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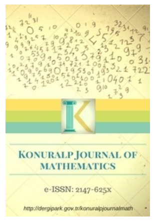

Triple Lacunary Δ-Statistical Convergence in Neutrosophic Normed Spaces

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Abstract

The aim of this article is to investigate triple lacunary Δ-statistically convergent and triple lacunary Δ-statistically Cauchy sequences in a neutrosophic normed space (NNS). Also, we present their feature utilizing triple lacunary density and derive the relationship between these notions.

Keywords: *triple sequence, lacunary sequence, difference sequence, neutrosophic normed space*
2010 Mathematics Subject Classification: *Primary: 40A05; Secondary: 40C05.*

1. Introduction

The notion neutrosophy suggests impartial knowledge of thought and then neutral describes the basic difference between neutral, fuzzy, intuitive fuzzy set and logic. The neutrosophic set (NS) was studied by F. Smarandache [1] who introduced the degree of indeterminacy (i) as indepedent component. In [2], neutrosophic logic was firstly examined. It is a logic where each proposition is determined to have a degree of truth (T), falsity (F), and indeterminacy (I). A Neutrosophic set (NS) is specified as a set where every component of the universe has a degree of T, F and I. Kirişçi and Şimşek [3] considered neutrosophic metric space (NMS) with continuous t-norms and continuous t-conorms. The theory of NNS and statistical convergence in NNS were first developed by Kirişci and Şimşek [4]. Neutrosophic set and neutrosophic logic has utilized by applied sciences and theoretical science for instance summability theory, decision making, robotics. Some remarkable results on this topic can be reviewed in [5, 6, 7, 8]. In [6], lacunary statistical convergence of sequences in NNS was investigated. Also, lacunary statistically Cauchy sequence in NNS was presented and lacunary statistically completeness in connection with a NNS was worked. In other study, Kişi [7] defined the notion of ideal convergence in NNS.

The concept of statistical convergence was defined under the name of almost convergence by Zygmund [9]. It was formally introduced by Fast [10]. Later the idea was associated with summability theory by Fridy [11], and many others (see [12, 13, 14, 15, 16]). The studies of triple sequences have seen rapid growth. The initial work on the statistical convergence of triple sequences was established by Şahiner et al. [17] and the other researches continued by [18, 19, 20, 21]. Utilizing lacunary sequence, Fridy and Orhan [22] considered lacunary statistical convergence. Some studies on lacunary statistical convergence can be examined in [23, 24]. The idea of difference sequences was given by Kızmaz [25] where $\Delta x = (\Delta x_k) = x_k - x_{k+1}$. Başarır [26] investigated the Δ-statistical convergence of sequences. Bilgin [27] presented the definition of lacunary strongly Δ-convergence of fuzzy numbers. Also, the generalized difference sequence spaces were worked by various authors [28, 29, 30, 31].

Since sequence convergence plays a very significant role in the fundamental theory of mathematics, there are many convergence notions in summability theory, in approximation theory, in classical measure theory, in probability theory, and the relationships between them are discussed. The interested reader may consult Hazarika et al. [32], the monographs [33] and [34] for the background on the sequence spaces and related topics. Inspired by this, in this study, a further investigation into the mathematical features of triple sequences will be thought. Section 2 recalls some definitions in summability theory and NNS. In Section 3, we study the concepts of lacunary statistical convergence and lacunary statistical Cauchy of triple difference sequences in a NNS and establish some fundamental properties of NNS.

2. Preliminaries

Now, we remember essential definitions required in this study.

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Let $A \subset \mathbb{N}$ and $r \in \mathbb{N}$. $\delta_\theta^r(A)$ is named the rth partial lacunary density of A, if

$$\delta_\theta^r(A) = \frac{|A \cap I_r|}{h_r},$$

where $I_r = (k_{r-1}, k_r]$.
The number $\delta_\theta(A)$ is indicated the lacunary density (θ-density) of A if

$$\delta_\theta(A) = \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : k \in A\}|, \ \left(i.e., \delta_\theta(A) = \lim_{r\to\infty} \delta_\theta^r(A)\right)$$

exists. Also, $\Lambda = \{A \subset \mathbb{N} : \delta_\theta(A) = 0\}$ is called to be zero density set.
A sequence (x_k) is named to be lacunary statistically convergent (or S_θ-convergent) to L if for every $\varepsilon > 0$,

$$\delta_\theta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0.$$

Triangular norms (t-norms) (TN) were considered by Menger [35]. TNs are utilized to generalise with the probability distribution of triangle inequality in metric space terms. Triangular conorms (t-conorms) (TC) recognized as dual operations of TNs. TNs and TCs are significant for fuzzy operations.

Definition 2.1. *([35]) Let $* : [0,1] \times [0,1] \to [0,1]$ be an operation. If $*$ provides subsequent cases, it is named continuous TN. Take $a,b,c,d \in [0,1]$,*

(a) $a * 1 = a$,
(b) *If* $a \leq c$ *and* $b \leq d$, *then* $a * b \leq c * d$,
(c) $*$ *is continuous,*
(d) $*$ *associative and commutative.*

Definition 2.2. *([35]) Let $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ be an operation. If $\diamondsuit$ provides subsequent cases, it is named to be continuous TC.*

(a) $a \diamondsuit 0 = a$,
(b) *If* $a \leq c$ *and* $b \leq d$, *then* $a \diamondsuit b \leq c \diamondsuit d$,
(c) $\diamondsuit$ *is continuous,*
(d) $\diamondsuit$ *associative and commutative.*

Definition 2.3. *([4]) Let F be a vector space, $\mathcal{N} = \{\langle \varpi, \mathcal{G}(\varpi), \mathcal{B}(\varpi), \mathcal{Y}(\varpi)\rangle : \varpi \in F\}$ be a normed space (NS) such that $\mathcal{N} : F \times \mathbb{R}^+ \to [0,1]$. While subsequent situations hold, $V = (F, \mathcal{N}, *, \diamondsuit)$ is called to be NNS. For each $\varpi, \kappa \in F$ and $\lambda, \mu > 0$ and for all $\sigma \neq 0$,*

(a) $0 \leq \mathcal{G}(\varpi,\lambda) \leq 1, 0 \leq \mathcal{B}(\varpi,\lambda) \leq 1, 0 \leq \mathcal{Y}(\varpi,\lambda) \leq 1 \ \forall \lambda \in \mathbb{R}^+$,
(b) $\mathcal{G}(\varpi,\lambda) + \mathcal{B}(\varpi,\lambda) + \mathcal{Y}(\varpi,\lambda) \leq 3$ *(for* $\lambda \in \mathbb{R}^+$*)*,
(c) $\mathcal{G}(\varpi,\lambda) = 1$ *(for* $\lambda > 0$*) iff* $\varpi = 0$,
(d) $\mathcal{G}(\sigma\varpi,\lambda) = \mathcal{G}\left(\varpi, \frac{\lambda}{|\sigma|}\right)$,
(e) $\mathcal{G}(\varpi,\mu) * \mathcal{G}(\kappa,\lambda) \leq \mathcal{G}(\varpi+\kappa, \mu+\lambda)$,
(f) $\mathcal{G}(\varpi,.)$ *is non-decreasing continuous function,*
(g) $\lim_{\lambda\to\infty} \mathcal{G}(\varpi,\lambda) = 1$,
(h) $\mathcal{B}(\varpi,\lambda) = 0$ *(for* $\lambda > 0$*) iff* $\varpi = 0$,
(i) $\mathcal{B}(\sigma\varpi,\lambda) = \mathcal{B}\left(\varpi, \frac{\lambda}{|\sigma|}\right)$,
(j) $\mathcal{B}(\varpi,\mu) \diamondsuit \mathcal{B}(\kappa,\lambda) \geq \mathcal{B}(\varpi+\kappa, \mu+\lambda)$,
(k) $\mathcal{B}(\varpi,.)$ *is non-decreasing continuous function,*
(l) $\lim_{\lambda\to\infty} \mathcal{B}(\varpi,\lambda) = 0$,
(m) $\mathcal{Y}(\varpi,\lambda) = 0$ *(for* $\lambda > 0$*) iff* $\varpi = 0$,
(n) $\mathcal{Y}(\sigma\varpi,\lambda) = \mathcal{Y}\left(\varpi, \frac{\lambda}{|\sigma|}\right)$,
(o) $\mathcal{Y}(\varpi,\mu) \diamondsuit \mathcal{Y}(\kappa,\lambda) \geq \mathcal{Y}(\varpi+\kappa, \mu+\lambda)$,
(p) $\mathcal{Y}(\varpi,.)$ *is non-decreasing continuous function,*
(r) $\lim_{\lambda\to\infty} \mathcal{Y}(\varpi,\lambda) = 0$,
(s) *If* $\lambda \leq 0$, *then* $\mathcal{G}(\varpi,\lambda) = 0, \mathcal{B}(\varpi,\lambda) = 1$ *and* $\mathcal{Y}(\varpi,\lambda) = 1$.
Then $\mathcal{N} = (\mathcal{G}, \mathcal{B}, \mathcal{Y})$ *is called Neutrosophic norm (NN).*
We recall the notions of convergence, statistical convergence, lacunary statistical convergence for single sequences in a NNS.

Definition 2.4. *([4]) Take V as an NNS. Let $\varepsilon \in (0,1)$ and $\lambda > 0$. Then, a sequence (x_k) is converges to $L \in F$ iff there is $N \in \mathbb{N}$ such that $\mathcal{G}(x_k - L, \lambda) > 1 - \varepsilon$, $\mathcal{B}(x_k - L, \lambda) < \varepsilon$, $\mathcal{Y}(x_k - L, \lambda) < \varepsilon$. That is,*

$$\lim_{k\to\infty} \mathcal{G}(x_k - L, \lambda) = 1, \lim_{k\to\infty} \mathcal{B}(x_k - L, \lambda) = 0 \ and \ \lim_{k\to\infty} \mathcal{Y}(x_k - L, \lambda) = 0$$

as $\lambda > 0$. The convergent in NNS is signified by $\mathcal{N}-\lim x_k = L$.

Definition 2.5. *([4]) A sequence (x_k) is named to be statistically convergent to $L \in F$ with regards to NN (SC-NN), provided that, for each $\lambda > 0$ and $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \mathcal{G}(x_k - L, \lambda) \leq 1 - \varepsilon \ or \ \mathcal{B}(x_k - L, \lambda) \geq \varepsilon, \ \mathcal{Y}(x_k - L, \lambda) \geq \varepsilon\}| = 0.$$

It is demonstrated by $S_{\mathcal{N}}$-$\lim x_k = L$.

Definition 2.6. *([6]) A sequence (x_k) is named to be lacunary statistically convergent to $L \in F$ with regards to NN (LSC-NN), provided that, for each $\lambda > 0$ and $\varepsilon > 0$ the set*

$$C_\varepsilon := \{k \in \mathbb{N} : \mathscr{G}(x_k - L, \lambda) \leq 1 - \varepsilon \text{ or } \mathscr{B}(x_k - L, \lambda) \geq \varepsilon, \mathscr{Y}(x_k - L, \lambda) \geq \varepsilon\}$$

has lacunary density zero. It is signified by $S_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim x_k = \xi$.

Now we introduce the following notions (see [17] and [18]).

Definition 2.7. *A subset K of $\mathbb{N}^3$ is said to have natural density $\delta_3(K)$ if*

$$\delta_3(K) = P - \lim_{n,l,k \to \infty} \frac{|K_{nlk}|}{nlk}$$

exists, where the vertical bars denote the number of (n,l,k) in K such that $p \leq n$, $q \leq l$, $r \leq k$. Then, a real triple sequence $x = (x_{pqr})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : p \leq n, q \leq l, r \leq k,\ \left|x_{pqr} - L\right| \geq \varepsilon\right\}\right) = 0.$$

The triple sequence $\theta_3 = \theta_{r,s,t} = \{(n_r, l_s, l_t)\}$ is named triple lacunary sequence if there exist three increasing sequences of integers such that

$$n_0 = 0,\ h_r = n_r - n_{r-1} \to \infty \text{ as } r \to \infty,$$

$$l_0 = 0,\ h_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty,$$

and

$$k_0 = 0,\ h_t = k_t - k_{t-1} \to \infty \text{ as } t \to \infty.$$

Let $n_{r,s,t} = n_r l_s k_t$, $h_{r,s,t} = h_r h_s h_t$ and $\theta_{r,s,t}$ is determined as

$$I_{r,s,t} = \{(n,l,k) : n_{r-1} < n \leq n_r, l_{s-1} < l \leq l_s \text{ and } k_{t-1} < k \leq k_t\},$$

$$s_r = \frac{n_r}{n_{r-1}}, s_s = \frac{l_s}{l_{s-1}},\ s_t = \frac{k_t}{k_{t-1}} \text{ and } s_{r,s,t} = s_r s_s s_t.$$

Let $D \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. The number

$$\delta_{\theta_3}(D) = \lim_{r,s,t} \frac{1}{h_{r,s,t}} |\{(n,l,k) \in I_{r,s,t} : (n,l,k) \in D\}|$$

is said to be the θ_3-density of D, provided the limit exists.

3. Main results

Now, we examine Δ-convergence and lacunary Δ-statistical convergence of triple sequences in NNS. Throughout the paper we consider V as an NNS.

Definition 3.1. *A triple sequence $x = (x_{nlk})$ in V is named to be Δ-convergent to $L \in F$ with respect to (w.r.t in short) NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ provided that for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is a positive integer k_0 such that*

$$\mathscr{G}(\Delta x_{nlk} - L, \lambda) > 1 - \varepsilon \text{ and } \mathscr{B}(\Delta x_{nlk} - L, \lambda) < \varepsilon,\ \mathscr{Y}(\Delta x_{nlk} - L, \lambda) < \varepsilon$$

for every $n \geq k_0, l \geq k_0, k \geq k_0$ where $n,l,k \in \mathbb{N}$ and $\Delta x_{nlk} = x_{nlk} - x_{n,l+1,k} - x_{n,l,k+1} + x_{n,l+1,k+1} - x_{n+1,l,k} + x_{n+1,l+1,k} + x_{n+1,l,k+1} - x_{n+1,l+1,k+1}$. We indicate $(\mathscr{G},\mathscr{B},\mathscr{Y}) - \lim \Delta x = L$ or $\Delta x \to L((\mathscr{G},\mathscr{B},\mathscr{Y}))$ as $n,l,k \to \infty$.

Definition 3.2. *A triple sequence $x = (x_{nlk})$ in V is said to be lacunary Δ-statistically convergent (or $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)$-convergent) to $L \in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ provided that for every $\lambda > 0$ and $\varepsilon \in (0,1)$*

$$\delta_{\theta_3}(\Delta)\left(\left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta x_{nlk} - L, \lambda) \leq 1 - \varepsilon \text{ or } \mathscr{B}(\Delta x_{nlk} - L, \lambda) \geq \varepsilon, \mathscr{Y}(\Delta x_{nlk} - L, \lambda) \geq \varepsilon\right\}\right) = 0,$$

or equivalently,

$$\delta_{\theta_3}(\Delta)\left(\left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta x_{nlk} - L, \lambda) > 1 - \varepsilon \text{ and } \mathscr{B}(\Delta x_{nlk} - L, \lambda) < \varepsilon,\ \mathscr{Y}(\Delta x_{nlk} - L, \lambda) < \varepsilon\right\}\right) = 1.$$

It is indicated by $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta) - \lim x = L$ or $x_{nlk} \to L(S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta))$. Using Definition 3.2 and features of the θ_3-density, we can simply achieve the following lemma.

Lemma 3.3. *For every $\varepsilon\in(0,1)$ and $\lambda>0$, the following cases are equivalent:*

(a) $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x=\xi$,

(b)

$$\begin{aligned}&\delta_{\theta_3}(\Delta)\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-L,\lambda)\leq 1-\varepsilon\right\}\right)\\&=\delta_{\theta_3}(\Delta)\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{B}(\Delta x_{nlk}-L,\lambda)\geq \varepsilon\right\}\right)\\&=\delta_{\theta_3}(\Delta)\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{Y}(\Delta x_{nlk}-L,\lambda)\geq \varepsilon\right\}\right)=0,\end{aligned}$$

(c)

$$\delta_{\theta_3}(\Delta)\left(\left\{\begin{array}{c}(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-L,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nlk}-L,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nlk}-L,\lambda)<\varepsilon\end{array}\right\}\right)=1,$$

(d)

$$\begin{aligned}&\delta_{\theta_3}(\Delta)\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-L,\lambda)> 1-\varepsilon\right\}\right)\\&=\delta_{\theta_3}(\Delta)\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{B}(\Delta x_{nlk}-L,\lambda)< \varepsilon\right\}\right)\\&=\delta_{\theta_3}(\Delta)\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{Y}(\Delta x_{nlk}-L,\lambda)< \varepsilon\right\}\right)=1,\end{aligned}$$

(e)

$$\begin{aligned}&S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim\mathscr{G}(\Delta x_k-L,\lambda)=1,\ S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim\mathscr{B}(\Delta x_{nlk}-L,\lambda)=0\\&\text{and } S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim\mathscr{Y}(\Delta x_{nlk}-L,\lambda)=0.\end{aligned}$$

Theorem 3.4. *If a triple sequence $x=(x_{nlk})$ in V is $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)$-convergent to $L\in F$ w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$, then $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x$ is unique.*

Proof. Let $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x=L_1$ and $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x=L_2$. For a given $\varepsilon\in(0,1)$, we select $\Theta\in(0,1)$ such that $(1-\Theta)*(1-\Theta)>1-\varepsilon$ and $\Theta\Diamond\Theta<\varepsilon$. Then, for any $\lambda>0$, we determine the following sets:

$$\begin{aligned}K_{\mathscr{G}1}(\Theta,\lambda)&=\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}\left(x_{nlk}-L_1,\tfrac{\lambda}{2}\right)\leq 1-\Theta\right\},\\K_{\mathscr{G}2}(\Theta,\lambda)&=\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}\left(x_{nlk}-L_2,\tfrac{\lambda}{2}\right)\leq 1-\Theta\right\},\\K_{\mathscr{B}1}(\Theta,\lambda)&=\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{B}\left(x_{nlk}-L_1,\tfrac{\lambda}{2}\right)\geq \Theta\right\},\\K_{\mathscr{B}2}(\Theta,\lambda)&=\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{B}\left(x_{nlk}-L_2,\tfrac{\lambda}{2}\right)\geq \Theta\right\},\\K_{\mathscr{Y}1}(\Theta,\lambda)&=\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{Y}\left(x_{nlk}-L_1,\tfrac{\lambda}{2}\right)\geq \Theta\right\},\\K_{\mathscr{Y}2}(\Theta,\lambda)&=\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{Y}\left(x_{nlk}-L_2,\tfrac{\lambda}{2}\right)\geq \Theta\right\}.\end{aligned}$$

Since $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x_{nlk}=L_1$, then utilizing Lemma 3.3, for every $\lambda>0$, we have

$$\delta_{\theta_3}(\Delta)(K_{\mathscr{G}1}(\Theta,\lambda))=\delta_{\theta_3}(\Delta)(K_{\mathscr{B}1}(\Theta,\lambda))=\delta_{\theta_3}(\Delta)(K_{\mathscr{Y}1}(\Theta,\lambda))=0.$$

Also, using $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x_{nlk}=L_2$, we get

$$\delta_{\theta_3}(\Delta)(K_{\mathscr{G}2}(\Theta,\lambda))=\delta_{\theta_3}(\Delta)(K_{\mathscr{B}2}(\Theta,\lambda))=\delta_{\theta_3}(\Delta)(K_{\mathscr{Y}2}(\Theta,\lambda))=0.$$

Now let

$$\begin{aligned}K_{\mathscr{N}}(\Theta,\lambda)\quad&:=\{K_{\mathscr{G}1}(\Theta,\lambda)\cup K_{\mathscr{G}2}(\Theta,\lambda)\}\cap\{K_{\mathscr{B}1}(\Theta,\lambda)\cup K_{\mathscr{B}2}(\Theta,\lambda)\}\\&\cap\{K_{\mathscr{Y}1}(\Theta,\lambda)\cup K_{\mathscr{Y}2}(\Theta,\lambda)\}.\end{aligned}$$

Then examine that $\delta_{\theta_3}(\Delta)(K_{\mathscr{N}}(\Theta,\lambda))=0$, which gives that $\delta_{\theta_3}(\Delta)\left(\mathbb{N}^3\setminus K_{\mathscr{N}}(\Theta,\lambda)\right)=1$. If $(n,l,k)\in\mathbb{N}^3\setminus K_{\mathscr{N}}(\Theta,\lambda)$, then we acquire three possible situations.
That is, $(n,l,k)\in\mathbb{N}^3\setminus(K_{\mathscr{G}1}(\Theta,\lambda)\cup K_{\mathscr{G}2}(\Theta,\lambda))$, $(n,l,k)\in\mathbb{N}^3\setminus(K_{\mathscr{B}1}(\Theta,\lambda)\cup K_{\mathscr{B}2}(\Theta,\lambda))$ or $(n,l,k)\in\mathbb{N}^3\setminus(K_{\mathscr{Y}1}(\Theta,\lambda)\cup K_{\mathscr{Y}2}(\Theta,\lambda))$.
First, contemplate that $(n,l,k)\in\mathbb{N}^3\setminus(K_{\mathscr{G}1}(\Theta,\lambda)\cup K_{\mathscr{G}2}(\Theta,\lambda))$. Then, we have

$$\mathscr{G}(L_1-L_2,\lambda)\geq\mathscr{G}\left(x_{nlk}-L_1,\frac{\lambda}{2}\right)*\mathscr{G}\left(x_{nlk}-L_2,\frac{\lambda}{2}\right)>(1-\Theta)*(1-\Theta)>1-\varepsilon.$$

For arbitrary $\varepsilon>0$, we get $\mathscr{G}(L_1-K_2,\lambda)=1$ for all $\lambda>0$, which yields $L_1=L_2$. At the same time, if we take $(n,l,k)\in\mathbb{N}^3\setminus(K_{\mathscr{B}1}(\Theta,\lambda)\cup K_{\mathscr{B}2}(\Theta,\lambda))$, then we can write

$$\mathscr{B}(L_1-L_2,\lambda)\leq\mathscr{B}\left(x_{nlk}-L_1,\frac{\lambda}{2}\right)\Diamond\mathscr{B}\left(x_{nlk}-L_2,\frac{\lambda}{2}\right)\leq\Theta\Diamond\Theta<\varepsilon.$$

Therefore, we can see that $\mathscr{B}(L_1-L_2,\lambda)<\varepsilon$. For all $\lambda>0$, we obtain $\mathscr{B}(L_1-L_2,\lambda)=0$, which indicates that $L_1=L_2$. Again, for the case $(n,l,k)\in\mathbb{N}^3\setminus(K_{\mathscr{Y}1}(\Theta,\lambda)\cup K_{\mathscr{Y}2}(\Theta,\lambda))$, then we can write

$$\mathscr{Y}(L_1-L_2,\lambda)\leq\mathscr{Y}\left(x_{nlk}-L_1,\frac{\lambda}{2}\right)\Diamond\mathscr{Y}\left(x_{nlk}-L_2,\frac{\lambda}{2}\right)\leq\Theta\Diamond\Theta<\varepsilon.$$

For all $\lambda>0$, we have $\mathscr{Y}(L_1-L_2,\lambda)=0$, which yields $L_1=L_2$. In all cases, we conclude that $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)$-limit of triple sequence is unique. □

Theorem 3.5. *If* $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim \Delta x = L$*, then* $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x = L$*, but not necessarily conversely.*

Proof. By hypothesis $x=(x_{nlk})$, Δ-converges to $L\in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. Therefore, for every $\lambda>0$ and $\varepsilon\in(0,1)$, there is a positive integer k_0 such that $\mathscr{G}(\Delta x_{nlk}-L,\lambda)>1-\varepsilon$ and $\mathscr{B}(\Delta x_{nlk}-L,\lambda)<\varepsilon$, $\mathscr{Y}(\Delta x_{nlk}-L,\lambda)<\varepsilon$ for all $n\geq k_0, l\geq k_0, k\geq k_0$. Thus the set

$$\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-L,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(\Delta x_{nlk}-L,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nlk}-L,\lambda)\geq\varepsilon\right\}$$

has finitely many terms. Since every finite subset of $\mathbb{N}^3$ has lacunary density zero, we see that

$$\delta_{\theta_3}(\Delta)\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-L,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(\Delta x_{nlk}-L,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nlk}-L,\lambda)\geq\varepsilon\right\}\right)=0.$$

This ends the proof. □

Theorem 3.6. *Take NNS as V. Then,* $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x_{nlk}=L$ *iff there is a subset*

$$K=\left\{(n,l,k)\in\mathbb{N}^3:n,l,k=1,2,3,...\right\}\subset\mathbb{N}^3$$

such that $\delta_{\theta_3}(\Delta)(K)=1$ *and* $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim_{(n,l,k)\in K,n,l,k\to\infty}\Delta x_{nlk}=L$.

Proof. Presume that $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x_{nlk}=L$. Then, for every $\lambda>0$ and $j\geq 1$,

$$K(j,\lambda)=\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-L,\lambda)>1-\frac{1}{j} \text{ and } \mathscr{B}(\Delta x_{nlk}-L,\lambda)<\frac{1}{j},\ \mathscr{Y}(\Delta x_{nlk}-L,\lambda)<\frac{1}{j}\right\}$$

and

$$M(j,\lambda)=\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-L,\lambda)\leq 1-\frac{1}{j} \text{ or } \mathscr{B}(\Delta x_{nlk}-L,\lambda)\geq\frac{1}{j},\ \mathscr{Y}(\Delta x_{nlk}-L,\lambda)\geq\frac{1}{j}\right\}.$$

Then $\delta_{\theta_3}(\Delta)(M(j,\lambda))=0$ since

$$K(j,\lambda)\supset K(j+1,\lambda) \tag{3.1}$$

and

$$\delta_{\theta_3}(\Delta)(K(j,\lambda))=1 \tag{3.2}$$

for $\lambda>0$ and $j\geq 1$. Now we need to show that for $(n,l,k)\in K(j,\lambda)$ the triple sequence $x=(x_{nlk})$ is Δ-convergent to $L\in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. Suppose $x=(x_{nlk})$ be not Δ-convergent to $L\in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. Therefore, there are $\beta>0$ and $k_0>0$ such that $\mathscr{G}(\Delta x_{nlk}-L,\lambda)\leq 1-\beta$ or $\mathscr{B}(\Delta x_{nlk}-L,\lambda)\geq\beta$, $\mathscr{Y}(\Delta x_{nlk}-L,\lambda)\geq\beta$ for all $n\geq k_0, l\geq k_0, k\geq k_0$. Let $\beta>\frac{1}{j}$ and

$$K(\beta,\lambda)=\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-L,\lambda)>1-\beta \text{ and } \mathscr{B}(\Delta x_{nlk}-L,\lambda)<\beta,\ \mathscr{Y}(\Delta x_{nlk}-L,\lambda)<\beta\right\}.$$

Then, we have $\delta_{\theta_3}(\Delta)(K(\beta,\lambda))=0$. Since $\beta>\frac{1}{j}$, by (3.1) we get $\delta_{\theta_3}(\Delta)(K(j,\lambda))=0$, which contradicts by (3.2). Therefore $x=(x_{nlk})$ is Δ-convergent to $L\in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$.
Conversely presume that there is a subset $K=\left\{(n,l,k)\in\mathbb{N}^3:n,l,k=1,2,3,...\right\}\subset\mathbb{N}^3$ such that $\delta_{\theta_3}(\Delta)(K)=1$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim_{(n,l,k)\in K,n,l,k\to\infty}\Delta x_{nlk}=L$. Then for every $\lambda>0$ and $\varepsilon\in(0,1)$, there is $k_0\in\mathbb{N}$ such that $\mathscr{G}(\Delta x_{nlk}-L,\lambda)>1-\varepsilon$ and $\mathscr{B}(\Delta x_{nlk}-L,\lambda)<\varepsilon$, $\mathscr{Y}(\Delta x_{nlk}-L,\lambda)<\varepsilon$ for all $n\geq k_0, l\geq k_0, k\geq k_0$. Let

$$\begin{aligned} M(\varepsilon,\lambda) &:= \left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-L,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(\Delta x_{nlk}-L,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nlk}-L,\lambda)\geq\varepsilon\right\} \\ &\subseteq \mathbb{N}^3-\left\{\left(n_{k_0+1},l_{k_0+1}k_{k_0+1}\right),\left(n_{k_0+2},l_{k_0+2}k_{k_0+2}\right),...\right\} \end{aligned}$$

and as a consequence $\delta_{\theta_3}(\Delta)(M(\varepsilon,\lambda))\leq 1-1=0$. Hence $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x_{nlk}=L$. Then, the desired result has been acquired. □

Theorem 3.7. *Let V be an NNS. Then* $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim\Delta x_{nlk}=L$ *iff there are sequences* $y=(y_{nlk})$ *and* $z=(z_{nlk})$ *in V such that* $\Delta x_{nlk}=\Delta y_{nlk}+\Delta z_{nlk}$ *for all* $n,k,l\in\mathbb{N}$ *where* $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim\Delta y_{nlk}=L$ *and* $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim\Delta z_{nlk}=L$.

Proof. Assume that $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)-\lim x=L$. By Theorem 3.6, there is an increasing sequence

$$K=\left\{(n,l,k)\in\mathbb{N}^3:n,l,k=1,2,3,...\right\}\subset\mathbb{N}^3$$

such that $\delta_{\theta_3}(\Delta)(K)=1$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim_{(n,l,k)\in K,n,l,k\to\infty}\Delta x_{nlk}=L$.
Determine the sequences $y=(y_{nlk})$ and $z=(z_{nlk})$ as follows:

$$\Delta y_{nlk}=\begin{cases}\Delta x_{nlk}, & \text{if } (n,l,k)\in K\\ L, & \text{otherwise}\end{cases}$$

and

$$\Delta z_{nlk}=\begin{cases}0, & \text{if } (n,l,k)\in K\\ \Delta x_{nlk}-L, & \text{otherwise.}\end{cases}$$

Then, $y=(y_{nlk})$ and $z=(z_{nlk})$ serves our aim.
Conversely if such two sequences $y=(y_{nlk})$ and $z=(z_{nlk})$ exist with the required features, then the consequence follows immediately from Theorem 3.5 and Lemma 3.3. □

Definition 3.8. *A triple sequence $x=(x_{nlk})$ in V is named to be Δ-Cauchy w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ provided that for every $\varepsilon\in(0,1)$ and $\lambda>0$, there exist positive integers k_0,k_1,k_2 such that $\mathscr{G}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)>1-\varepsilon$ and $\mathscr{B}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)<\varepsilon$, $\mathscr{Y}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)<\varepsilon$ for all $n,m\geq k_0$, $l,p\geq k_1$, $k,q\geq k_2$.*

Definition 3.9. *A triple sequence $x=(x_{nlk})$ in V is named to be lacunary Δ-statistically Cauchy or $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)$-Cauchy w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ provided that, for every $\varepsilon\in(0,1)$ and $\lambda>0$, there exist positive integers N,M,P such that*

$$\delta_{\theta_3}(\Delta)\begin{pmatrix} (n,l,k)\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\leq 1-\varepsilon \text{ or} \\ \mathscr{B}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\geq\varepsilon,\ \mathscr{Y}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\geq\varepsilon\end{pmatrix}=0$$

for all $n,m\geq N$, $l,p\geq M$, $k,q\geq P$.

Theorem 3.10. *If a triple sequence $x=(x_{nlk})$ is lacunary Δ-statistically convergent w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ iff it is lacunary Δ-statistically Cauchy w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$.*

Proof. Let $x=(x_{nlk})$ be a lacunary Δ-statistically convergent sequence which converges to L. For a given $\varepsilon\in(0,1)$ select $s>0$ such that $(1-\varepsilon)*(1-\varepsilon)>1-s$ and $\varepsilon\Diamond\varepsilon<s$. Let

$$A(\varepsilon,\lambda)=\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{nlk}-L,\frac{\lambda}{2}\right)\leq 1-\varepsilon \text{ or } \mathscr{B}\left(\Delta x_{nlk}-L,\frac{\lambda}{2}\right)\geq\varepsilon,\ \mathscr{Y}\left(\Delta x_{nlk}-L,\frac{\lambda}{2}\right)\geq\varepsilon\right\}.$$

Then, for any $\lambda>0$,

$$\delta_{\theta_3}(\Delta)(A(\varepsilon,\lambda))=0, \tag{3.3}$$

which gives that $\delta_{\theta_3}(\Delta)(A^c(\varepsilon,\lambda))=1$.
Let $(m,p,q)\in A^c(\varepsilon,\lambda)$. Then

$$\mathscr{G}\left(\Delta x_{mpq}-L,\frac{\lambda}{2}\right)>1-\varepsilon \text{ and } \mathscr{B}\left(\Delta x_{mpq}-L,\frac{\lambda}{2}\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{mpq}-L,\frac{\lambda}{2}\right)<\varepsilon.$$

Now, take

$$B(s,\lambda)=\left\{\begin{array}{c}(n,l,k)\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\leq 1-s \text{ or} \\ \mathscr{B}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\geq s,\ \mathscr{Y}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\geq s\end{array}\right\}.$$

We have to prove that $B(s,\lambda)\subset A(\varepsilon,\lambda)$. Let $(n,l,k)\in B(s,\lambda)\cap A^c(\varepsilon,\lambda)$.
Hence $\mathscr{G}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\leq 1-s$, $\mathscr{G}\left(\Delta x_{nlk}-L,\frac{\lambda}{2}\right)\geq 1-\varepsilon$, in particular, $\mathscr{G}\left(\Delta x_{mpq}-L,\frac{\lambda}{2}\right)\geq 1-\varepsilon$. Then

$$1-s\geq\mathscr{G}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\geq\mathscr{G}\left(\Delta x_{nlk}-L,\frac{\lambda}{2}\right)*\mathscr{G}\left(\Delta x_{mpq}-L,\frac{\lambda}{2}\right)>(1-\varepsilon)*(1-\varepsilon)>1-s$$

which is not possible. On the other hand, $\mathscr{B}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\geq s$ and $\mathscr{B}\left(\Delta x_{nlk}-L,\frac{\lambda}{2}\right)<\varepsilon$, $\mathscr{B}\left(\Delta x_{mpq}-L,\frac{\lambda}{2}\right)<\varepsilon$. Hence,

$$s\leq\mathscr{B}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\leq\mathscr{B}\left(\Delta x_{nlk}-L,\frac{\lambda}{2}\right)\Diamond\mathscr{B}\left(\Delta x_{mpq}-L,\frac{\lambda}{2}\right)<\varepsilon\Diamond\varepsilon<s,$$

which is not possible. Hence $B(s,\lambda)\subset A(\varepsilon,\lambda)$ and by (3.3), we acquire $\delta_{\theta_3}(\Delta)(B(s,\lambda))=0$. In the last case, again we obtain $B(s,\lambda)\subset A(\varepsilon,\lambda)$. This proves that $x=(x_{nlk})$ is lacunary Δ-statistically Cauchy with regards to the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$.
Conversely, let $x=(x_{nlk})$ is lacunary Δ-statistically Cauchy but not lacunary Δ-statistically convergent w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. For a given $\varepsilon\in(0,1)$, select $s>0$ such that $(1-\varepsilon)*(1-\varepsilon)>1-s$ and $\varepsilon\Diamond\varepsilon<s$. Since x is not lacunary Δ-statistically convergent

$$\begin{aligned}&\mathscr{G}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\geq\mathscr{G}\left(\Delta x_{nlk}-L,\tfrac{\lambda}{2}\right)*\mathscr{G}\left(\Delta x_{mpq}-L,\tfrac{\lambda}{2}\right)>(1-\varepsilon)*(1-\varepsilon)>1-s,\\ &\mathscr{B}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\leq\mathscr{B}\left(\Delta x_{nlk}-L,\tfrac{\lambda}{2}\right)\Diamond\mathscr{B}\left(\Delta x_{mpq}-L,\tfrac{\lambda}{2}\right)<\varepsilon\Diamond\varepsilon<s,\\ &\mathscr{Y}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\leq\mathscr{Y}\left(\Delta x_{nlk}-L,\tfrac{\lambda}{2}\right)\Diamond\mathscr{Y}\left(\Delta x_{mpq}-L,\tfrac{\lambda}{2}\right)<\varepsilon\Diamond\varepsilon<s.\end{aligned}$$

Therefore $\delta_{\theta_3}(\Delta)(B^c(s,\lambda))=0$, where

$$B(s,\lambda)=\left\{\begin{array}{c}(n,l,k)\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\leq 1-s \text{ or} \\ \mathscr{B}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\geq s,\ \mathscr{Y}\left(\Delta x_{nlk}-\Delta x_{mpq},\lambda\right)\geq s\end{array}\right\}$$

and so $\delta_{\theta_3}(\Delta)(B(s,\lambda))=1$, which is a contradiction, since x was lacunary Δ-statistically Cauchy w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. Hence, x have to be lacunary Δ-statistically convergent w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. □

Theorem 3.11. *For any triple sequence $x=(x_{nlk})$ in NNS, the subsequent cases are equivalent:*
(i) x is $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)$-convergent w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$.
(ii) x is $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)$-Cauchy sequence w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$.
(iii) There is an increasing index sequence $K=\{(k_1,k_2,k_3)\}$ of $\mathbb{N}^3$ such that $\delta_{\theta_3}(\Delta)(K)=1$ and the subsequence $\left\{\left(x_{k_1,k_2,k_3}\right)\right\}_{(k_1,k_2,k_3)\in K}$ is a $S_{\theta_3}^{(\mathscr{G},\mathscr{B},\mathscr{Y})}(\Delta)$-Cauchy sequence w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$.

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