilbert spaces and an application of our method to solve the *LASSO* problem in the field of compressed sensing are given.

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**Keywords.** split equilibrium problem, strong convergence, uniformly convex Banach space, uniformly smooth Banach space

## 1. Introduction

Assume that $C$ and $Q$ are nonempty, convex and closed subsets of real Banach spaces $X$ and $Y$, respectively. The *split equilibrium problem*, $(SEP)$, is defined as follows:

$$\begin{aligned} &\text{“Find} \quad x^* \in C \quad \text{such that} \quad f(x^*, x) \geq 0, \quad \forall x \in C, \quad \text{and} \\ &\qquad\quad \text{such that} \quad y^* = Ax^* \in Q \quad \text{solves} \quad g(y^*, y) \geq 0, \quad \forall y \in Q\text{”}, \end{aligned} \tag{1.1}$$

where $A : X \to Y$ is a bounded linear operator and $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ are equilibrium bifunctions, i.e., $f(x, x) = 0$ for every $x \in C$ and $g(y, y) = 0$ for every $y \in Q$. In general, the equilibrium bifunctions $f$ and $g$ need not be convex. This problem is introduced in 2011 by Moudafi [32] (see more [11,14,45]). It is well known that $(SEP)$ is a generalization of a multiple set split feasibility problem. So, it generalizes the split variational inequality problem [5], which is the generalization of split zero problems and split feasibility problems [5,31,32].

In the case of $g = 0$ and $Q = Y$, the $(SEP)$ reduces to the following classical equilibrium problem, $(EP)$:

$$\text{“Find} \quad x^* \in C \quad \text{such that} \quad f(x^*, y) \geq 0, \quad \forall y \in C,\text{”} \tag{1.2}$$

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and its solution set is denoted by $EP(f, C)$.

Note that the solution set of such problems may be empty. In particular, if $f(x, y) = \langle Ax, y - x\rangle$, where $A : C \to X^*$, ($X^*$ is the dual space of Banach space $X$), then the problem (1.2) reduces to the following classical variational inequality problem (shortly, $VIP$) which is initially investigated by Kinderlehrer and Stampacchia [24, 26]:

$$\text{“Find} \quad x^* \in C \quad \text{such that} \quad \langle Ax^*, y - x^*\rangle \geq 0 \quad \forall y \in C\text{”}. \tag{1.3}$$

The solution set of the problem (1.3) is denoted by $VI(A, C)$.

Equilibrium problem ($EP$) plays an important role in many fields of mathematics such as nonlinear analysis and optimization, because many mathematical models such as fixed point problems, optimization problems and variational inequality problems can be formulated as an ($EP$) [4, 33]. Thus, its theory and applications have been extensively studied by many researchers. In particular, a great number of methods are introduced by mathematicians to solve the ($SEP$) and ($EP$); see for instance [9, 11, 12, 14–16, 19–21, 23, 39] and references therein.

Moreover, the ($SEP$) is composed of a pair of equilibrium problems, and aims at finding a solution $x^*$ of an equilibrium problem such that its image $y^* = Ax^*$ under a given bounded linear operator $A$ also solves another equilibrium problem. The solution set of ($SEP$) is denoted by

$$\Omega = \{z \in EP(f, C) : Az \in EP(g, Q)\}.$$

Suppose that $H$ is a Hilbert space and $C$ is a nonempty, closed and convex subset of $H$. In 2005, Combettes and Hirstoaga [9], assuming some conditions on bifunction $g$, showed that for any $r > 0$ and $x \in H$, the mapping $T_r^g(x) : H \to C$ defined by

$$T_r^g(x) = \{z \in C : g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\},$$

is single valued. Using the mapping $T_r^g(x)$, mathematicians introduced some algorithms in the framework of Hilbert and Banach spaces [1, 9, 43]. In particular, in one of the steps of those algorithms, the sequence $\{x_n\}$ is generated as follows:

$$\begin{cases} x_0 \in H \ \text{ chosen arbitrarily,} \quad \& \quad \{r_n\} \subset (0, \infty) \ \text{ satisfies } \ \liminf_{n\to\infty} r_n > 0, \\ x_{n+1} \in H \text{ such that } \quad f(x_{n+1}, y) + \frac{1}{r_n}\langle y - x_{n+1}, x_{n+1} - x_n\rangle \geq 0, \quad \forall\, y \in C, \end{cases}$$

One of the most important tools for solving optimization problems is Proximal Point Algorithms ($PPA$), which are introduced by Martinet [27]. Rockafellar [38] extended the ($PPA$) to finding a zero of the maximal monotone operator and proved his algorithm is weakly convergent. This extension made many mathematicians interested in the ($PPA$), and many modifications of ($PPA$) are introduced in the framework of Hilbert and Banach spaces; see for example [28, 37].

In 1995, for a bifunction $f(x, y)$ which is convex with respect to $y \in H$, for each fixed $x \in H$, Antipin [3] introduced the following algorithm for equilibrium problems in finite-dimensional vector spaces:

$$\begin{cases} x_0 \in C, \\ y_n := \operatorname{argmin}_{y\in C}\{\lambda f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ x_{n+1} := \operatorname{argmin}_{y\in C}\{\lambda f(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \quad n \geq 0, \end{cases}$$

The above algorithm used the proximity operator which can be computed by the Matlab Optimization Toolbox in practice. The advantage of Antipin's algorithm is that two strongly convex programming problems are solved at each iteration. His method is known under the name of the extragradient method.

If the bifunction $f$ satisfies the following pseudomonotoncity condition on $C$:

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0 \quad \forall\, x, y \in C,$$

then the equilibrium problem is called pseudomonotone equilibrium problem. In 2008, Quoc et. al. [35] applied the Antipin's algorithm for pseudomonotone equilibrium problems in Hilbert spaces and presented the following algorithm:

Step 0 Take $x_0 \in C$, $\rho > 0$ and set $k = 0$.

Step 1 Solve the strongly convex program

$$\min_{y \in C}\{\rho f(x_k, y) + G(y) - \langle \nabla G(x_k), y - x_k\rangle\}$$

to obtain its unique optimal solution $y_k$.

If $y_k = x_k$, then stop; $x_k$ is a solution to the equilibrium problem. Otherwise, go to Step 2.

Step 2 Solve the strongly convex program

$$\min_{y \in C}\{\rho f(y_k, y) + G(y) - \langle \nabla G(x_k), y - x_k\rangle\}$$

to obtain its unique optimal solution $x_{k+1}$.

Step 3 Set $k := k + 1$, and go back to Step 1.

where $G : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex and continuously differentiable function. Under mild conditions, they obtained the weak convergence of the sequences generated by this algorithm. The Antipin's algorithm attracted a lot of attention and its variants have been extensively studied by researchers, see [18, 20, 34, 42] and references therein.

Recently, Lyashko and Semenov [25] proposed an algorithm for pseudomonotone equilibrium problems which is called the two-step proximal point algorithm. This algorithm can be summarized as follows: choose $x_0 = y_0 \in C, \epsilon > 0$ and $0 < \lambda < \frac{1}{2(2c_1+c_2)}$, where $c_1, c_2$ are positive constants.

Step 1 For $x_n$ and $y_n$, compute

$$x_{n+1} = \text{argmin}_{y\in C}\Big\{\lambda f(y_n, y) + \frac{1}{2}\|y - x_n\|^2\Big\}$$

Step 2 If $\max\{\|x_{n+1} - x_n\|, \|y_n - x_n\| < \epsilon$ then stop, else compute

$$y_{n+1} = \text{argmin}_{y\in C}\Big\{\lambda f(y_n, y) + \frac{1}{2}\|y - x_{n+1}\|^2\Big\}$$

Step 3 Set $n := n + 1$, and go to Step 1.

Lyashko and Semenovs algorithm and most other algorithms must either solve two strongly convex programming problems or solve one strongly convex programming problem and compute one projection onto the feasible set. Therefore, their computations are expensive if the bifunctions and the feasible sets have complicated structures.

The bifunction $f : H \times H \to (-\infty, +\infty]$ is proper if $\text{dom} f = \{(x, y) \in H \mid f(x, y) < +\infty\}$ is nonempty. Suppose that $f : H \times H \to (-\infty, +\infty]$ is a proper bifunction and $f(x, .)$ is a convex function, i.e., $f(x, y)$ is convex respect to the second argument $y \in H$, for each fixed $x \in H$, the subdifferential of $f(x, .)$ at $y_0$ as the subset of $H$ is given by

$$\partial_2 f(x, y_0) = \Big\{x \in H : \; f(x, y) \geq f(x, y_0) + \langle x, y - y_0\rangle, \; \forall y \in H\Big\}.$$

If $\partial_2 f(x, y_0) \neq \emptyset$, then $f$ is called subdifferentiable respect to the second argument $y_0$. The normal cone of $C$ at $\nu \in C$ is defined by $N_C(\nu) := \Big\{x \in H : \langle x, \nu - y\rangle \geq 0, \; \forall\, y \in C\Big\}$.

In 2018, Kassay et.al. [22] introduced an algorithm for solving the equilibrium problems in Hilbert spaces which only required solving the subprogram over a half-space instead of over the feasible set as Lyashko and Semenovs algorithm. This algorithm is as follows:

Step 1 For $x_0$ and $y_0$, compute

$$x_1 = \text{argmin}_{y\in C}\Big\{\lambda f(y_0, y) + \frac{1}{2}\|y - x_0\|^2\Big\}$$
$$y_1 = \text{argmin}_{y\in C}\Big\{\lambda f(y_0, y) + \frac{1}{2}\|y - x_1\|^2\Big\}$$

Step 2 given $x_n, y_n$ and $y_{n-1}$, let $\omega_n \in \partial_2 f(y_{n-1}, y_n)$ such that there exists an element $q_n \in N_C(y_n)$ (see Lemma 2.4 in subsection 2.5) satisfying

$$0 = \lambda\omega_n + y_n - x_n + q_n,$$

and construct the half-space

$$H_n = \{z \in H : \langle x_n - \lambda\omega_n - y_n, z - y_n\rangle \leq 0\}.$$

and compute

$$x_{n+1} = \text{argmin}_{y\in H_n}\Big\{\lambda f(y_n, y) + \frac{1}{2}\|y - x_n\|^2\Big\}$$
$$y_{n+1} = \text{argmin}_{y\in C}\Big\{\lambda f(y_n, y) + \frac{1}{2}\|y - x_{n+1}\|^2\Big\}$$

Step 3 If $x_{n+1} = x_n$ and $y_n = y_{n-1}$ then stop. Otherwise, set $n := n + 1$, and return to step 2.

They proved weak and strong convergence theorems under some suitable assumptions.

It should be noted that usually only weak convergence is deduced for the proposed iteration processes. One of the methods that researchers have used to obtain strong convergence for the generated iterates is the hybrid projection method, see for instance [1,2].

In this paper, motivated by Kassay et.al. [22], combining the hybrid method and the proximal point method, we propose a hybrid proximal algorithm for solving $(SEP)$ in Banach spaces. The advantage of our approach is that we only require to solve a convex subprogram on a half-space instead of on the feasible set. The solution of a convex optimization problem on a closed sets usually needs to be computed numerically by means of an iterative algorithm. So, many powerful algorithms such as the proximal algorithm have been introduced for finding the solution of a convex optimization problem. Recently, many researchers have presented the various improvements of the proximal algorithm in different ways. One of these ways is replacing the closed convex set in the second projection with a half-space. This change simplifies the calculations and may increase the performance of the algorithm.

This paper is organized as follows : In section 2, we recall some definitions and preliminary results for the further use. Section 3 deals with proposing a new algorithm and proving it's convergence. In section 4, we present an application of the presented algorithm in Hilbert spaces. Then, in section 5, we provide an example to show the efficiency of our results. Also, we give some comparative results to verify that the proposed method is more effective than the other existing methods in the literature. Furthermore, the efficiency of our algorithm has been illustrated on the $LASSO$ problem in the field of compressed sensing.

## 2. Preliminaries

Suppose that $X^*$ is the dual of a Banach space $X$ and $S(X)$ is the unit sphere centered at the origin of $X$. For $x^* \in X^*$ and $x \in X$ the notation $\prec x^*, x \succ$ means $x^*(x)$. Suppose $B$ is the closed unit ball of a Banach space $X$. The norm of $X^*$ define by

$$\|x^*\|_{X^*} = \sup\{\prec x^*, x \succ \ |x \in B\},$$

for every $x^* \in X^*$. This norm makes $X^*$ into a Banach space. Assume that $\{x_n\}$ be a sequence in $X$, we denote strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

### 2.1. Strict convexity

A Banach space $X$ is strictly convex if $\|\frac{x+y}{2}\| < 1$, whenever $x, y \in S(X)$ and $x \neq y$. The function $\delta_X(\epsilon) : [0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\Big\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \geqslant \epsilon\Big\},$$

is called the *modulus of convexity* of $X$. Also, $X$ is said to be uniformly convex if $\delta_X(0) = 0$ and $\delta_X(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. If a Banach space $X$ is uniformly convex, then $X$ is reflexive and strictly convex [8, 40]. Let $p > 1$, the Banach space $X$ is called to be $p$-uniformly convex [7] if there exists a constant $c > 0$ such that $\delta_X(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in [0,2]$.

For $0 < p < \infty$, a $\mu$-measurable function $f : X \to \mathbb{R}$ is $p$-integrable if $|f|^p$ is an integrable function. The set of all $p$-integrable functions is denoted by $L_p(\mu)$, or for convenience by $L_p$. If $f \in L_p$, then the $L_p$-norm of $f$ is defined by $\|f\|_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}$. It is well known that for $1 < p \leq 2$, $L_p$ is 2-uniformly convex and for $p \geq 2$, $L_p$ is $p$-uniformly convex.

### 2.2. Smoothness

A Banach space $X$ is called smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}, \tag{2.1}$$

exists for all $x, y \in S(X)$. The *modulus of smoothness* of $X$ is the function $\rho_X(\tau) : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(\tau) = \sup\Big\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\Big\}.$$

The Banach space $X$ is said to be uniformly smooth if $\dfrac{\rho_X(t)}{t} \to 0$ as $t \to 0$. For instance, for $p > 1$, $L_p$ is uniformly smooth. It well known that every uniformly smooth Banach space $X$ is smooth.

Let $q > 1$. If there exists a fixed constant $c > 0$ such that $\rho_X(t) \leq ct^q$, then $X$ is said to be $q$-uniformly smooth. If $X$ is $q$-uniformly smooth, then $q \leq 2$ and $X$ is uniformly smooth.

### 2.3. Duality mappings

Let $p > 1$ be a real number, the generalized duality mapping $J_p^X : X \to 2^{X^*}$ is defined by

$$J_p^X x = \{x^* \in X^* : \prec x^*, x \succ = \|x\|\|x^*\|_{X^*},\ \|x^*\|_{X^*} = \|x\|^{p-1}\} \quad \forall\, x \in X. \tag{2.2}$$

Let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $X$ is $p$-uniformly convex and uniformly smooth, then $J_p^X$ is single valued, monotone, one-to-one, onto and $(J_p^X)^{-1} = J_q^{X^*}$ where $J_q^{X^*}$ is the generalized duality mapping on $X^*$ [8, 40].

If $X$ is uniformly smooth, then $J_p^X$ is uniformly norm-to-norm continuous on bounded sets of $X$ [8] i.e., for every bounded set $M$ of $X$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|J_p^X x - J_p^X y\|_{X^*} < \varepsilon,$$

for all $x, y \in M$ such that $\|x - y\| < \delta$. If $p = 2$, then $J_2^X$ is called the normalized duality mapping.

### 2.4. Bregman distance

Let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $X$ is a $p$-uniformly convex, uniformly smooth real Banach space, we define the Bregman distance $\Delta_p^X : X \times X \to \mathbb{R}$ by

$$\begin{aligned} \Delta_p^X(x,y) &= \frac{1}{q}\|x\|^p - \prec J_p^X x, y \succ + \frac{1}{p}\|y\|^p, \\ &= \frac{1}{p}(\|y\|^p - \|x\|^p) + \prec J_p^X x, x-y \succ, \\ &= \frac{1}{q}(\|x\|^p - \|y\|^p) - \prec J_p^X x - J_p^X y, y \succ, \end{aligned} \tag{2.3}$$

for all $x, y \in X$. Using the definition of $\Delta_p^X$, we can easily conclude that

$$\Delta_p^X(x,y) = \Delta_p^X(x,z) + \Delta_p^X(z,y) + \prec J_p^X x - J_p^X z, z-y \succ, \quad \forall\, x,y,z \in X, \tag{2.4}$$

and

$$\Delta_p^X(x,y) + \Delta_p^X(y,x) = \prec J_p^X x - J_p^X y, x-y \succ, \qquad \forall\, x,y \in X. \tag{2.5}$$

Moreover, in a $p$-uniformly convex space $X$, we have the property [41]

$$\tau\|x-y\|^p \leq \Delta_p^X(x,y), \tag{2.6}$$

for all $x, y \in X$ and for some constant $\tau > 0$. It is worth noting that, in a Hilbert space $H$, $\Delta_2^X(x,y) = \frac{1}{2}\|x-y\|^2$ for all $x, y \in X$.

Now, suppose that $C$ is a nonempty closed convex subset of a Banach space $X$. Recall that, metric projection of $X$ onto $C$ is defined by

$$P_C x = \mathrm{argmin}_{y \in C}\|x-y\|, \quad x \in X.$$

Here, $P_C$ is characterized by the following variational inequality:

$$\prec J_p^X(x - P_C x), z - P_C x \succ \leq 0, \quad z \in C.$$

The Bregman projection [41] is defined as the unique minimizer of the Bregman distance and is denoted by $\Pi_C x$, so

$$\Pi_C x = \mathrm{argmin}_{y \in C}\Delta_p^X(x,y), \quad \forall x \in X.$$

Some useful properties of the Bregman projection are expressed in the following lemma.

**Lemma 2.1** ([41])**.** *Let $C$ be a nonempty closed subset of a $p$-uniformly convex and uniformly smooth real Banach space $X$ and let $(x,z) \in X \times C$. Then the following propositions hold:*

(i) $z = \Pi_C x$ *if and only if* $\prec J_p^X x - J_p^X z, y - z \succ \leq 0$ *for all* $y \in C$,
(ii) $\Delta_p^X(\Pi_C x, z) + \Delta_p^X(x, \Pi_C x) \leq \Delta_p^X(x,z)$,
(iii) $\Delta_p^X(x, \Pi_C x) = \min_{y \in C} \Delta_p^X(x,y)$.

Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $X$, let $T$ be a mapping from $C$ into itself. A point $p \in C$ is said to be an asymptotic fixed point of $T$ if there exists $\{x_n\}$ in $C$ which converges weakly to $p$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote the set of all fixed points and all asymptotic fixed points of $T$ by $F(T)$ and $\widehat{F}(T)$ respectively. Following Matsushita and Takahashi [29,30], a mapping $T$ of $C$ into itself is said to be relatively nonexpansive (see also [36]) if the following conditions are satisfied:

(i) $F(T)$ is nonempty,
(ii) $\Delta_p^X(Tx,u) \leq \Delta_p^X(x,u)$, $\forall u \in F(T)$, $x \in C$,
(iii) $\widehat{F}(T) = F(T)$.

**Lemma 2.2** ([29]). *Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $X$ and let $T$ be a relatively nonexpansive mapping from $C$ into itself. Then $F(T)$ is closed and convex.*

### 2.5. Subdifferentiability of convex functions

To present our algorithm in the next section we need the following definitions and lemmas.

The function $f : X \to (-\infty, +\infty]$ is proper if dom$f$= $\{x \in X \mid f(x) < +\infty\}$ is nonempty. Let $f : X \to (-\infty, +\infty]$ be a proper and convex function. For $x_0 \in Dom(f)$, the subdifferential of $f$ at $x_0$ as the subset of $X$ is given by

$$\partial f(x_0) = \Big\{x^* \in X^* : \ f(x) \geq f(x_0) + \prec x^*, x - x_0 \succ, \ \forall x \in X\Big\}.$$

If $\partial f(x_0) \neq \emptyset$, then $f$ is called subdifferentiable at $x_0$. If $\partial f(x_0)$ is single valued, then $f$ is said to be Gâteaux differentiable at $x_0$ which is denoted by $\nabla f(x_0)$.
Let $f : X \times X \to (-\infty, +\infty]$ be a proper bifunction and $f(x, .)$ be a convex function, i.e., $f(x, y)$ be convex respect to the second argument $y \in X$, for each fixed $x \in X$, the subdifferential of $f(x, .)$ at $y_0$ as the subset of $X$ is given by

$$\partial_2 f(x, y_0) = \Big\{x^* \in X^* : \ f(x, y) \geq f(x, y_0) + \prec x^*, y - y_0 \succ, \ \forall y \in X\Big\}.$$

If $\partial_2 f(x, y_0) \neq \emptyset$, then $f$ is called subdifferentiable respect to the second argument $y_0$. If $\partial_2 f(x, y_0)$ is single valued, then $f(x, .)$ is said to be Gâteaux differentiable respect to the second argument $y_0$ which is denoted by $\nabla_2 f(x, y_0)$.

Let $f : X \to (-\infty, \infty]$ be proper, and let $x_0 \in X$. Then $x_0$ is a minimizer of $f$ if $f(x_0) = \inf_{x \in X} f(x)$. The set of minimizers of $f$ is denoted by Argmin$f$. If Argmin$f$ is a singleton, its unique element is denoted by $\text{argmin}_{x \in X} f(x)$.

**Lemma 2.3** ([20]). *Let $X$ be a reflexive Banach space. If $f : X \to (-\infty, +\infty]$ and $g : X \to (-\infty, +\infty]$ are proper, convex, and lower semicontinuous functions and if $0 \in Int(Domf - Domg)$, then $\partial(f + g)(x) = \partial f(x) + \partial g(x)$.*

**Lemma 2.4** ([20]). *Let $C$ be a nonempty convex subset of a Banach space $X$ and $f : X \to \mathbb{R}$ be a convex and subdifferentiable function. Then $\hat{x}$ is the solution of the convex problem*

$$min\{f(x) : x \in C\}$$

*if and only if*

$$0 \in \partial f(\hat{x}) + N_C(\hat{x}),$$

*where $N_C(\nu) := \Big\{x^* \in X^* : \prec x^*, \nu - y \succ \geq 0, \ \forall \ y \in C\Big\}$ is normal cone of $C$ at $\nu \in C$.*

If $X$ is a real Banach space, a closed half-space in $X$ is a set of the form

$$\{x \in X : \prec x^*, x \succ \leq a\},$$

where $x^* \in X^*$ and $a \in \mathbb{R}$.

The bifunction $f : X \times X \to (-\infty, +\infty]$ is said to be

(i) monotone on $C$, if

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C,$$

(ii) pseudomonotone on $C$, if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0 \quad \forall \, x, y \in C.$$

## 2.6. Monotone mappings

Next, we present the monotone, pseudomonotone and Lipschitz continuous concepts of a mapping $A : H \to H$, where $H$ is a real Hilbert space with the inner product $\langle . \rangle$ and the associated norm $\|.\|$. Assume that $C \subseteq H$ is a nonempty closed convex subset. The mapping $A : H \to H$ is said to be

(i) monotone on $C$ if

$$\langle Ax - Ay, x - y\rangle \geq 0, \qquad \forall x, y \in C,$$

(ii) pseudomonotone on $C$ if

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, x - y\rangle \leq 0, \qquad \forall x, y \in C,$$

(iii) $L$-Lipschitz continuous on $C$ if there exists $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \qquad \forall x, y \in C.$$

It is well known that the monotonicity of mapping $A$ (bifunction $f$) on $C$ imply that $A$ ($f$) is pseudomonotone on $C$. Also, It is clear that if $A : C \to H$ is monotone on $C$ then the corresponding bifunction $f(x, y) = \langle Ax, y - x\rangle$ is monotone on $C$.

## 2.7. Proximity operator

For a proper, convex and lower semicontinuous function $g : H \to (-\infty, \infty]$ and $\gamma > 0$, the Moreau envelope of $g$ of parameter $\gamma$ is the convex function

$$\gamma_g(x) = \inf_{y\in H}\{g(y) + \frac{1}{2\gamma}\|y - x\|^2\} \;\; \forall x \in H.$$

For all $x \in H$, the function

$$y \to g(y) + \frac{1}{2\gamma}\|y - x\|^2$$

is proper, strongly convex and lower semicontinuous, thus the infimum is attained.
The unique minimum of

$$y \to g(y) + \frac{1}{2}\|y - x\|^2$$

is called proximal point of $g$ at $x$ and it is denoted by $prox_g(x)$. The operator

$$prox_g(x) : H \to H$$

$$x \to \text{argmin}_{y\in H}\{g(y) + \frac{1}{2\gamma}\|y - x\|^2\}$$

is well-defined and is said to be the proximity operator of $g$. When $g = i_C$ (the indicator function of the convex set $C$), one has

$$prox_{i_C}(x) = P_C(x)$$

for all $x \in H$.

# 3. Main results

The goal of this section is presenting a new hybrid proximal point algorithm which generates a sequence that strongly converges to the solution of the $(SEP)$ under some mild conditions in Banach spaces. In fact, to get our algorithm, we combine the hybrid method and the proximal point method. One of the convex subprograms of our algorithm only needs to be solved in a half-space instead of on the feasible set. This change simplifies the calculations and may increase the performance of the algorithm.

Throughout this paper, assume that $C$ and $Q$ are nonempty, convex and closed subsets of $p$-uniformly convex and uniformly smooth real Banach spaces $X$ and $Y$, respectively and $A : X \to Y$ is a bounded linear operator. Let $g : Q \times Q \to \mathbb{R}$ be a bifunction, which satisfies the following :

**Condition A:**
(A1) $g(y,y)=0$, for all $y\in Q$,
(A2) $g$ is monotone on $Q$,
(A3) $g(y,.)$ is convex, lower semicontinuous on $Q$ for all $y\in Q$,
(A4) for each $x,y,z\in Q$,

$$\lim_{t\to 0^+} g(tz+(1-t)x,y)\leq g(x,y).$$

Moreover, suppose that the bifunction $f: X\times X\to\mathbb{R}$ satisfies the following :
**Condition B:**
(B1) $f(x,x)=0$, for all $x\in X$,
(B2) $f$ is pseudomonotone on $X$,
(B3) $f(x,.)$ is convex, lower semicontinuous and subdifferentiable on $X$, for all $x\in X$,
(B4) $f$ is jointly weakly continuous on $X\times X$, i.e., if $x,y\in X$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in $X$ converging weakly to $x$ and $y$, respectively, then $f(x_n,y_n)\to f(x,y)$,
(B5) $f$ satisfies $\Delta$-Lipschitz-type condition:

$$\exists c_1,c_2>0 \text{ s.t. } f(x,y)+f(y,z)\geq f(x,z)-c_1\Delta_p^X(x,y)-c_2\Delta_p^X(y,z),\quad \forall x,y,z\in X.$$

From now on, we denote the $\Delta_p^X$ and $J_p^X$ by $\Delta_p$ and $J_p$, respectively, for convenience.

**Remark 3.1.** If $H$ is a Hilbert space, then the bifunction $f: H\times H\to\mathbb{R}$ satisfies Lipschitz-type condition if

$$\exists c_1,c_2>0 \text{ s.t. } f(x,y)+f(y,z)\geq f(x,z)-c_1\|x-y\|-c_2\|y-z\|,\quad \forall x,y,z\in H. \tag{3.1}$$

**Remark 3.2.** Let $A: C\to H$ be Lipschitz continuous with constant $L>0$, if we define $f(x,y):=\langle Ax,y-x\rangle$, then $f$ is a Lipschitz-type mapping with constants $c_1=c_2=\frac{L}{2}$.

Indeed, for all $x,y,z\in C$ we have

$$\begin{aligned}
f(x,y)+f(y,z)-f(x,z) &= \langle Ax,y-x\rangle+\langle Ay,z-y\rangle-\langle Ax,z-x\rangle\\
&= -\langle Ay-Ax,y-z\rangle\\
&\geq -\|Ax-Ay\|\|y-z\|\\
&\geq -L\|x-y\|\|y-z\|\\
&\geq \frac{-L}{2}\|x-y\|^2-\frac{L}{2}\|y-z\|^2\\
&= c_1\|x-y\|^2-c_2\|y-z\|^2.
\end{aligned}$$

Thus, $f$ satisfies the inequality (3.1).

**Remark 3.3.** If there exists $\lambda>0$ such that

$$|f(z,w)-f(x,w)-f(z,y)+f(x,y)|\leq\lambda\|z-x\|\|w-y\|\ \forall x,y,z,w\in C, \tag{3.2}$$

then it is easy to see that $f$ also satisfies the inequality (3.1). The inequality (3.2) is called Lipschitz type inequality and has been introduced by Antibin [3]. In the framework of a finite dimensional space, he showed that if $f$ is a differentiable function whose partial derivative with respect to the first variable satisfies the Lipschitz type inequality, then the inequality (3.2) holds. Therefore, the class of these functions also satisfies the inequality (3.1).

**Remark 3.4.** Assume that $A: H\to H$ is weak to strong continuous mapping on $H$, that is, for each sequence $\{x_n\}\subseteq H$ such that $x_n\rightharpoonup\hat{x}$, then $Ax_n\to A\hat{x}$. Define $f(x,y)=\langle Ax,y-x\rangle$, then $f$ is jointly weakly continuous on $H\times H$, because if $x,y\in X$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in $X$ converging weakly to $x$ and $y$, respectively, then

$$\lim_{n\to\infty} f(x_n,y_n)=\lim_{n\to\infty}\langle Ax_n,y_n-x_n\rangle=\langle Ax,y-x\rangle=f(x,y).$$

**Lemma 3.5.** *Let $Q$ be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space $Y$. Suppose that the bifunction $g : Q \times Q \to \mathbb{R}$ satisfies the condition $A$ and let $r > 0$ and $x \in Y$. Then, there exists $z \in Q$ such that*

$$g(z, y) + \frac{1}{r} \prec J_p^Y z - J_p^Y x, y - z \succ \geq 0, \forall y \in Q.$$

***Proof.*** Using a similar argument such as in Corollary 1 of [4], we can prove the statement. □

**Lemma 3.6.** *Let $Q$ be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space $Y$. Suppose that the bifunction $g : Q \times Q \to \mathbb{R}$ satisfies the condition $A$ and let $r > 0$ and $x \in Y$. Define a mapping $T_r^g : Y \to 2^Q$ as follows:*

$$T_r^g(x) = \{z \in Q : g(z, y) + \frac{1}{r} \prec J_p^Y z - J_p^Y x, y - z \succ \geq 0, \forall y \in Q\}.$$

*Then, the following statements hold:*

(i) *$T_r^g(x)$ is single-valued for all $x \in Y$,*

(ii) *$T_r^g$ is firmly nonexpansive-type, i.e. for all $x, y \in Y$,*

$$\prec J_p^Y T_r^g x - J_p^Y T_r^g y, T_r^g x - T_r^g y \succ \leq \prec J_p^Y x - J_p^Y y, T_r^g x - T_r^g y \succ,$$

(iii) *$F(T_r^g) = EP(g, Q) \neq \emptyset$, where $F(T_r^g)$ is the fixed point set of $T_r^g$,*

(iv) *$\Delta_p^Y(T_r^g x, z) \leq \Delta_p^Y(x, z)$, $\forall z \in F(T_r^g)$, $\forall x \in Y$,*

(v) *$EP(g, Q)$ is closed and convex.*

***Proof.*** (i) We claim that $T_r^g(x)$ is single-valued. Indeed, for $x \in C$ and $r > 0$ let $z_1, z_2 \in T_r^g(x)$. Then,

$$g(z_1, z_2) + \frac{1}{r} \prec J_p^Y z_1 - J_p^Y x, z_2 - z_1 \succ \geq 0$$

and

$$g(z_2, z_1) + \frac{1}{r} \prec J_p^Y z_2 - J_p^Y x, z_1 - z_2 \succ \geq 0.$$

Adding the two inequalities, we have

$$g(z_1, z_2) + g(z_2, z_1) + \frac{1}{r} \prec J_p^Y z_1 - J_p^Y z_2, z_2 - z_1 \succ \geq 0.$$

From (A2) and $r > 0$, we have

$$\prec J_p^Y z_1 - J_p^Y z_2, z_2 - z_1 \succ \geq 0.$$

Therefore, since (2.5), we have $z_1 = z_2$.
(ii) Next, we claim that $T_r^g(x)$ is a firmly nonexpansive-type mapping. Indeed, for $x, y \in Q$, we have

$$g(T_r^g(x), T_r^g(y)) + \frac{1}{r} \prec J_p^Y T_r^g(x) - J_p^Y x, T_r^g(y) - T_r^g(x) \succ \geq 0,$$

and

$$g(T_r^g(y), T_r^g(x)) + \frac{1}{r} \prec J_p^Y T_r^g(y) - J_p^Y y, T_r^g(x) - T_r^g(y) \succ \geq 0.$$

Adding the two inequalities, we have

$$g(T_r^g(x), T_r^g(y)) + g(T_r^g(y), T_r^g(x)) + \frac{1}{r} \prec J_p^Y T_r^g(x) - J_p^Y T_r^g(y) - J_p^Y x + J_p^Y y, T_r^g(y) - T_r^g(x) \succ \geq 0,$$

also, from (A2) and $r > 0$, we have

$$\prec J_p^Y T_r^g(x) - J_p^Y T_r^g(y) - J_p^Y x + J_p^Y y, T_r^g(y) - T_r^g(x) \succ \geq 0,$$

therefore, we get

$$\prec J_p^Y T_r^g(x) - J_p^Y T_r^g(y), T_r^g(x) - T_r^g(y) \succ \leq \prec J_p^Y x - J_p^Y y, T_r^g(x) - T_r^g(y) \succ .$$

(iii) Next, we claim that $F(T_r^g) = EP(g,Q)$. Indeed we have the following:

$$u \in F(T_r^g) \Leftrightarrow u = T_r^g(u) \Leftrightarrow g(u,y) + \frac{1}{r} \prec J_p^Y u - J_p^Y u, y - u \succ \geq 0, \forall y \in Q$$

$$\Leftrightarrow g(u,y) \geq 0, \forall y \in Q \Leftrightarrow u \in EP(g,Q).$$

(iv) From (ii) we have, for $x, y \in Q$,

$$\prec J_p^Y T_r^g(x) - J_p^Y T_r^g(y), T_r^g(x) - T_r^g(y) \succ \leq \prec J_p^Y x - J_p^Y y, T_r^g(x) - T_r^g(y) \succ .$$

Moreover, we have

$$\Delta_p(T_r^g(x), T_r^g(y)) + \Delta_p(T_r^g(y), T_r^g(x)) = \prec J_p^Y T_r^g(x) - J_p^Y T_r^g(y), T_r^g(x) - T_r^g(y) \succ$$

and from (2.3) we get

$$\Delta_p^Y(y, T_r^g(x)) + \Delta_p^Y(x, T_r^g(y)) - \Delta_p^Y(x, T_r^g(x)) - \Delta_p^Y(y, T_r^g(y)) = \prec J_p^Y x - J_p^Y y, T_r^g(x) - T_r^g(y) \succ .$$

Hence from (ii) we have

$$\begin{aligned} \Delta_p^Y(T_r^g(x), T_r^g(y)) + \Delta_p^Y(T_r^g(y), T_r^g(x)) \leq \Delta_p^Y(y, T_r^g(x)) + \Delta_p^Y(x, T_r^g(y)) \\ - \Delta_p^Y(x, T_r^g(x)) - \Delta_p^Y(y, T_r^g(y)), \end{aligned} \tag{3.3}$$

so, we conclude, for $x, y \in Q$,

$$\Delta_p^Y(T_r^g(x), T_r^g(y)) + \Delta_p^Y(T_r^g(y), T_r^g(x)) \leq \Delta_p^Y(y, T_r^g(x)) + \Delta_p^Y(x, T_r^g(y)).$$

Taking $y = z \in F(T_r^g)$, we have

$$\Delta_p^Y(T_r^g(x), z) \leq \Delta_p^Y(x, z)$$

(v) Next, we claim that $EP(g,Q)$ is closed and convex. Indeed, from (iii) we have $EP(g,Q) = F(T_r^g)$. Now, we show that $\widehat{F}(T_r^g) = EP(g,Q)$. Let $p \in \widehat{F}(T_r^g)$. Then, there exists $\{z_n\} \subseteq Y$ such that $z_n \rightharpoonup p$ and $\lim_{n\to\infty}(z_n - T_r^g z_n) = 0$. Moreover, we get $T_r^g z_n \rightharpoonup p$. Hence we have $p \in Q$. Since $J_p^Y$ is uniformly continuous on bounded sets, we have

$$\lim_{n\to\infty} \frac{1}{r} \| J_p^Y z_n - J_p^Y T_r^g z_n \|_{Y^*} = 0.$$

From the definition of $T_r^g$, we have

$$g(T_r^g z_n, y) + \frac{1}{r} \prec J_p^Y T_r^g z_n - J_p^Y z_n, y - T_r^g z_n \succ \geq 0.$$

Since

$$\frac{1}{r} \prec J_p^Y T_r^g z_n - J_p^Y z_n, y - T_r^g z_n \succ \geq -g(T_r^g z_n, y) \geq g(y, T_r^g z_n),$$

and $g$ is lower semicontinuous and convex in the second variable. So, we have

$$g(y,p) \leq \liminf_{n\to\infty} g(y, T_r^g z_n) \leq 0.$$

Therefore, we have

$$g(p,y) \geq 0, \ \forall y \in Q.$$

Hence $p \in EP(g,Q)$. So, we get $F(T_r^g) = EP(g,Q) = \widehat{F}(T_r^g)$. Therefore, we have $T_r^g$ is a relatively nonexpansive mapping. From Lemma 2.2, $EP(g,Q) = F(T_r^g)$ is closed and convex. □

**Remark 3.7.** In the rest of the paper, for convenience, we denote $J_p^X$ by $J_p$, $J_q^{X^*}$ by $J_q^*$ and $\Delta_p^X$ by $\Delta_p$.

Now we present a new algorithm and show that it generates iterates that converge strongly to a solution of the $(SEP)$ under some mild assumptions.

---

**Algorithm** 1 **(Hybrid proximal point algorithm for SEPs):**

**Step 0:** Suppose that $0 < a \leq \lambda \leq b < \min\{\frac{1}{c_1}, \frac{1}{c_2}\}$, $\{r_n\} \subseteq (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$, $\{\alpha_n\} \subseteq [0,d]$ for some $d<1$ and $\{\beta_n\} \subseteq [0,1]$.
**Step 1:** Let $x_0 \in C$ and $n=0$,
**Step 2:** Compute $y_n$ such that

$$y_n = \operatorname{argmin}_{y\in C}\{\lambda f(x_n,y) + \Delta_p(x_n,y)\},$$

**Step 3:** Let $\omega_n \in \partial_2 f(x_n,y_n)$ such that there exists an element $q_n \in N_C(y_n)$ satisfying

$$q_n = J_p x_n - J_p y_n - \lambda\omega_n \tag{3.4}$$

and construct the half-space

$$H_n = \{z \in X : \prec J_p x_n - J_p y_n - \lambda\omega_n, z - y_n \succ \leq 0\},$$

**Step 4:** Compute $z_n$ such that

$$z_n = \operatorname{argmin}_{y\in H_n}\{\lambda f(y_n,y) + \Delta_p(x_n,y)\},$$

**Step 5:** Let $t_n = J_q^*(\beta_n J_p x_n + (1-\beta_n)J_p z_n)$ and $v_n = \Pi_Q(At_n)$, where $\Pi_Q$ is the Bregman projection from $X$ onto $Q$.
**Step 6:** Let $u_n \in Q$ *such that* $g(u_n,y) + \frac{1}{r_n} \prec J_p^Y u_n - J_p^Y v_n, y - u_n \succ \geq 0\,,\ \forall y \in Q$, and compute

$$w_n = J_q^{Y*}\Big(\alpha_n J_p^Y v_n + (1-\alpha_n)J_p^Y u_n\Big),$$

**Step 7:** Compute $x_{n+1} = \Pi_{D_n\cap A^{-1}(E_n)}(x_n)$, where $\Pi_{D_n\cap A^{-1}(E_n)}$ is the Bregman projection from $X$ onto $D_n \cap A^{-1}(E_n)$ and

$$E_n = \Big\{z \in Q : \Delta_p^Y(w_n,z) \leq \Delta_p^Y(v_n,z)\Big\},$$

$$D_n = \Big\{z \in C : \Delta_p(z_n,z) \leq \Delta_p(x_n,z)\Big\},$$

**Step 8:** put $n = n+1$ and go to Step 2.

---

**Remark 3.8.** The existence of $w_n \in \partial_2 f(x_n,y_n)$ and $q_n \in N_C(y_n)$ satisfying (3.4) is guaranted by Lemma 2.3. Hence, Algorithm 1 is well-defined.

**Remark 3.9.** It should be noted that the definitions $y_n$ and $z_n$ in Step 2 and step 4 are well-defined because $f(x,.)$ is proper, convex and lower semicontinuous . Also lemmas 2.4 and 3.6 guarantee the existence of $\omega_n$ and $u_n$, respectively. Moreover, since $Q$ and $D_n \cap A^{-1}(E_n)$ are nonempty closed convex, by the definition of Bregman projection $\Pi_Q$ and $\Pi_{D_n\cap A^{-1}(E_n)}(x_n)$ are uniquely determined.

We claim that the iterates generated by Algorithm 1 converge to $\lim_{n\to\infty}\Pi_\Omega x_n$ strongly, where $\Omega$ is the solution set of (SEP) and $\Pi_\Omega$ is the Bregman projection onto $\Omega$.
At first, we prove some lemmas which are useful in the proof of our main results in this paper.

**Lemma 3.10.** *For each $n \in \mathbb{N}\cup\{0\}$, $C \subseteq H_n$.*

***Proof.*** Utilizing lemmas 2.3 and 2.4, we get

$$y_n = \text{argmin}_{y \in C}\{\lambda f(x_n, y) + \Delta_p(x_n, y)\} \Leftrightarrow 0 \in \lambda \partial_2 f(x_n, y_n) + \nabla_2 \Delta_p(x_n, y_n) + N_C(y_n).$$

Using Proposition 4.9 of [7], we have $\partial(\frac{\|\cdot\|^p}{p}) = J_p(\cdot)$ and so from (2.3) , we deduce that $\nabla_2 \Delta_p(x_n, y_n) = J_p y_n - J_p x_n$. Thus $\omega_n \in \partial_2 f(x_n, y_n)$ and $q_n \in N_C(y_n)$ exist such that

$$q_n = J_p x_n - J_p y_n - \lambda\omega_n, \quad \forall n \geq 1.$$

Since $N_C(y_n) = \Big\{q \in X : \prec q, y - y_n \succ \leq 0 \;\; , \forall y \in C\Big\}$, we deduce

$$\prec J_p x_n - J_p y_n - \lambda\omega_n, y - y_n \succ \leq 0 \;\; \forall y \in C, \;\; \forall n \geq 1.$$

This shows that $C \subseteq H_n, \quad \forall n \in \mathbb{N} \cup \{0\}$. □

**Lemma 3.11.** *Assume that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are the sequences generated by Algorithm 1 and $x^* \in \Omega$, where $\Omega$ is the solution set of $(SEP)$. Let $c_1$ and $c_2$ be the $\Delta$-Lipschitz-type constants of the bifunction $f$. Then*
*(i) $\prec J_p x_n - J_p z_n, y - z_n \succ \leq \lambda[f(y_n, y) - f(y_n, z_n)]$, for all $y \in H_n$,*
*(ii) $\Delta_p(z_n, x^*) \leq \Delta_p(x_n, x^*) - (1 - \lambda c_1)\Delta_p(x_n, y_n) - (1 - \lambda c_2)\Delta_p(y_n, z_n)$,*

***Proof.*** Since $z_n = \text{argmin}_{y \in H_n}\Big\{\lambda f(y_n, y) + \Delta_p(x_n, y)\Big\}$, using lemmas 2.3 and 2.4, we have

$$0 = \lambda\omega'_n + J_p z_n - J_p x_n + q'_n, \quad q'_n \in N_{H_n}(z_n), \quad \omega'_n \in \partial_2 f(y_n, z_n). \tag{3.5}$$

From the definitions of $\partial_2 f(y_n, z_n)$ and $N_{H_n}(z_n)$ we get

$$f(y_n, y) - f(y_n, z_n) \geq \prec \omega'_n, y - z_n \succ, \;\; \forall y \in X, \tag{3.6}$$

and

$$\lambda \prec \omega'_n, y - z_n \succ \geq \prec J_p x_n - J_p z_n, y - z_n \succ \quad \forall y \in H_n. \tag{3.7}$$

Therefore we conclude

$$\prec J_p x_n - J_p z_n, y - z_n \succ \leq \lambda(f(y_n, y) - f(y_n, z_n)) \,, \;\; \forall y \in H_n. \tag{3.8}$$

Putting $y = x^*$ in (3.8), we have

$$\prec J_p x_n - J_p z_n, x^* - z_n \succ \leq \lambda(f(y_n, x^*) - f(y_n, z_n)). \tag{3.9}$$

It follows from (2.4), (2.5) and (3.9), that

$$\begin{aligned}
\Delta_p(z_n, x^*) &= \Delta_p(z_n, x_n) + \Delta_p(x_n, x^*) + \prec J_p z_n - J_p x_n, x_n - x^* \succ \\
&= \Delta_p(x_n, x^*) - \Delta_p(x_n, z_n) + \prec J_p z_n - J_p x_n, z_n - x_n \succ \\
&\quad + \prec J_p z_n - J_p x_n, x_n - x^* \succ \\
&= \Delta_p(x_n, x^*) - \Delta_p(x_n, z_n) + \prec J_p z_n - J_p x_n, z_n - x^* \succ \\
&\leq \Delta_p(x_n, x^*) - \Delta_p(x_n, z_n) + \lambda(f(y_n, x^*) - f(y_n, z_n)) \\
&= \Delta_p(x_n, x^*) - \Delta_p(x_n, z_n) + \lambda(f(x_n, y_n) - f(x_n, z_n)) \\
&\quad + \lambda(f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n)) + \lambda f(y_n, x^*).
\end{aligned} \tag{3.10}$$

Since $z_n \in H_n$, we have $\prec J_p x_n - J_p y_n, z_n - y_n \succ \leq \lambda \prec \omega_n, z_n - y_n \succ$, where $\omega_n \in \partial_2 f(x_n, y_n)$ and therefore

$$f(x_n, y) - f(x_n, y_n) \geq \prec \omega_n, y - y_n \succ \quad \forall\, y \in X. \tag{3.11}$$

Setting $y = z_n$ in (3.11), we conclude

$$\lambda(f(x_n, y_n) - f(x_n, z_n)) \leq \lambda \prec \omega_n, y_n - z_n \succ \leq \prec J_p y_n - J_p x_n, z_n - y_n \succ .$$

On the other hand

$$\prec J_p y_n - J_p x_n, z_n - y_n \succ = \Delta_p(x_n, z_n) - \Delta_p(x_n, y_n) - \Delta_p(y_n, z_n). \tag{3.12}$$

Replacing, $x, y$ and $z$ by $x_n, y_n$ and $z_n$ in $(B5)$, respectively, we get

$$\lambda(f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n)) \leq \lambda c_1 \Delta_p(x_n, y_n) + \lambda c_2 \Delta_p(y_n, z_n). \tag{3.13}$$

Therefore, using (3.10), (3.12) and (3.13), we can derive that

$$\begin{aligned}\Delta_p(z_n, x^*) &\leq \Delta_p(x_n, x^*) - \Delta_p(x_n, y_n) - \Delta_p(y_n, z_n) + \lambda c_1 \Delta_p(x_n, y_n) + \lambda c_2 \Delta_p(y_n, z_n) \\ &= \Delta_p(x_n, x^*) - (1 - \lambda c_1)\Delta_p(x_n, y_n) - (1 - \lambda c_2)\Delta_p(y_n, z_n).\end{aligned}$$

So, the proof is complete. □

**Lemma 3.12.** *Let $f$ and $g$ be the bifunctions which satisfy conditions A and B, respectively. Then $\Omega$ is convex and closed.*

***Proof.*** To show the closedness of $\Omega$, Suppose that $x_n \in \Omega$, for all $n \in \mathbb{N} \cup \{0\}$, such that $x_n \to \hat{x}$. This implies that $x_n \in C$ for all $n \in \mathbb{N} \cup \{0\}$ such that $f(x_n, y) \geq 0$ for all $y \in C$ and all $n \in \mathbb{N} \cup \{0\}$ and $Ax_n \in Q$ for all $n \in \mathbb{N} \cup \{0\}$ such that $g(Ax_n, z) \geq 0$ for all $z \in Q$ and all $n \in \mathbb{N} \cup \{0\}$. Also, $Ax_n \to A\hat{x}$, because $A$ is bounded linear. Closedness of $C$ and $Q$ implies that $\hat{x} \in C$ and $A\hat{x} \in Q$ and so from condition (B4), we obtain $f(\hat{x}, y) \geq 0$ for all $y \in C$. It follows from Lemma 3.6(v) that $EP(g, Q)$ is closed, so $g(A\hat{x}, z) \geq 0$ for all $z \in Q$. Therefore $\hat{x} \in \Omega$.

For proving convexity of $\Omega$, assume that $x_1, x_2 \in \Omega$ and $0 \leq \lambda \leq 1$. So, $\lambda x_1 + (1-\lambda)x_2 \in C$ and $\lambda Ax_1 + (1 - \lambda)Ax_2 \in Q$, since of $C$ and $Q$ are convex. Utilising conditions (B2) and (B3), we get

$$f(y, \lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(y, x_1) + (1 - \lambda)f(y, x_2) \leq 0, \tag{3.14}$$

for all $y \in C$ and using the conditions (A2) and (A3), we have

$$g(z, \lambda Ax_1 + (1 - \lambda)Ax_2) \leq \lambda g(z, Ax_1) + (1 - \lambda)g(z, Ax_2) \leq 0, \tag{3.15}$$

for all $z \in Q$. Let $y \in C$, $z \in Q$ and $0 < t < 1$ and $y_t = ty + (1 - t)(\lambda x_1 + (1 - \lambda)x_2)$ and $z_t = tz + (1 - t)(\lambda Ax_1 + (1 - \lambda)Ax_2)$. Using (B1) and (B3) and (3.14), we can conclude that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)f(y_t, \lambda x_1 + (1 - \lambda)x_2) \leq tf(y_t, y)$$

and also utilizing conditions (A1) and (A3) and (3.15), we get

$$0 = g(z_t, z_t) \leq tg(z_t, z) + (1 - t)g(z_t, \lambda Ax_1 + (1 - \lambda)Ax_2) \leq tg(z_t, z).$$

Hence, $f(y_t, y) \geq 0$ and $g(z_t, z) \geq 0$. Letting as $t \to 0$ and using (A4) and (B4), we yield that $f(\lambda x_1 + (1 - \lambda)x_2, y) \geq 0$ and $g(\lambda Ax_1 + (1 - \lambda)Ax_2, z) \geq 0$. So $\lambda x_1 + (1 - \lambda)x_2 \in \Omega$. Since $y \in C$ and $z \in Q$ had been selected arbitrarily, we derive $\Omega$ is convex. □

**Lemma 3.13.** *The generated sequence $\{x_n\}$ in Algorithm 1 is well defined.*

***Proof.*** For any $n \in \mathbb{N} \cup \{0\}$, we prove $D_n \cap A^{-1}(E_n)$ is nonempty, convex and closed. It is readily seen that $E_n$ and $D_n$ are closed for all $n \in \mathbb{N} \cup \{0\}$. Since

$$\Delta_p(w_n, z) \leq \Delta_p(v_n, z) \Leftrightarrow \prec J_p^Y v_n - J_p^Y w_n, z \succ \leq \frac{1}{q}(\|v_n\|^p - \|w_n\|^p),$$

and

$$\Delta_p(z_n, z) \leq \Delta_p(x_n, z) \Leftrightarrow \prec J_p x_n - J_p z_n, z \succ \leq \frac{1}{q}(\|x_n\|^p - \|z_n\|^p),$$

so $E_n$ and $D_n$ are the half-spaces and therefore, are convex. Thus $D_n \cap A^{-1}(E_n)$ is convex and closed for all $n \in \mathbb{N} \cup \{0\}$, due to the assumption that $A$ is bounded and linear.

Now we prove that $D_n \cap A^{-1}(E_n)$ is nonempty. Since $\Omega$ is nonempty, it is sufficient to show that $\Omega \subseteq D_n \cap A^{-1}(E_n)$ for all $n \in \mathbb{N} \cup \{0\}$. To do this, let $x^* \in \Omega$ and $n$ be a fixed positive integer number. At first, we show that $Ax^* \in E_n$. Using Lemma 3.6(iv) we have

$$\Delta_p^Y(u_n, Ax^*) = \Delta_p^Y(T_{r_n}^g v_n, Ax^*) \leq \Delta_p^Y(v_n, Ax^*), \tag{3.16}$$

since, $Ax^* \in F(T^g_{r_n})$. Now, it follows from (2.3) and (3.16), that

$$\begin{aligned}
&\Delta_p^Y(w_n, Ax^*)\\
&= \frac{1}{q}\|w_n\|^p - \prec J_p w_n, Ax^* \succ + \frac{1}{p}\|Ax^*\|^p\\
&= \frac{1}{q}\|\alpha_n J_p^Y v_n + (1-\alpha_n)J_p^Y u_n\|^q_{Y^*} - \prec \alpha_n J_p^Y v_n + (1-\alpha_n)J_p^Y u_n, Ax^* \succ + \frac{1}{p}\|Ax^*\|^p\\
&\leq \frac{1}{q}\Big(\alpha_n\|v_n\|^p + (1-\alpha_n)\|u_n\|^p\Big) - \alpha_n \prec J_p^Y v_n, Ax^* \succ - (1-\alpha_n) \prec J_p^Y u_n, Ax^* \succ\\
&\quad + \frac{1}{p}\|Ax^*\|^p\\
&\leq \alpha_n\Big(\frac{1}{q}\|v_n\|^p - \prec J_p^Y v_n, Ax^* \succ + \frac{1}{p}\|Ax^*\|^p\Big) + (1-\alpha_n)(\frac{1}{q}\|u_n\|^p - \prec J_p^Y u_n, Ax^* \succ\\
&\quad + \frac{1}{p}\|Ax^*\|^p)\\
&= \alpha_n\Delta_p^Y(v_n, Ax^*) + (1-\alpha_n)\Delta_p^Y(u_n, Ax^*) \leq \Delta_p^Y(v_n, Ax^*).
\end{aligned} \tag{3.17}$$

Therefore, $Ax^* \in E_n$, due to the definition of $E_n$, i.e., $\Omega \subseteq A^{-1}(E_n)$, for all $n \in \mathbb{N} \cup \{0\}$. On the other hand, utilizing lemma 3.11(ii) we get

$$\Delta_p(z_n, x^*) \leq \Delta_p(x_n, x^*).$$

So, a glance at the definition of $D_n$, yields, $x^* \in D_n$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $\Omega \subseteq A^{-1}(E_n) \cap D_n$, for all $n \in \mathbb{N} \cup \{0\}$. □

**Theorem 3.14.** *Assume that $\Omega \neq \phi$, then the generated sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$, in Algorithm 1 are strongly convergent to the same solution $u \in \Omega$, where $u = \lim_{n\to\infty} \Pi_\Omega x_n$. Moreover, the sequences $\{v_n\}$, $\{u_n\}$ and $\{w_n\}$ are strongly convergent to $Au$.*

***Proof.*** Suppose that $x^* \in \Omega \subseteq A^{-1}(E_n) \cap D_n$. Since $x_{n+1} = \Pi_{A^{-1}(E_n)\cap D_n}(x_n)$ and $x^* \in D_n$ for all $n \in \mathbb{N} \cup \{0\}$, utilizing Lemma 2.1(ii), we get

$$\Delta_p(x_{n+1}, x^*) \leq \Delta_p(x_n, x^*) - \Delta_p(x_n, x_{n+1}).$$

This yields $\lim_{n\to\infty} \Delta_p(x_n, x^*)$ exists. Therefore $\lim_{n\to\infty} \Delta_p(x_n, x_{n+1}) = 0$ and so (2.6) implies that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, i.e., $\{x_n\}$ is a cauchy sequence. Consequently $x_n$ converges strongly to $u \in C$ such that $Au \in Q$, since $D_n \cap A^{-1}(E_n) \subseteq C \cap A^{-1}Q$, for all $n \in \mathbb{N} \cup \{0\}$ and $C \cap A^{-1}Q$ is closed. Furthermore, using uniform norm-to-norm continuity of $J_p$ on bounded sets, we have

$$\lim_{n\to\infty} \Delta_p(z_n, x_{n+1}) \leq \lim_{n\to\infty} \Delta_p(x_n, x_{n+1}) = 0, \tag{3.18}$$

since $x_{n+1} \in D_n$. Thus, utilizing (2.6) we conclude $\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0$. This means $\{z_n\}$ converges strongly to $u$. Moreover, we have

$$\Delta_p(t_n, u) \leq \beta_n\Delta_p(x_n, u) + (1-\beta_n)\Delta_p(z_n, u) \leq \beta_n\Delta_p(x_n, u) + (1-\beta_n)\Delta_p(x_n, u) = \Delta_p(x_n, u),$$

due to $u \in D_n$. So,

$$\lim_{n\to\infty} \Delta_p(t_n, u) \leq \lim_{n\to\infty} \Delta_p(x_n, u) = \lim_{n\to\infty} \Big(\frac{1}{q}\|x_n\|^p - \prec J_p x_n, u \succ + \frac{1}{p}\|u\|^p\Big) = 0. \tag{3.19}$$

Hence, (3.19) implies that $\{t_n\}$ is bounded and $t_n \to u$. Therefore $At_n \to Au$, since $A$ is bounded linear. So,

$$\lim_{n\to\infty} \Delta_p^Y(v_n, Au) \leq \lim_{n\to\infty} \Delta_p^Y(At_n, Au) = \lim_{n\to\infty} (\frac{1}{q}\|At_n\|^p - \prec J_p^Y At_n, Au \succ + \frac{1}{p}\|Au\|^p) = 0,$$

due to $v_n = \Pi_Q(At_n)$. Therefore, it follows from (2.6) that $\lim_{n\to}\|v_n - Au\| = 0$. Furthermore, since $x_{n+1} \in A^{-1}(E_n)$ for all $n \in \mathbb{N} \cup \{0\}$ and $A^{-1}(E_n)$ is closed, we have $u \in A^{-1}(E_n)$ for all $n \in \mathbb{N} \cup \{0\}$, i.e., $Au \in E_n$ for all $n \in \mathbb{N} \cup \{0\}$. So, a glance at the definition of $E_n$, yields that

$$\Delta_p^Y(w_n, Au) \leq \Delta_p^Y(v_n, Au),$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore,

$$\lim_{n\to\infty} \Delta_p^Y(w_n, Au) \leq \lim_{n\to\infty} \Delta_p^Y(v_n, Au) = 0.$$

Thus, utilizing (2.6), we conclude $\lim_{n\to\infty}\|w_n - Au\| = 0$ and hence, $\lim_{n\to\infty}\|w_n - v_n\| = 0$. Moreover, from $J_p^Y w_n = \alpha_n J_p^Y v_n + (1-\alpha_n)J_p^Y u_n$ and $0 \leq \alpha_n \leq d$, we derive

$$\|J_p^Y u_n - J_p^Y v_n\|_{Y^*} = \frac{1}{1-\alpha_n}\|J_p^Y w_n - J_p^Y v_n\|_{Y^*} \leq \frac{1}{1-d}\|J_p^Y w_n - J_p^Y v_n\|_{Y^*}.$$

Thus,

$$\lim_{n\to\infty}\|J_p^Y u_n - J_p^Y v_n\|_{Y^*} \leq \lim_{n\to\infty}\frac{1}{1-d}\|J_p^Y w_n - J_p^Y v_n\|_{Y^*} = 0.$$

Since, $\liminf_{n\to\infty} r_n > 0$, we derive that

$$\lim_{n\to\infty}\frac{1}{r_n}\|J_p^Y u_n - J_p^Y v_n\|_{Y^*} = 0. \tag{3.20}$$

Also, because $J_q^{Y^*}$ is uniformly norm-to-norm continuous on bounded sets, we can conclude that $\lim_{n\to\infty}\|u_n - v_n\| = 0$.

Now, we show that $u \in \Omega$. Utilizing Lemma 3.11(ii) we obtain

$$(1-\lambda c_1)\Delta_p(x_n, y_n) \leq \Delta_p(x_n, x^*) - \Delta_p(z_n, x^*).$$

It follows from (2.6) that

$$\tau(1-\lambda c_1)\|x_n - y_n\| \leq (1-\lambda c_1)\Delta_p(x_n, y_n) \leq \Delta_p(x_n, x^*) - \Delta_p(z_n, x^*), \tag{3.21}$$

where $\tau > 0$ is a fixed number. Letting $n \to \infty$ in (3.21), we conclude that $\lim_{n\to\infty}\|x_n - y_n\| = 0$, since $(1-\lambda c_1) > 0$, therefore, $y_n \to u$. On the other hand, using Lemma 3.11(i), we yields

$$\prec J_p x_n - J_p z_n, y - z_n \succ \leq \lambda f(y_n, y) - \lambda f(y_n, z_n), \tag{3.22}$$

for all $y \in H_n$. Taking the limits as $n \to \infty$ in (3.22), it follows from conditions (B1), (B4) and uniform continuity of $J_p$ on bounded sets that $f(u, y) \geq 0$, $\forall y \in H_n$. Since $C \subseteq H_n$, we can conclude that $u \in EP(f, C)$. Since $u_n = T_{r_n} v_n$, we get

$$g(u_n, y) + \frac{1}{r_n} \prec J_p^Y u_n - J_p^Y v_n, y - u_n \succ \geq 0,$$

for all $y \in Q$. So, the monotonicity of $g$ implies that

$$\frac{1}{r_n} \prec J_p^Y u_n - J_P^Y v_n, y - u_n \succ \geq -g(u_n, y) \geq g(y, u_n),$$

for all $y \in Q$. Taking limit $n \to \infty$, in above inequality and using (3.20) and the condition (A3), we can conclude that $g(y, Au) \leq 0$ for all $y \in Q$. Since, $g$ is monotone, so we derive $g(Au, y) \geq 0$, for all $y \in Q$. In other words, $u \in \Omega$.

Now, we prove that $u = \lim_{n\to\infty}\Pi_\Omega(x_n)$. To do this, assume that $k_n = \Pi_\Omega(x_n)$. Since $x_{n+1} = \Pi_{D_n \cap A^{-1}(E_n)}(x_n)$, we derive from Lemma 2.1(ii) that

$$\Delta_p(x_{n+1}, k_n) \leq \Delta_p(x_n, k_n). \tag{3.23}$$

Therefore, due to $k_n \in \Omega$, we obtain

$$\Delta_p(x_{n+1}, k_{n+1}) = \Delta_p(x_{n+1}, \Pi_\Omega(x_{n+1})) \leq \Delta_p(x_{n+1}, k_n) \leq \Delta_p(x_n, k_n).$$

So, $\{\Delta_p(x_n, k_n)\}$ is bounded and decreasing and therefore $\lim_{n\to\infty} \Delta_p(x_n, k_n)$ exists. On the other hand, we deduce from Lemma 2.1(ii) and (3.23) that

$$\Delta_p(k_{n+m}, k_n) + \Delta_p(k_{n+m}, x_{n+m}) \leq \Delta_p(x_{n+m}, k_n) \leq \Delta_p(x_n, k_n),$$

because $k_{n+m} = \Pi_\Omega(x_{n+m})$. Hence, utilizing above inequality and (2.6), there exists $\tau > 0$ such that

$$\tau\|k_{n+m} - k_n\| \leq \Delta_p(k_{n+m}, k_n) \leq \Delta_p(x_n, k_n) - \Delta_p(x_{n+m}, k_{n+m}) \to 0, \ as\ n \to \infty.$$

Therefore $\{k_n\} \subseteq \Omega$ is a Cauchy sequence and therefore converges strongly to $q \in \Omega$, since according to Lemma 3.12, $\Omega$ is closed. Using Lemma 2.1(i), we get

$$\prec J_p x_n - J_p k_n, y - k_n \succ \leq 0, \quad \forall y \in \Omega.$$

Putting $y = u \in \Omega$ and letting $n \to \infty$ on both sides of above inequality and using uniform continuity of $J_p$ on bounded sets, we conclude, $\prec J_p u - J_p q, u - q \succ \leq 0$. Moreover, the monotonicity of $J_p$ implies that $\prec J_p u - J_p q, u - q \succ \geq 0$. Consequently, $u = q$, because of $J_p$ is one to one. Therefore $x_n \to u$, where $u = \lim_{n\to\infty} \Pi_\Omega(x_n)$. □

The following remark follows from Theorem 3.14 immediately.

**Remark 3.15.** If $X$ and $Y$ are the Hilbert spaces $H_1$ and $H_2$, respectively, then Steps 2-8 in Algorithm 1 reduces to the following form :

Step 2 : Compute $y_n$

$$y_n = \operatorname{argmin}_{y\in C}\{\lambda f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\},$$

Step 3 : Let $\omega_n \in \partial_2 f(x_n, y_n)$ such that there exists an element $q_n \in N_C(y_n)$ satisfying

$$q_n = x_n - y_n - \lambda\omega_n$$

and construct the half-space

$$H_n = \{z \in X : \prec x_n - y_n - \lambda\omega_n, z - y_n \succ \leq 0\},$$

Step 4 : Compute $z_n$

$$z_n = \operatorname{argmin}_{y\in H_n}\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\},$$

Step 5 : Put $t_n = \beta_n x_n + (1 - \beta_n)z_n$ and $v_n = \frac{1}{2}P_Q(At_n)$,

Step 6 : Put $u_n \in Q$ *such that* $g(u_n, y) + \frac{1}{r_n} \prec y - u_n, u_n - v_n \succ \geq 0$, $\forall y \in Q$, and compute

$$w_n = \alpha_n v_n + (1 - \alpha_n)u_n,$$

Step 7 : Compute $x_{n+1} = \frac{1}{2}P_{D_n\cap A^{-1}(E_n)}(x_n)$, where $P_{D_n\cap A^{-1}(E_n)}$ is the metric projection from $X$ onto $D_n \cap A^{-1}(E_n)$ in which

$$E_n = \{z \in Q : \|w_n - z\| \leq \|v_n - z\|\},$$

$$D_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\},$$

Step 8 : Put $n = n + 1$ and go to Step 2.

## 4. Application to the split variational inequality problems

Now, we give an application of our algorithm to solving the split variational inequality problems $(SVIP)$ in Hilbert spaces. For this purpose, following [6], we briefly introduce $(SVIP)$. Throughout this section, assume that $H_1$ and $H_2$ are two real Hilbert spaces and $A : H_1 \to H_2$ is a bounded linear operator. Also, let $B : H_1 \to H_1$ and $F : H_2 \to H_2$ be two operators and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex subsets. So, a point $x^* \in C$ is a solution of the $(SVIP)$, if

$$\langle B(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C$$

and also if the point $y^* = Ax^* \in Q$ satisfies

$$\langle F(y^*), y - y^* \rangle \geq 0, \quad \forall y \in Q.$$

The solution set of (SVIP) is denoted by

$$\Gamma = \{p \in VI(B, C) : \; Ap \in VI(F, Q)\}.$$

Now, considering $f(x, y) := \langle B(x), y - x \rangle$ and $g(u, v) := \langle F(u), u - v \rangle$ in Remark 3.15 and using the above assumptions, we can present an algorithm for finding the solution of $(SVIP)$.

---

**Algorithm 2 (Hybrid proximal point algorithm for SVIPs):**

**Step 0:** : Suppose that $0 < a \leq \lambda \leq b < \frac{2}{L}$, $\{r_n\} \subseteq (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$, $\{\alpha_n\} \subseteq [0, d]$ for some $d < 1$ and $\{\beta_n\} \subseteq [0, 1]$.
**Step 1:** : Let $x_0 \in C$ and $n = 0$,
**Step 2:** : Compute $y_n$ such that

$$y_n = P_C(x_n - \lambda B x_n),$$

**Step 3:** : Construct the half-space

$$T_n = \{z \in H_1 : \langle x_n - y_n - \lambda B x_n, z - y_n \rangle \leq 0\},$$

**Step 4:** : Compute $z_n$ such that

$$z_n = P_{T_n}(x_n - \lambda B y_n),$$

**Step 5:** : Put $t_n = \beta_n x_n + (1 - \beta_n) z_n$ and $v_n = \frac{1}{2} P_Q(At_n)$ where $P_Q$ is the metric projection from $X$ on to $Q$.
**Step 6:** : Put $u_n \in Q$ *such that* $\langle F(u_n) + \frac{1}{r_n}(u_n - v_n), y - u_n \rangle \geq 0$ , $\forall y \in Q$, and compute

$$w_n = \alpha_n v_n + (1 - \alpha_n) u_n,$$

**Step 7:** : Compute $x_{n+1} = \frac{1}{2} P_{D_n \cap A^{-1}(E_n)}(x_n)$, where $P_{D_n \cap A^{-1}(E_n)}$ is the metric projection from $X$ on to $D_n \cap A^{-1}(E_n)$ and

$$E_n = \{z \in Q : \|w_n - z\| \leq \|v_n - z\|\},$$

$$D_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\},$$

**Step 8:** : Put $n = n + 1$ and return to Step 2.

---

**Corollary 4.1.** *Assume that $B$ is $L$-Lipschitz continuous and pseudomonotone on $H_1$ and $F$ is monotone on $Q$. Also, let $B$ be weak to strong continuous mapping on $H_1$, that is, for each sequence $\{x_n\} \subseteq H_1$ such that $x_n \rightharpoonup \hat{x}$, then $Bx_n \to B\hat{x}$. Moreover, $F$ is weak to strong continuous mapping on $Q$ and $\Gamma \neq \emptyset$. Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ generated by Algorithm 2 converge strongly to the same point $x^* \in \Gamma$. Moreover, the sequences $\{v_n\}$, $\{u_n\}$ and $\{w_n\}$ converge strongly to $Ax^*$. where $\Gamma$ is the solution set of (SVIP) and $P$ is the metric projection.*

***Proof.*** Define

$$f(x, y) = \langle Bx, y - x \rangle, \qquad \forall x, y \in H_1,$$

and

$$g(u, v) = \langle Fu, v - u \rangle, \qquad \forall u, v \in Q.$$

It is readily seen that $f$ is a Lipschitz-type mapping with constants $c_1 = c_2 = \frac{L}{2}$. Using the assumptions, it is easy to check that conditions A and B hold. Also, Step 2 and Step 4 in Remark 3.15 reduce to

$$y_n = \operatorname{argmin}_{y\in C}\{\lambda\langle Bx_n, y - x_n\rangle + \frac{1}{2}\|y - x_n\|^2\},$$

$$z_n = \operatorname{argmin}_{y\in T_n}\{\lambda\langle By_n, y - y_n\rangle + \frac{1}{2}\|y - x_n\|^2\}.$$

It follows that

$$y_n = \operatorname{argmin}_{y\in C}\{\frac{1}{2}\|y - (x_n - \lambda Bx_n)\|^2\} = P_C(x_n - \lambda Bx_n),$$

$$z_n = \operatorname{argmin}_{y\in T_n}\{\frac{1}{2}\|y - (x_n - \lambda By_n)\|^2\} = P_{T_n}(x_n - \lambda By_n).$$

Also, in this case $\partial_2 f(x_n, y_n) = Bx_n$, so the definition of $T_n$ in Step 3 of Remark 3.15 reduces to

$$T_n = \{z \in H_1 : \langle x_n - y_n - \lambda Bx_n, z - y_n\rangle \le 0\}.$$

By Theorem 3.14, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ converge strongly to $x^* \in \Omega$ and $\{v_n\}$, $\{u_n\}$ and $\{w_n\}$ converge strongly to $Ax^*$. It is easy to see that by definition $f$ and $g$ as above, $\Omega$ reduces to $\Gamma$. □

## 5. Numerical illustrations

In this section, first we present some numerical examples and investigate the behavior of the generated sequences by hybrid proximal point Algorithm 1. we compare our algorithm with other ones in the literature to show the efficiency of it. The optimization subproblems in these examples are solved by the FMINCON and QUADPROG optimization toolbox in MATLAB software. For more details about the notations which are used in this section, we refer readers to [5].

### 5.1. Numerical examples

**Example 5.1.** Consider $(\mathbb{R}^3, \|x\|_2)$ and let $C = [-6, 5] \times [-6, 6] \times [-4, 6] = \{x \in \mathbb{R}^3 | a \le x \le b\}$ where $a = (-6, -6, -4)$, $b = (5, 6, 6)$. Moreover, let $Q = [-7, 5] \times [-5, 4] \times [-7, 6] = \{x \in \mathbb{R}^3 | e \le x \le d\}$ where $e = (-7, -5, -7)$ and $d = (5, 4, 6)$. Furthermore, let $f(x, y) = 9\|y\|_2^2 + \langle x, y\rangle - 10\|x\|_2^2$, $g(x, y) = 6\|y\|_2^2 - 4\langle x, y\rangle - 2\|x\|_2^2$ and let $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{3n+4}$, $\lambda = 1$, $r_n = 1$. Also, assume that $A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 4 & 1 \\ 1 & 4 & 2 \end{bmatrix}$, hence $A$ is bounded linear operator and $A^* = A^T$. So, by this assumption our algorithm is converted to the following form

$$\begin{cases} y_n = \operatorname{argmin}_{y\in C}\{\frac{19}{2}\|y\|_2^2 - \frac{19}{2}\|x_n\|_2^2\}, \\ z_n = \operatorname{argmin}_{y\in H_n}\{\frac{19}{2}\|y\|_2^2 - 10\|y_n\|_2^2 + \frac{1}{2}\|x_n\|_2^2 + \langle y_n, y\rangle - \langle x_n, y\rangle\}, \\ t_n = \beta_n x_n + (1 - \beta_n)z_n, \\ v_n = \frac{1}{2}P_Q(At_n), \\ u_n = \frac{v_n}{9}, \\ w_n = \alpha_n v_n + (1 - \alpha_n)u_n, \\ D_n = \{x \in C : \|z_n - x\|_2 \le \|x_n - x\|_2\}, \\ E_n = \{z \in Q : \|w_n - z\|_2 \le \|v_n - z\|_2\}, \\ H_n = \{z \in X : \langle y_n, z - y_n\rangle \le 0\}, \\ x_{n+1} = \frac{1}{2}P_{D_n\cap A^{-1}(E_n)}(x_n). \end{cases}$$

| Table 1 | | | | | | | |
|---|---|---|---|---|---|---|---|
| $n$ | $\|x_n\|_2$ | $\|y_n\|_2$ | $\|z_n\|_2$ | $\|t_n\|_2$ | $\|v_n\|_2$ | $\|u_n\|_2$ | $\|w_n\|_2$ |
| 1 | 5.9161 | 0.1517 | 0.4473 | 0.8983 | 2.3125 | 0.2569 | 1.2847 |
| 2 | 2.9552 | 0.0758 | 0.2234 | 0.2234 | 0.6753 | 0.0750 | 0.2751 |
| 3 | 1.3419 | 0.0344 | 0.0616 | 0.0616 | 0.2797 | 0.0311 | 0.0932 |
| 4 | 0.7901 | 0.0203 | 0.0204 | 0.0204 | 0.0335 | 0.0037 | 0.0097 |
| 5 | 0.4303 | 0.0110 | 0.0162 | 0.0162 | 0.0260 | 0.0029 | 0.0067 |
| 6 | 0.2225 | 0.0057 | 0.0074 | 0.0074 | 0.0068 | 0.0008 | 0.0016 |
| 7 | 0.1146 | 0.0029 | 0.0039 | 0.0001 | 0.0031 | 0.0003 | 0.0007 |
| 8 | 0.0590 | 0.0015 | 0.0020 | 0.0000 | 0.0015 | 0.0002 | 0.0003 |
| 9 | 0.0304 | 0.0008 | 0.0010 | 0.0000 | 0.0008 | 0.0001 | 0.0002 |
| ... | ... | .... | ..... | ..... | ..... | ..... | ..... |
| 27 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

**Table 1.** The sequences generated by Algorithm 1 for Example 5.1 with starting point $x_0 = (-5, 3, -1)$ on Example 5.1.

| | $x_0 = (-5, 3, -1)$ | $x_0 = (2, -4, 6)$ | $x_0 = (1, 1, 1)$ |
|---|---|---|---|
| Algorithms | Iter | Iter | Iter |
| *Algorithm*1 | 27 | 28 | 25 |
| *Hieu*1 | 45 | 45 | 45 |
| *Hieu*2 | 38 | 38 | 38 |
| *Hieu*3 | 44 | 45 | 44 |

**Table 2.** A comparison of the results for Algorithm 1, Hieu 1, Hieu 2 and Hieu 3 with different starting points on Example 5.2.

Numerical results show that $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{t_n\}$ converges strongly to $x^* = (0,0,0)$ and $\{v_n\}, \{u_n\}$ and $\{w_n\}$ converges strongly to $Ax^* = (0,0,0)$. Table 1 is numerical results for $\|x_n\|_2, \|y_n\|_2, \|z_n\|_2, \|t_n\|_2, \|v_n\|_2, \|u_n\|_2$ and $\|w_n\|_2$ with the starting point $x_0 = (-5, 3, -1)$ and the stopping criterion $\|x_n\|_2 < 10^{-10}$ . We obtain the approximate solution after 27 iteration.

Now, we will compare the effectiveness of our algorithm with the algorithms [16, Algorithm 3.1], [16, Algorithm 4.1] and [17, Algorithm 1], which we refer as Hieu 1, Hieu 2 and Hieu 3, respectively, in the subsequent discussions.

## 5.2. Comparable results

**Example 5.2.** To get a comparative result, we let $C := \{x \in \mathbb{R}^3 : -1 \le x_i \le 5, i = 1,2,3\}$ and $Q := \{x \in \mathbb{R}^3 : -2 \le x_i \le 5, i = 1,2,3\}$. Also, assume that $f$, $g$, $A$, $\alpha_n$, $\beta_n$, $\lambda$ and $r_n$ are the same as in Example 5.1. Moreover, in Hieu 1, Hieu 2 and Hieu 3 assume that $\xi_n = 0$, $\rho_n = 1$, $\beta_n = \frac{1}{(n+1)^{0.75}}$, $\mu_n = \frac{1}{\|A\|^2}$, $\partial_2 f(x_n, x_n) = 19x_n$ and $\partial_2 g(u_n, u_n) = 8u_n$. In all algorithms, we will consider the same starting point $x_0$ and the same stopping rule $\|x_{n+1} - x_n\|_2 < 10^{-10}$. The number of iterations in Table 2 and Figure 1 show that our Algorithm reaches to the stopping condition faster than other schemes.

**Example 5.3.** The test problem here can be considered as an extension of the Nash-cournot oligopolistic equilibrium model in [10, 13] to the split equilibrium model in [16]. Let $X = (\mathbb{R}^m, \|x\|_2)$, $Y = (\mathbb{R}^k, \|x\|_2)$, $C := \{x \in \mathbb{R}^m : -1 \le x_i \le 5, i = 1, ..., m\}$ and $Q := \{x \in \mathbb{R}^k : -2 \le x_i \le 5, i = 1, ..., k\}$. Assume that the bifunction $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is of the form $f(x,y) = \langle Px + Gy + p, y - x\rangle$, where $p$ is a vector in $\mathbb{R}^m$, $P$ and $G$ are two

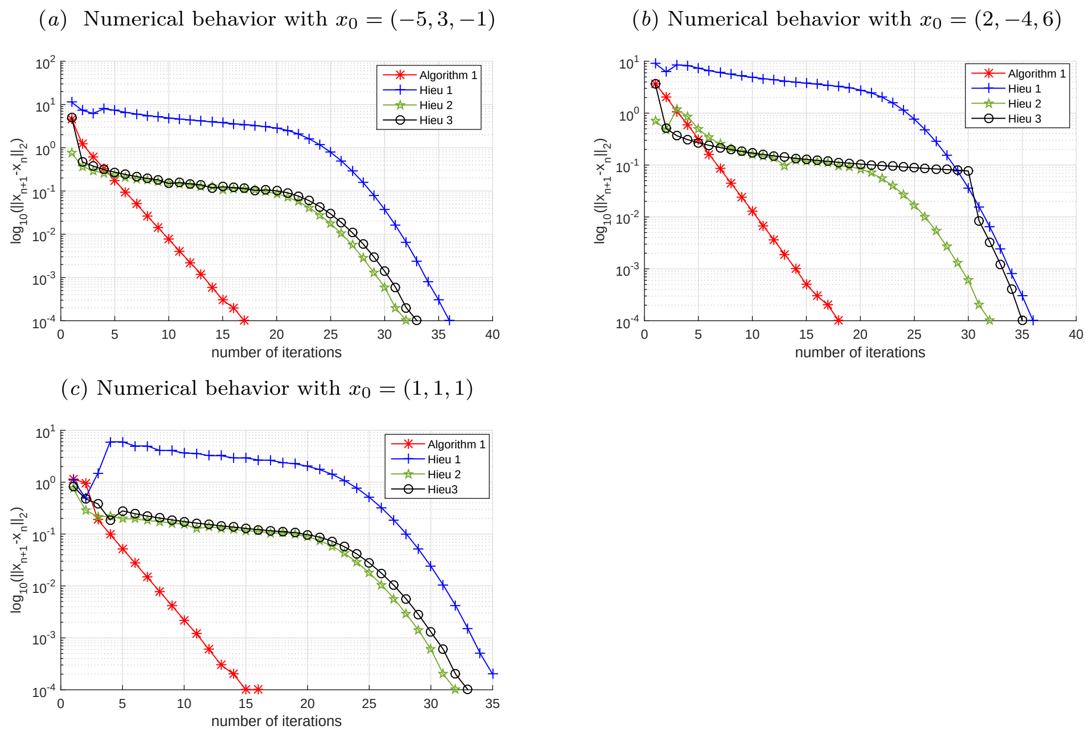


**Figure 1.** Numerical behavior of Algorithm 1, Hieu 1, Hieu 2 and Hieu 3 with different starting points on Examples 5.2.

matrices of order $m$ such that $G$ is symmetric positive semidefinite and $G - P$ is negative semidefinite. As in [15, section 5], the bifunction $f$ satisfies the Lipschitz-type condition that means

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|_2^2 - c_2\|y - z\|_2^2, \quad \forall x, y, z \in \mathbb{R}^m,$$

where $c_1 = c_2 = \frac{\|P-G\|}{2}$, The bifunction $g : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ defined by $g(x, y) = h(y) - h(x)$ where $h(x) = \frac{1}{2}x^T Nx + q^T x$, with $q \in \mathbb{R}^k$ and $N$ being a symmetric positive definite matrix of order $k$. Moreover, consider that the operator $A : \mathbb{R}^m \to \mathbb{R}^k$ is defined by a matrix of size $k \times m$. In this case, the mapping $T_r^g$ coincides with the proximal mapping of the function $g$ with the constant $r > 0$, i.e., $T_r^g(x) = prox_{rg}(x)$, where

$$prox_{rg}(x) = \text{argmin}_{y\in Q}\{g(y) + \frac{1}{2r}\|y - x\|_2^2\}.$$

To get the non empty solution set for the problem and to attain all steps of the algorithms, we choose the two vectors p and q equal to zero vectors in $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively. We choose the parameters $m$ and $k$ as follow:

$$(m = 25, k = 15), \quad (m = 50, k = 30) \quad \text{and} \quad (m = 200, k = 150).$$

The matrices $A$ and $N$ are generated randomly and uniformly with their entries in $[-m, m]$ and the two matrices $P$ and $G$ are also generated randomly as follows:

Consider two diagonal matrices $A_1$ and $A_2$ whose entries are chosen randomly from $[0, m]$ and $[-m, 0]$, respectively. Two random orthogonal matrices $B_1$ and $B_2$ are used to generate a positive semidefinite matrix $M_1 = B_1A_1B_1^T$ and a negative semidefinite matrix $M_2 = B_2A_2B_2^T$. Finally, set $G = M_1 + M_1^T$, $S = M_2 + M_2^T$ and $P = G - S$. Moreover, we will use the starting point $x_0 = (1, 1, ..., 1) \in \mathbb{R}^m$ and the stopping rule $\|x_{n+1} - x_n\|_2 < 10^{-10}$ for all algorithms. Note that, the solution of the (SEP) in this case is $x^* = 0$. Observe that in Algorithm 1, $y_n = prox_{\lambda f(x_n,.)}(x_n)$, which means we have to

| | $m = 25, k = 15$ | $m = 50, k = 30$ | $m = 200, k = 150$ |
|---|---|---|---|
| Algorithms | Iter | Iter | Iter |
| *Algorithm*1 | 75 | 94 | 125 |
| *Hieu*1 | 153 | 189 | 452 |
| *Hieu*2 | 84 | 136 | 1612 |
| *Hieu*3 | 2223 | 4165 | 8313 |

**Table 3.** A comparison of the results for Algorithm 1, Hieu 1, Hieu 2 and Hieu 3 with starting point $x_0 = (1, 1, ..., 1) \in \mathbb{R}^m$ on Example 5.3.

solve the following quadratic minimization problems at each iteration:

$$y_n = \text{argmin}\{\frac{1}{2}y^T H_n y + c_n y : -1 \leq y_i \leq 5 \quad i = 1, 2, \cdots, m\},$$

and

$$z_n = \text{argmin}\{\frac{1}{2}y^T H_n y + d_n y : Ey \leq b\},$$

where $y^T$ stands for the transpose of $y$, $H_n = 2\lambda G + I$ ($I$ is the identity matrix), $c_n = \lambda[Px_n - Gx_n] - x_n$, $d_n = \lambda[Py_n - Gy_n] - x_n$, $E = x_n - \lambda\omega_n - y_n$ and $b = x_n y_n - \lambda\omega_n y_n - y_n^2$. Note that $\omega_n \in \partial_2 f(x_n, y_n) = Px_n + Gy_n$. Since the feasible set $C$ is a box in $\mathbb{R}^m$, the projection of a point $x \in \mathbb{R}^m$ onto $C$ can be calculated as follows:

$$[P_C(x)]_i = \begin{cases} x_i & x_i \in [-1, 5], \\ -1 & x_i < -1, \\ 5 & x_i > 5. \end{cases}$$

Similarly, because the feasible set $Q$ is a box in $\mathbb{R}^k$, the projection of a point $x \in \mathbb{R}^k$ onto $Q$ can be calculated as follows:

$$[P_Q(x)]_i = \begin{cases} x_i & x_i \in [-2, 5], \\ -2 & x_i < -2, \\ 5 & x_i > 5. \end{cases}$$

Also, in Algorithm 1, assume that $\alpha_n$, $\beta_n$, $\lambda$ and $r_n$ are the same as in Example 5.1. Moreover, in Hieu 1, Hieu 2 and Hieu 3 assume that $\xi_n = 0$, $\rho_n = 1$, $\beta_n = \frac{1}{n+1}$, $\mu_n = \frac{1}{\|A\|^2}$, $\partial_2 f(x_n, x_n) = Px_n + Gx_n$ and $\partial_2 g(u_n, u_n) = \frac{1}{2}Nu_n$. The number of iterations in Table 3 and Figure 2 show that our algorithm reaches to the stopping condition faster than other schemes.

## 5.3. *LASSO* problem in compressed sensing

In statistics and machine learning, least absolute selection and shrinkage and selection operator (*LASSO*) is a regression analysis method that performs both variable selection and regularization in order to enhance the prediction accuracy and interpretability of the statistical model it produces. It was originally introduced by [44] who coined out the term and provided further insights into the observed performance. Subsequently, a number of (*LASSO*) variants have been created in order to remedy certain limitations of the original technique and to make the method more useful for particular problems. More specifically, the (*LASSO*) is a regularized regression method with the $l_1$ penalty. Here the $l_1$ penalty is defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$.

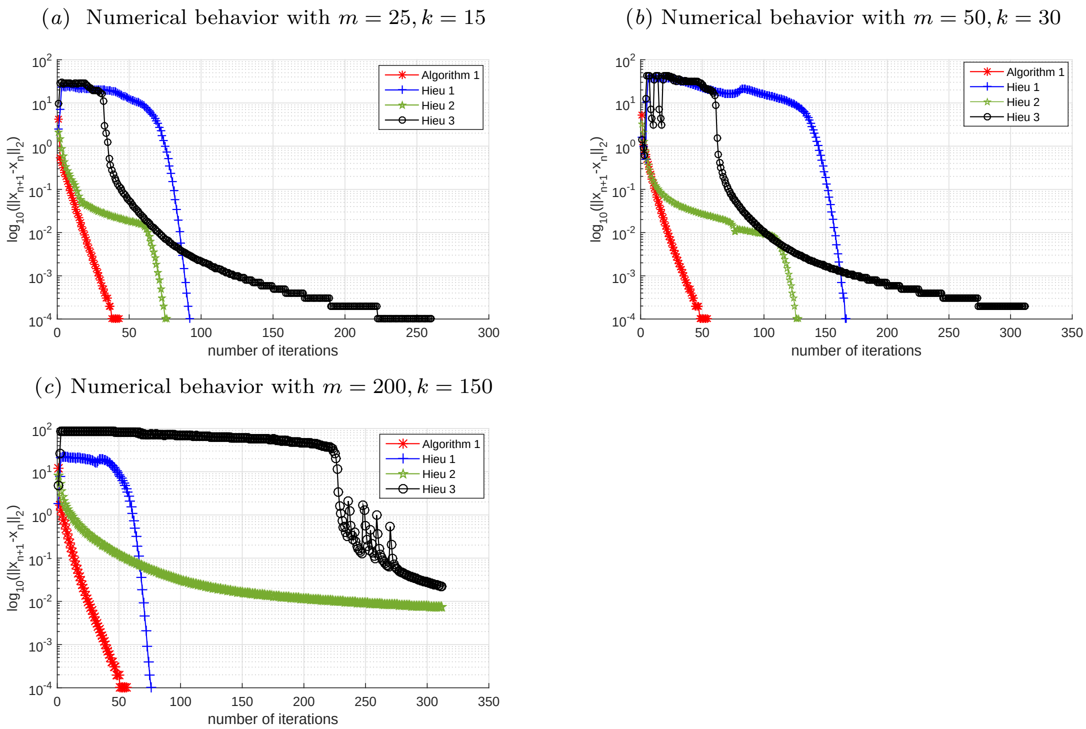


**Figure 2.** Numerical behavior of Algorithm 1, Hieu 1, Hieu 2 and Hieu 3 with the starting point $x_0 = (1, 1, ..., 1) \in \mathbb{R}^m$ on Example 5.3.

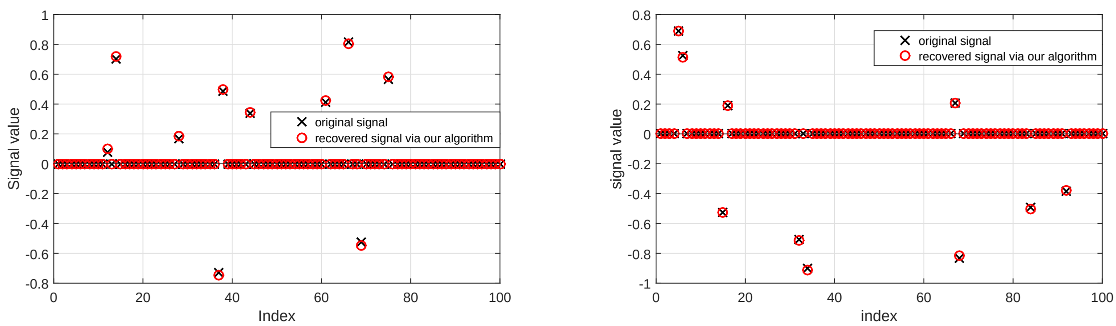


**Figure 3.** The recovered sparse signal versus the original $k$-sparse signal, when m=50, n=100 and k=10.

In this section, we apply our algorithm to solve the $(LASSO)$ problem which is of the form

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2, \quad s.t. \quad \|x\|_1 \leq t, \tag{5.1}$$

where $A \in \mathbb{R}^{m\times n}(m \ll n), b \in \mathbb{R}^m$, $t > 0$ is a arbitrary constant and $\|x\|_2$ is the Euclidean norm of $x$ and $\|x\|_1 = \sum_{i=1}^n |x_i|$ is the $l_1$ norm of $x$. Compressed sensing is very important when it comes to the problem of efficiently acquiring and reconstructing a signal. Note that (5.1) is a convex constrained minimization problem which appears in compressed sensing and image reconstruction where the original signal (or image) is sparse in some orthogonal basis by the process $b = Ax + e$ where $x$ is orthogonal signal (or image), $A$ is the blurring operator, $e$ is a noise and $b$ is the degraded or blurred data which needs to be recovered. The minimization problem (5.1) is reduced to the $(SEP)$, if we consider $f = g \equiv 0$, $C = \{x \in \mathbb{R}^n : \|x\|_1 \leq t\}$ and $Q = \{b\}$. To present a sparse signal recovery

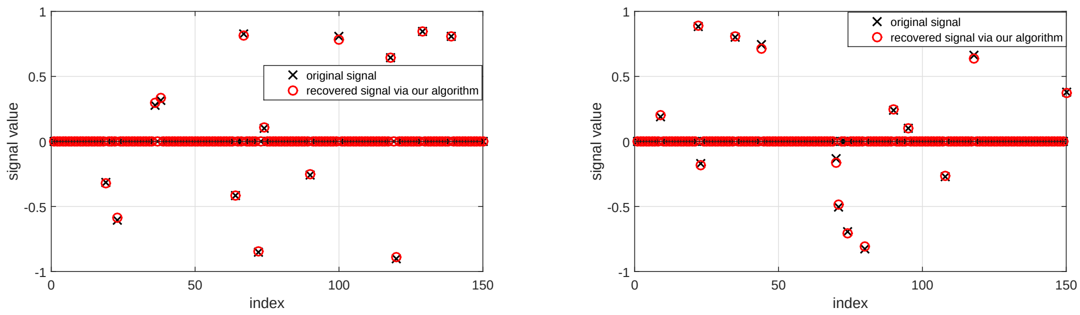


**Figure 4.** The recovered sparse signal versus the original $k$-sparse signal, when m=70, n=150 and k=14.

illustration, we assume that the vector $x \in \mathbb{R}^n$ is a $K$-sparse signal which is generated from uniform distribution in the interval $[-1, 1]$ with $K$ non-zero elements. Furthermore, $A \in \mathbb{R}^{m\times n}$ is a sampling matrix which is generated from a normal distribution and let $b = Ax$. In such case, we assume that the observed data has no noise. The task is then to recover the signal $x$ from the data $b$ by solving the $(LASSO)$ problem.

We apply our algorithm for this purpose by setting $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{1}{3n+4}$. Also in our algorithm the stopping criterion is considered $\|y_{n+1} - y_n\|_2^2 < 10^{-10}$. In Figures 3 and 4 we present the exact k-sparse signal and the recovered signals obtained by our method.

## 6. Conclusions

Using Bregman distance, we have introduced a new hybrid proximal point algorithm for finding a solution of split equilibrium problems, which is a generalization of some other problems in mathematics such as the multiple set split feasibility problems and the split variational inequality problems. We have shown that the generated iterates by our algorithm converge strongly. Also, to show the efficiency of our algorithm, we have provided a numerical example and have compared the results of our algorithm with other methods in the literature. We have demonstrated that the obtained iterates by our algorithm converge to the solution of split equilibrium problems faster than the other methods. Furthermore, we have presented an application of our algorithm for solving the split variational inequality problems in Hilbert spaces. Also, we have applied our algorithm to a problem that arises from compressed sensing, namely the $LASSO$ problem.

## References

[1] S. Alizadeh and F. Moradlou, *Strong convergence theorems for m-generalized hybrid mappings in Hilbert spaces,* Topol. Methods Nonlinear Anal. **46**, 315-328, 2015.

[2] S. Alizadeh and F. Moradlou, *A strong convergence theorem for equilibrium problems and generalized hybrid mappings,* Mediterr. J. Math. **13**, 379-390, 2016.

[3] A.S. Antipin, *The convergence of proximal methods to fixed points of extremal mappings and estimates of their rate of convergence*, Comput. Math. Math. Phys. **35**, 539-551, 1995.

[4] E. Blum and W. Oettli, *From Optimization and variational inequalities to equilibrium problems*, Math. Stud. **63**, 123-145, 1994.

[5] A. Cegielski, *Iterative methods for fixed point problems in Hilbert spaces*, Lecture Notes in Mathematics, Vol. 2057, Springer, 2012.

[6] Y. Censor, A. Gibali and S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms **59**, 301-323, 2012.

[7] C. Chidume, *Geometric properties of Banach spaces and nonlinear iterations*, in: Lecture Notes in Mathematics, vol. 1965, Springer, Berlin, 2009.

[8] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

[9] P.L. Combettes and S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6**, 117-136, 2005.

[10] J. Contreras, M. Klusch and J.B. Krawczyk, *Numerical solution to Nash-Cournot equilibria in coupled constraint electricity markets*, EEE Trans. Power. Syst. **19**, 195-206, 2004.

[11] J. Deepho, W. Kumm and P. Kumm, *A new hybrid projection algorithm for solving the split generalized equilibrium problems and the system of variational inequality problems*, J. Math. Model. Algorithms **13**, 405-423, 2014.

[12] B.V. Dinh, D.X. Son and T.V. Anh, *Extragradient-Proximal Methods for Split Equilibrium and Fixed Point Problems in Hilbert Spaces*, Vietnam J. Math. **45**, 651-668, 2015.

[13] F. Facchinei and J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer, Berlin, 2002.

[14] Z. He, *The split equilibrium problem and its convergence algorithms*, J. Inequal. Appl. **162**, 2012.

[15] D.V. Hieu, *Parallel Extragradient-Proximal Methods for Split Equilibrium Problems*, Math. Model. Anal. **21**, 478-501, 2016.

[16] D.V. Hieu, *Two hybrid algorithms for solving split equilibrium problems*, Int. J. Comput. Math. **95**, 561-583, 2018.

[17] D.V. Hieu, *Projection methods for solving split equilibrium problems*, J. Ind. Manag. Optim. **16**, 2331-2349, 2020.

[18] D.V. Hieu, L.D. Muu and P. K. Anh, *Parallel hybrid extragradient methods for pseudomonotone equilibrium problems and nonexpansive mappings*, Numer. Algorithms **73**, 197-217, 2016.

[19] Z. Jouymandi and F. Moradlou, *Extragradient methods for solving equilibrium problems, variational inequalities and fixed point problems*, Numer. Funct. Anal. Optim. **38**, 1391-1409, 2017.

[20] Z. Jouymandi and F. Moradlou, *Retraction algorithms for solving variational inequalities, pseudomonotone equilibrium problems and fixed point problems in Banach spaces*, Numer. Algorithms **78**, 1153-1182, 2018.

[21] Z. Jouymandi and F. Moradlou, *Extragradient and linesearch algorithms for solving equilibrium problems, variational inequalities and fixed point problems in Banach spaces*, Fixed Point Theory **20**, 523-540, 2019.

[22] G. Kassay, T.N. Hai and N.T. Vinh, *Coupling Popov's algorithm with subgradient extragradient method for solving equilibrium problems*, J. Nonlinear Convex Anal. **19**, 959-986, 2018.

[23] D.S. Kim and B.V. Dinh, *Parallel extragradient algorithms for multiple set split equilibrium problems in Hilbert spaces*, Numer. Algorithms **77**, 741-761, 2018.

[24] D. Kinderlehrar and D. Stampacchia, *An introduction to variational inequality and their application,* Academic Press, New York, 1980.

[25] S.I. Lyashko and V.V. Semenov, *A new two-step proximal algorithm of solving the problem of equilibrium programming,* in: Optimization and Applications in Control and Data Sciences **115**, 315-326, Springer, 2016.

[26] Y. Malitsky, *Projected reflected gradient methods for monotone variational inequalities,* SIAM J. Optim. **25**, 502-520, 2015.

[27] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Rev Française Informat Recherche Opérationnelle **4**, 154-158, 1970.

[28] S. Matsushita and L. Xu, *On convergence of the proximal point algorithm in Banach spaces*, Proc. Amer. Math. Soc. **139**, 4087-4095, 2011.

[29] S. Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. **2004** 37-47, 2004.

[30] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory **134**, 257-266, 2005.

[31] A. Moudafi, *The split common fixed-point problem for demicontractive mappings*, Inverse Problems **26**, (Article ID 055007), 2010 .

[32] A. Moudafi, *Split monotone variational inclusions*, J. Optim. Theory Appl. **150**, 275-283, 2011.

[33] L.D. Muu and W. Oettli, *Convergence of an adaptive penalty scheme for finding constrained equilibria*, Nonlinear Anal. **18**, 1159-1166, 1992.

[34] T.D. Quoc, P. N. Anh and L. D. Muu, *Dual extragradient algorithms to equilibrium Problems,* J. Glob. Optim. **52**, 139-159, 2012.

[35] T.D. Quoc, L.D. Muu and V.H. Nguyen, *Extragradient algorithms extended to equilibrium problems*, Optimization **57**, 749-776, 2008.

[36] S. Reich, *A weak convergence theorem for the alternating method with Bregman distance*, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996.

[37] S. Reich and S. Sabach, *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal. **10**, 471-485, 2009.

[38] R.T. Rockafellar, *Maximal monotone operators and proximal point algorithm*, SIAM J. Control Optim. **14**, 877-898, 1976.

[39] M. Safari and F. Moradlou, *Shrinking hybrid method for multiple-sets split feasibility problems and variational inequality problems*, Ric. Mat., accepted, doi:10.1007/s11587-021-00676-z.

[40] D. Sahu, D. O'Regan and R. P. Agarwal, Fixed point theory for Lipschitzian-type mappings with Applications, Springer, New York, 2009.

[41] F. Schöpfer, T. Schuster and A. K. Louis, *An iterative regularization method for the solution of the split feasibility problem in Banach spaces*, Inverse Problems **24**, (Article ID 055008), 2008.

[42] J.J. Strodiot, P.T. Vuong and N.T.T. Van, *A class of shrinking projection extragradient methods for solving non-monotone equilibrium problems in Hilbert spaces,* J. Global Optim. **64**, 159-178, 2016.

[43] S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space,* Nonlinear Anal. **69**, 1025-1033, 2008.

[44] R. Tibshirani, *Regression shrinkage and selection via LASSO*, J. R. Stat. Soc. Ser. B. Stat. Methodol. **58**, 267-288, 1996.

[45] S. Wang, X. Gong, A.N. Abdou and Y.J. Cho, *Iterative algorithm for a family of split equilibrium problems and fixed point problems in Hilbert spaces with applications*, Fixed Point Theory Appl. **2016**, 1-22, 2016.