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Semi-analytic solution of time-fractional Korteweg-de Vries equation using fractional residual power series method

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Abstract

In this paper, we solve the non-linear Korteweg-de Vries equation by considering the time-fraction derivative in Caputo sense and offered intrinsic properties of solitary waves. The fractional residual power series method is used to obtain the approximate solution of the aforesaid equation and compared the obtained results with Adomian Decomposition Method. Obtained results are efficient, reliable, and simple to execute on most of the non-linear fractional partial differential equations, which arise in various dynamical systems.

Keywords: Fractional differential equation, KdV equation, Residual Power series method, Caputo derivative
2020 MSC: 35R11, 33E50, 35Q51, 35L05

1. Introduction

In 1877, Joseph Valentin Boussinesq[1] initiated theoretical investigations on solitary waves induced on shallow water; later in 1895, Diederik Korteweg and Gustav de Vries (Dutch Mathematicians) has retrieved the weakly non-linear partial differential equation (popularly known as KdV equation) and presented a

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mathematical model illustrating the wave of shallow water surfaces[2]. In a classical sense, the Korteweg-de Vries (KdV) is a non-linear partial differential equation of order three, given as

$$\psi_\tau + 6\psi\psi_\xi + \psi_{\xi\xi\xi} = 0\,, \tag{1}$$

where $\psi(\xi, \tau)$ denotes the elongation of the wave at place ξ and time τ.

The Korteweg-de Vries (KdV) is an equation, which is widely accepted and used in various branches of physical sciences and engineering during the study of fluid dynamics as it describes weakly non-linear long waves. The KdV equation gives information about small amplitudes of long waves on the free surface of the water.

Historically the KdV equation was first developed in the study of shallow-water waves in canals[3]; since then it has been discovered to be involved in a variety of physical processes, especially those exhibiting shock waves, travelling waves, and solitons. The KdV model is used to explain many theoretical physical phenomena in the solitons[4], aerodynamics[5], turbulence[6], fluid dynamics[7] etc. Thus the KdV equation has been studied and applied for many decades.

The KdV equation exhibits several properties, including discovery due to Gardner[8], viz. asymptotic stability in the energy space, total energy of the solution, etc. Gardener revealed the fact that "*The KdV equation can be solved exactly, as an initial value problem, starting with arbitrary initial data in a suitable space.*" Thus, the KdV equation is to be considered as a classical one with its theoretical significance, demonstrating that the PDEs governing physical phenomena may also be solved exactly. Thus, unlike the majority of the PDEs which require a numerical approach, KdV can be solved analytically. As a noticeable characteristic of the KdV equation, Xiang[3] discussed the existence and uniqueness of this celebrated equation in his work.

Here in the present work, we consider the KdV equation in the form of a fractional partial differential equation (FPDE) and use the Fractional Residual Power Series Method (FRPSM) to obtain the semi-analytic solution. The significance of studying this celebrated equation in the fractional form lies in the fact that using the fractional approach we get more realistic results in real-time situations, as compared to conventional derivatives of integer order. And, FRPSM intends to determine an exact and estimated solution in a semi-analytical way for the fractional physical equations. FRPSM leads to a closed-form solution of several well-known functions as it is working in a well-organized and competent way to solve ordinary and partial differential equations. To solve non-linear time-dependent FPDEs, the FRPSM is one of the dominant techniques, which is established by the generalized formula of the Taylor series. The different types and orders of non-linear FDEs can be solved by the FRPSM effectively. It constructs Residual power series expansion without doing discretization, linearization, or perturbation.

In many fields, the FRPSM introduced by some authors, viz. Lane-Emden Equations[9], Boussinesq-Burgers Equations[10], Diffusion Equations[11], Burger Types Equations[12], Multipantograph Differential Equations[13], Whitham-Broer-Kaup Equations[13], Fredholm Integral Equations[16]. Reaction Diffusion Model is one of the applications discussed by modeling fractal-fractional partial differential equations and oil pollution[17] are another real-life problem studied by the analytic solution of diffusion equations. Recursion relation in this method is not important and coefficients comparison of the corresponding terms do not need over the classical power series method.

FRPSM was introduced by Alquran[18] in 2015. He has solved the equation of the drainage problem in fractional form. In addition, in 2015, Wang and Chen[19] have obtained the solution of Whitham-Broer-Kaup equations using FRPSM. As the solution obtained using FRPSM was accurate, convergent and does not need outsize computer memory compared to other numerical methods like Jacobi elliptic function expansion method[20], spectral collocation method[21], Petrov-Galerkin method[22], some iterative methods[23], Bernstein basis polynomials[24], Laplace Adomian Decomposition Method[25], Haar wavelet collocation method[26], etc. due to that reason, several researchers have gained interest in it. Thereafter, Khalil et al.[27] in 2021 have studied long-wave equations, Modanli et al.[13] in 2020 have studied pseudo hyperbolic differential equation, Kumar et al.[28] in 2021 have discussed the bi-Hamiltonian Boussinesq System.

To stimulate more interest in the subject and to show its utility, the present article focuses on the new application of FRPSM and comparison with ADM. The article comprises different sections. Besides the first section, the upcoming sections are devoted to the formulation of FRPSM for the KdV equation and its solution. Furthermore, we have also compared the solution with the exact solution and demonstrated the results using graphs and tables. In conclusion, we have discussed the geometrical interpretation and accuracy of the solution.

2. Preliminaries

Definition 2.1. *Riemann-Liouville Fractional Integral[29]*
The Riemann-Liouville version of fractional integral operator of order α is given by

$$ {}_aD_\xi^{-\alpha}\psi(\xi) = \frac{1}{\Gamma(\alpha)}\int_a^\xi (\xi-\tau)^{\alpha-1}\phi(\tau)d\tau, \quad n-1<\alpha\le n. \tag{2}$$

Definition 2.2. *Caputo Fractional Derivative[30]*
The Caputo time-fractional derivative of order $\alpha > 0$ of $\psi(\xi,\tau)$ is defined as

$$ D_\tau^\alpha[\psi(\xi,\tau)] = \begin{cases} \frac{1}{\Gamma(\eta-\alpha)}\int_o^\tau (\tau-t)^{\eta-a-1}\frac{\partial^\eta\psi(\xi,t)}{\partial t^\eta}dt, & \eta-1<\alpha<\eta \\ \frac{\partial^\eta\psi(\xi,t)}{\partial t^\eta}, & \alpha=\eta\in N. \end{cases} \tag{3}$$

Definition 2.3. *Fractional Residual Power Series[31]*
A fractional residual power series expansion about $\tau=\tau_0$ is expressed in the form

$$ \sum_{\nu=0}^{\infty} c_\nu(\tau-\tau_0)^{\nu\alpha} = c_0 + c_1(\tau-\tau_0)^\alpha + c_2(\tau-\tau_0)^{2\alpha}+..., 0\le\eta-1<\alpha\le\eta,\ \tau\ge\tau_0. \tag{4}$$

Theorem 2.4. *[31] Let the FRPS representation at $\tau=\tau_0$ for the function ψ of the form*

$$ \psi(\tau) = \sum_{\nu=0}^{\infty} c_\nu(\tau-\tau_0)^{\nu\alpha}, \tau_0\le\tau<\tau_0+R \tag{5}$$

where R is the radius of convergence.
If $D^{\nu\alpha}\psi(\tau)$, $\nu=0,1,2,3,...$ are continuous on (τ_0,τ_0+R) then the coefficient c_ν are given by the formula $c_\nu = \frac{D^{\nu\alpha}\psi(\tau_0)}{\Gamma(1+\nu\alpha)}$, $\nu=0,1,2,...$ where $D^{\nu\alpha}=D^\alpha.\,D^\alpha.\,D^a...\,D^a(\nu\,\text{times})$.

Theorem 2.5. *[31] Let the FRPS representation at $\tau=\tau_0$ for the function ψ is expressed by the form*

$$ \psi(\xi,\tau) = \sum_{\nu=0}^{\infty} \phi_\nu(\xi)(\tau-\tau_0)^{\nu\alpha}, \text{where } \xi\in\mathbb{R}, \tau_0\le\tau<\tau_0+R, 0\le\nu-1<\alpha\le\nu. \tag{6}$$

If $D^{\nu\alpha}\psi(\xi,\tau)$, $\nu=0,1,2,3,...$ are continuous on $\mathbb{R}\times(\tau_0,\tau_0+R)$ then the coefficient c_ν are given by $\phi_\nu(\xi) = \frac{D^{\nu\alpha}\psi(\xi,\tau_0)}{\Gamma(1+\nu\alpha)}$, $\nu=0,1,2,...$ where

$$ D^{\nu\alpha} = D^\alpha\cdot D^\alpha\cdot D^a\cdot\dots\cdot D^a\ (\nu\,\text{times}).$$

Thus, the generalized Taylor series formula for fractional residual power series at $\tau = \tau_0$ can be expressed as

$$\psi(\xi,\tau) = \sum_{\nu=0}^{\infty} \frac{D^{\nu\alpha}\psi(\xi,\tau_0)}{\Gamma(1+\nu\alpha)}(\tau-\tau_0)^{\nu\alpha}, \tag{7}$$

where $\xi \in \mathbb{R}$, $\tau_0 \le \tau < \tau_0 + R$, $0 \le \nu - 1 < \alpha \le \nu$.

If $\alpha = 1$, then classical Taylor series is obtained as

$$\psi(\xi,\tau) = \sum_{\nu=0}^{\infty} \frac{D^{\nu}\psi(\xi,\tau_0)}{\Gamma(1+\nu)}(\tau-\tau_0)^{\nu}, \text{where } \xi \in \mathbb{R}, \ \tau_0 \le \tau < \tau_0 + R. \tag{8}$$

Corollary 2.6. *Suppose that $\psi(\xi,\zeta,\tau)$ has a multiple fractional residual power series representation at $\tau = \tau_0$ of the form*

$$\psi(\xi,\zeta,\tau) = \sum_{\nu=0}^{\infty} \phi_\nu(\xi,\zeta)(\tau-\tau_0)^{\nu\alpha}, \tag{9}$$

where $(\xi,\zeta) \in \mathbb{R}\times\mathbb{R}, \tau_0 \le \tau < \tau_0 + R, 0 \le \nu - 1 < \alpha \le \nu$.

If $D^{\nu\alpha}\psi(\xi,\zeta,\tau)$, $\nu = 0, 1, 2, 3, ...$are continuous on $\mathbb{R}\times\mathbb{R}\times(\tau_0, \tau_0 + R)$ then the coefficient c_ν are given by

$$\phi_\nu(\xi,\zeta) = \frac{D^{\nu\alpha}\psi(\xi,\zeta,\tau_0)}{\Gamma(1+\nu\alpha)}, \ \nu = 0, 1, 2, ...$$

3. Analysis of Fractional Residual Power Series Method

To illustrate the essential concept of FRPSM, the generalized fractional differential equation of non-linear form is considered as follows:

$$D_\tau^\alpha[\psi(\xi,\tau)] = R(\psi) + N(\psi), 0 \le \nu - 1 < \alpha \le \nu. \tag{10}$$

where $R(\psi)$ and $N(\psi)$ are linear and non-linear terms respectively, subject to initial conditions,

$$\psi(\xi,0) = \phi_0(\xi) = \phi(\xi) \text{ and } D_\tau^{(\eta-1)\alpha}[u(\xi,0)] = \phi_{\eta-1}(\xi). \tag{11}$$

The FRPSM presents the solution for (10) at t = 0,

$$\psi(\xi,\tau) = \sum_{\eta=0}^{\infty} \phi_\eta(\xi)\frac{\tau^{\eta\alpha}}{\Gamma(1+\eta\alpha)}, \text{ where } \xi \in \mathbb{R}, \ 0 \le \tau < R, \ 0 < \alpha \le 1. \tag{12}$$

Let $\psi_\kappa(\xi,\tau)$ denote as $\kappa^{th}-$ truncated series

$$\psi_\kappa(\xi,\tau) = \sum_{\eta=0}^{\kappa} \phi_\eta(\xi)\frac{\tau^{\eta\alpha}}{\Gamma(1+\eta\alpha)}, \tag{13}$$

where $\xi \in \mathbb{R}, \ 0 \le \tau < R, \ 0 < \alpha \le 1, \ \kappa = 1, 2, 3, ...$

The solution of FDE (10) namely $\psi(\xi,\tau)$ satisfies the initial conditions as given in (11). Moreover, applying $\tau = 0$ in equation (12), we obtain

$$\psi_0(\xi,0) = \psi(\xi,0) = \phi_0(\xi) = \phi(\xi). \tag{14}$$

Using series (13) for $\kappa = 1$, we have

$$\psi_1(\xi,\tau) = \phi_0(\xi) + \phi_1(\xi)\frac{\tau^\alpha}{\Gamma(1+\alpha)}, \tag{15}$$

and in general

$$\psi_\kappa(\xi,\tau) = \phi_0(\xi) + \phi_1(\xi)\frac{\tau^\alpha}{\Gamma(1+\alpha)} + \sum_{\eta=2}^{\kappa}\phi_\eta(\xi)\frac{\tau^{\eta\alpha}}{\Gamma(1+\eta\alpha)}, \tag{16}$$

where $\kappa = 2,3,4,...$
Subsequently, using FRPSM we can evaluate $\phi_\eta(\xi),\ \eta = 1,2,3,...,\kappa$ in the equation (16).
Now, we define the residual function to generalized FDE (10) as,

$$\text{Res}\ \psi(\xi,\tau) = D_\tau^\alpha[\psi(\xi,\tau)] - R(\psi) - N(\psi). \tag{17}$$

Thus, $\kappa^{th}-$ Residual function is

$$\text{Res}\ \psi_\kappa(\xi,\tau) = D_\tau^\alpha[\psi_\kappa(\xi,\tau)] - R(\psi_\kappa) - N(\psi_\kappa). \tag{18}$$

As mentioned in El-Ajouh et al.[31], we can easily see that

$$\lim_{\kappa\to\infty}\text{Res}\ \psi_\kappa(\xi,\tau) = \text{Res}\ \psi(\xi,\tau) = 0.$$

$$D_\tau^{\eta\alpha}\left[\text{Res}\,\psi(\xi,\tau)\right] = 0. \tag{19}$$

In Caputo sense, fractional differentiation is

$$\mathrm{D}_\tau^{\eta\alpha}\left[\text{Res}\ \psi(\xi,0)\right] = \mathrm{D}_\tau^{\eta\alpha}\left[\text{Res}\ \psi_\kappa(\xi,0)\right] = 0; \quad \eta = 0,1,2,...,\kappa. \tag{20}$$

To evaluate $\phi_i(\xi)$ where $i = 1,2,...,$ we calculate for $\kappa = 1,2,...,$ in (16) then replace it in (18), taking $D_\tau^{(\kappa-1)\alpha}$on both the sides, we have $\phi_i(\xi)$ where $i = 1,2,...,$ using

$$D_\tau^{(\kappa-1)\alpha}\left[\text{Res}\ \psi_\kappa(\xi,0)\right] = 0, \quad \kappa = 1,2,3,... \tag{21}$$

Substitution of $\phi_i(\xi)$, $i = 1,2,...,$ in (12) provide us with the series solution of (10).

3.1. Convergence Analysis of FRPSM

Theorem 3.1. *[31] For any $\sum_{\nu=0}^{\infty}\phi_\nu(\xi)\tau^{\nu\alpha}, \tau \geq \tau_0$, there exist three possibilities*

(i) The series converges only when $\tau = 0$,

(ii) The series converges for each $\tau \geq 0$,

(iii) There is a positive real number R such that the series converges whenever $0 \leq \tau < R$ and diverges whenever $\tau > R$.

The number R in case-3 is Radius of Convergence of the fractional residual power series (FRPS). By convention $R = 0$ in case-1 and $R \to \infty$ in case-2.

Theorem 3.2. *[31] The power series $\sum_{\nu=0}^{\infty}\phi_\nu(\xi)\tau^{\nu\alpha}, -\infty < \tau < \infty$ has radius of convergence R, if and only if the FRPS, $\sum_{\nu=0}^{\infty}\phi_\nu(\xi)\tau^{\nu\alpha}, \tau \geq 0$ has radius of convergence $R^{1/\alpha}$. Here, radius of convergence $R = \lim_{\nu\to\infty}\left|\frac{\phi_\nu}{\phi_{\nu+1}}\right|$.*

4. Solution of Fractional KdV Equation

To demonstrate the one-dimensional non-linear homogeneous time fractional KdV (Korteweg-de Vries) equation, we used the concept of FRPSM, let us examine the question.

$$D_\tau^\alpha[\psi(\xi,\tau)] + 6\psi\psi_\xi + \psi_{\xi\xi\xi} = 0, \tag{22}$$

with initial conditions,

$$\psi(\xi,0) = \frac{1}{2}\text{sech}^2\left(\frac{x}{2}\right). \tag{23}$$

The exact solution[3] of KdV equation is given by,

$$\psi(\xi,\tau) = \frac{1}{2}\text{sech}^2\left(\frac{\tau+\xi}{2}\right). \tag{24}$$

Explicating the residual function for (22) as

$$\text{Res}\ \psi(\xi,\tau) = D_\tau^\alpha\psi(\xi,\tau) + 6\psi\psi_\xi + \psi_{\xi\xi\xi}. \tag{25}$$

Thus, $\kappa^{th}-$ Residual function Res $\psi_\kappa(\xi,\tau)$,

$$\text{Res}\ \psi_\kappa(\xi,\tau) = D_\tau^\alpha\psi_\kappa(\xi,\tau) + 6\psi_\kappa\psi_{\kappa\xi} + \psi_{\kappa\xi\xi\xi}. \tag{26}$$

For $\kappa = 1$, equations (16) and (26) yields,

$$\begin{aligned}\text{Res}\ \psi_1(\xi,\tau) =& \phi_1 + 6\phi\phi_\xi + \phi_{\xi\xi\xi} + (6\phi\phi_{1\xi} + 6\phi_1\phi_\xi + \phi_{1\xi\xi\xi})\frac{\tau^\alpha}{\Gamma(1+\alpha)} \\ &+ 6\phi_1\phi_{1\xi}\frac{\tau^{2\alpha}}{\Gamma(1+\alpha)^2}.\end{aligned} \tag{27}$$

Using initial condition (21), we have

$$\phi_1(\xi) = \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\text{sech}^2\left(\frac{\xi}{2}\right). \tag{28}$$

Similarly, for $\kappa = 2$ we have,

$$\text{Res}\ \psi_2(\xi,\tau) = D_\tau^\alpha\psi_2 + 6\psi_2\psi_{2\xi} + \psi_{2\xi\xi\xi}. \tag{29}$$

Now, from (16) at $\kappa = 2$,

$$\psi_2(\xi,\tau) = \phi(\xi) + \phi_1(\xi)\frac{\tau^\alpha}{\Gamma(1+\alpha)} + \phi_2(\xi)\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{30}$$

Residual function Res $\psi_2(\xi,\tau)$ is given by

$$\begin{aligned}\text{Res}\ \psi_2(\xi,\tau) =& \phi_1 + 6\phi\phi_\xi + \phi_{\xi\xi\xi} + (\phi_2 + 6\phi\phi_{1\xi} + 6\phi_1\phi_\xi + \phi_{1\xi\xi\xi})\frac{\tau^\alpha}{\Gamma(1+\alpha)} \\ &+ (6\phi\phi_{2\xi} + 6\phi_2\phi_\xi + \phi_{2\xi\xi\xi})\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} + 6\phi_1\phi_{1\xi}\frac{\tau^{2\alpha}}{\Gamma(1+\alpha)^2} \\ &+ (6\phi_1\phi_{2\xi} + 6\phi_2\phi_{1\xi})\frac{\tau^{3\alpha}}{\Gamma(1+\alpha)(1+2\alpha)} + \phi_2\phi_{2\xi}\frac{\tau^{4\alpha}}{\Gamma(1+2\alpha)^2}.\end{aligned} \tag{31}$$

Taking D_τ^α on both side and calculating the equation $D_\tau^\alpha\left[\text{Res}\psi_2(\xi,0)\right] = 0$,

$$\phi_2 = -6\phi\phi_{1\xi} - 6\phi_1\phi_\xi - \phi_{1\xi\xi\xi}. \tag{32}$$

with,

$$\phi_2(\xi) = \frac{1}{4}\text{sech}^2\left(\frac{\xi}{2}\right)\left(2 - 3\text{sech}^2\left(\frac{\xi}{2}\right)\right). \tag{33}$$

And, for $\kappa = 3$,

$$\text{Res}\left[\psi_3(\xi,\tau)\right] = D_\tau^\alpha \psi_3 + 6\psi_3\psi_{3\xi} + \psi_{3\xi\xi\xi}. \tag{34}$$

Now, from (16) at $\kappa = 3$,

$$\psi_3(\xi,\tau) = \phi(\xi) + \phi_1(\xi)\frac{\tau^\alpha}{\Gamma(1+\alpha)} + \phi_2(\xi)\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} + \phi_3(\xi)\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)}. \tag{35}$$

Residual function Res $\psi_3(\xi,\tau)$ is given by

$$\begin{aligned}\text{Res}\ \psi_3(\xi,\tau) =& \phi_1 + 6\phi\phi_\xi + \phi_{\xi\xi\xi} + (\phi_2 + 6\phi\phi_{1\xi} + 6\phi_1\phi_\xi + \phi_{1\xi\xi\xi})\frac{\tau^\alpha}{\Gamma(1+\alpha)} \\ &+ (\phi_3 + 6\phi\phi_{2\xi} + 6\phi_2\phi_\xi + \phi_{2\xi\xi\xi})\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ 6\phi_1\phi_{1\xi}\frac{\tau^{2\alpha}}{\Gamma(1+\alpha)^2} + (6\phi\phi_{3\xi} + 6\phi_3\phi_\xi + \phi_{3\xi\xi\xi})\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)} \\ &+ (6\phi_1\phi_{2\xi} + 6\phi_2\phi_{1\xi})\frac{\tau^{3\alpha}}{\Gamma(1+\alpha)(1+2\alpha)} + 6\phi_2\phi_{2\xi}\frac{\tau^{4\alpha}}{\Gamma(1+2\alpha)^2} \\ &+ (6\phi_3\phi_{1\xi} + 6\phi_1\phi_{3\xi})\frac{\tau^{4\alpha}}{\Gamma(1+\alpha)(1+3\alpha)} \\ &+ (6\phi_2\phi_{3\xi} + 6\phi_3\phi_{2\xi})\frac{\tau^{5\alpha}}{\Gamma(1+2\alpha)(1+3\alpha)} + 6\phi_3\phi_{3\xi}\frac{\tau^{6\alpha}}{\Gamma(1+3\alpha)^2}.\end{aligned} \tag{36}$$

Taking $D_\tau^{2\alpha}$ on both side and calculating the equation $D_\tau^{2\alpha}\left[\text{Res}\psi_3(\xi,0)\right] = 0$, then we get

$$\phi_3 = -6\phi\phi_{2\xi} - 6\phi_2\phi_\xi - \phi_{2\xi\xi\xi} - 6\phi_1\phi_{1\xi}. \tag{37}$$

with,

$$\phi_3(\xi) = \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\text{sech}^2\left(\frac{\xi}{2}\right)\left(1 - 3\text{sech}^2\left(\frac{\xi}{2}\right)\right). \tag{38}$$

Similarly, for $\kappa = 4$ we have,

$$\phi_4 = -6\phi\phi_{3\xi} - 18\phi_2\phi_{1\xi} - \phi_{3\xi\xi\xi} - 18\phi_1\phi_{2\xi} - 6\phi_3\phi_\xi. \tag{39}$$

with,

$$\phi_4(\xi) = \frac{1}{2}\text{sech}^2\left(\frac{\xi}{2}\right) - \frac{15}{4}\text{sech}^4\left(\frac{\xi}{2}\right) + \frac{15}{4}\text{sech}^6\left(\frac{\xi}{2}\right). \tag{40}$$

Subsequently, values of $\phi_5, \phi_6, ...$can be obtained,

Now, substituting values of $\phi, \phi_1, \phi_2, ...$ in equation (16), $\phi(\xi,\tau)$ is expressed in terms of series as,

$$\begin{aligned}\phi(\xi,\tau) =& \frac{1}{2}\text{sech}^2\left(\frac{\xi}{2}\right) + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\text{sech}^2\left(\frac{\xi}{2}\right)\frac{\tau^\alpha}{\Gamma(1+\alpha)} \\ &+ \frac{1}{4}\text{sech}^2\left(\frac{\xi}{2}\right)\left(2 - 3\text{sech}^2\left(\frac{\xi}{2}\right)\right)\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\text{sech}^2\left(\frac{\xi}{2}\right)\left(1 - 3\text{sech}^2\left(\frac{\xi}{2}\right)\right)\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)} \\ &+ \left(\frac{1}{2}\text{sech}^2\left(\frac{\xi}{2}\right) - \frac{15}{4}\text{sech}^4\left(\frac{\xi}{2}\right) + \frac{15}{4}\text{sech}^6\left(\frac{\xi}{2}\right)\right)\frac{\tau^{4\alpha}}{\Gamma(1+4\alpha)} + ...\end{aligned} \tag{41}$$

By considering the different values of $\alpha \in (0,1]$, corresponding values of $\psi(\xi,\tau)$ are discussed in Table (1).

5. Results

The present paper shows the approximate analytical solution obtained by using FRPSM for the time-fractional KdV equation. Comparison of obtained semi-analytical solution from FRPSM with exact solution using conventional method and ADM is demonstrated in Table-1 to Table-5. The behaviour of different fractional order α is shown in Figure-1. The dynamic solution of the KdV equation using different fractional orders i.e $\alpha = 0.2$, 0.4, 0.6, 0.8 along with $\alpha = 1$ and exact solution are graphically shown in Figure(2a to 2f). Further, Table-6 represents different values of $\psi(\xi, \tau)$ for $\xi = 2$ and $\alpha \in (0, 1]$. Further, the hereditary properties for different values of x and t with different fractional order α can be used for further study. The results so obtained clearly states the convergence of the method.

6. Conclusion

The present paper exhibits the solution of the well-known KdV equation in fractional order. We proposed the semi-analytic solution using the fractional residual power series method. FRPSM can be employed to various non-linear and linear FPDE. The results discussed in this article demonstrate good accuracy and are going to be useful to several complicated non-linear physical problems. The calculations of this technique are straightforward. In addition, we found improved accuracy of the proposed method by calculating the absolute error (Tables - 1 to 5) and compared it to the other iterative method. These errors turned out to be very small for both exact solutions and earlier approximate solutions. Figure (2a to 2f) shows that while changing fractional order α, three-dimensional graph increases the intelligibility of dynamic behaviour of the system. The method is also capable of solving the KdV type equation with other types of boundary conditions and initial conditions. Consequently, this method is a semi-analytic technique with high accuracy, exponential convergence rates, and is easy to use for different boundary conditions and can be useful to solve the similar type of space time-fractional differential equations.

Conflict of interest

The authors declare that they have no conflict of interest.

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Table 1: The Absolute Error in Solution of KdV Equation by RPSM Method and ADM Method[32] when $\xi = -20$ and $\alpha \to 1$.

τ	$\lvert\psi_{exact} - \psi_{rpsm}\rvert$	$\lvert\psi_{exact} - \psi_{adm}\rvert$
0.1	-8.25131E-10	3.73002E-09
0.2	-1.65416E-09	3.37506E-09
0.3	-2.49062E-09	3.05388E-09
0.4	-3.33775E-09	2.76327E-09
0.5	-4.19844E-09	2.50031E-09

Table 2: When $\xi = -10$ and $\alpha \to 1$.

τ	$\lvert\psi_{exact} - \psi_{rpsm}\rvert$	$\lvert\psi_{exact} - \psi_{adm}\rvert$
0.1	-2.86113E-05	8.21524E-05
0.2	-3.64285E-05	7.43351E-05
0.3	-5.48495E-05	6.72617E-05
0.4	-7.35051E-05	6.08613E-05
0.5	-9.24593E-05	5.50699E-05

Table 3: When $\xi = 0$ and $\alpha \to 1$.

τ	$\lvert\psi_{exact} - \psi_{rpsm}\rvert$	$\lvert\psi_{exact} - \psi_{adm}\rvert$
0.1	-2.90000E-09	2.08040E-06
0.2	-1.87933E-07	3.31454E-05
0.3	-2.12660E-06	1.66623E-04
0.4	-1.18419E-05	5.21491E-04
0.5	-4.46589E-05	1.25742E-03

Table 4: When $\xi = 10$ and $\alpha \to 1$.

τ	$\lvert\psi_{exact} - \psi_{rpsm}\rvert$	$\lvert\psi_{exact} - \psi_{adm}\rvert$
0.1	1.59158E-05	1.00339E-04
0.2	3.64406E-05	1.10891E-04
0.3	5.49109E-05	1.22552E-04
0.4	7.36995E-05	1.35439E-04
0.5	9.29355E-05	1.49681E-04

Table 5: When $\xi = 20$ and $\alpha \to 1$.

τ	$\lvert\psi_{exact} - \psi_{rpsm}\rvert$	$\lvert\psi_{exact} - \psi_{adm}\rvert$
0.1	8.25166E-10	4.55585E-09
0.2	1.65471E-09	5.03500E-09
0.3	2.49341E-09	5.56453E-09
0.4	3.34659E-09	6.14976E-09
0.5	4.22009E-09	6.79654E-09

Table 6: Value of $\psi(\xi, \tau)$ with different fractional order α when $\xi = 2$.

τ	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0.1	0.354750978	0.29496648	0.259395686	0.238566145	0.226368188
0.2	0.382232355	0.327692525	0.28835476	0.261507702	0.243526238
0.3	0.400996543	0.353185341	0.313530228	0.283441821	0.261461322
0.4	0.415702913	0.375006055	0.336761221	0.305029944	0.280173439
0.5	0.427988593	0.394504163	0.358769892	0.326536438	0.299662589

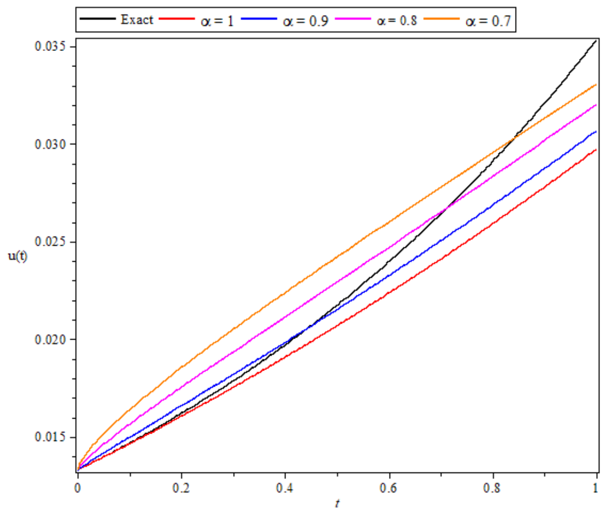
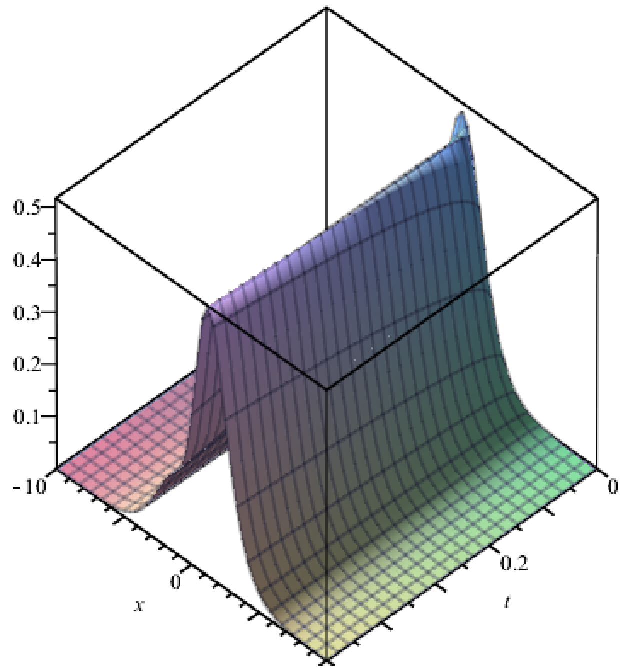
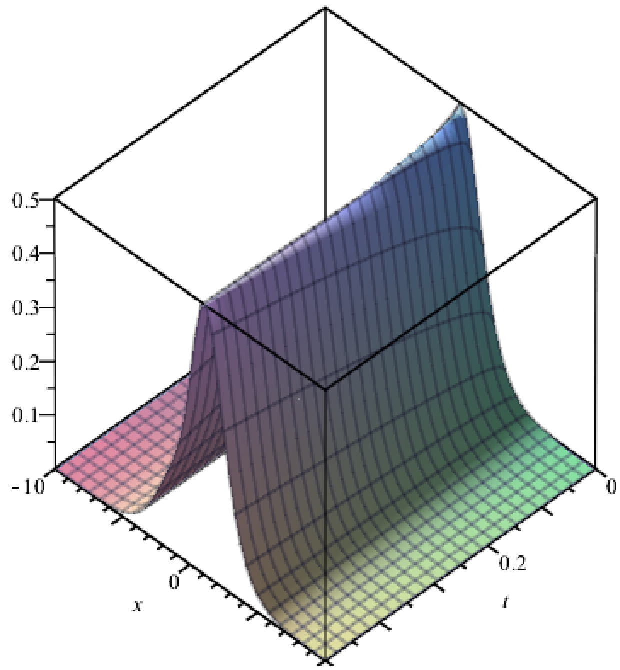


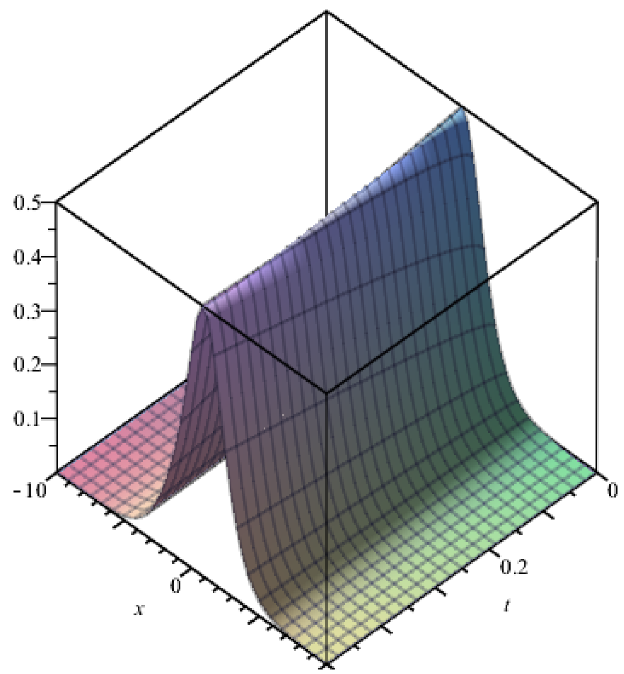
Figure 1: Comparison of fractional KdV equation with different value of α with exact solution.



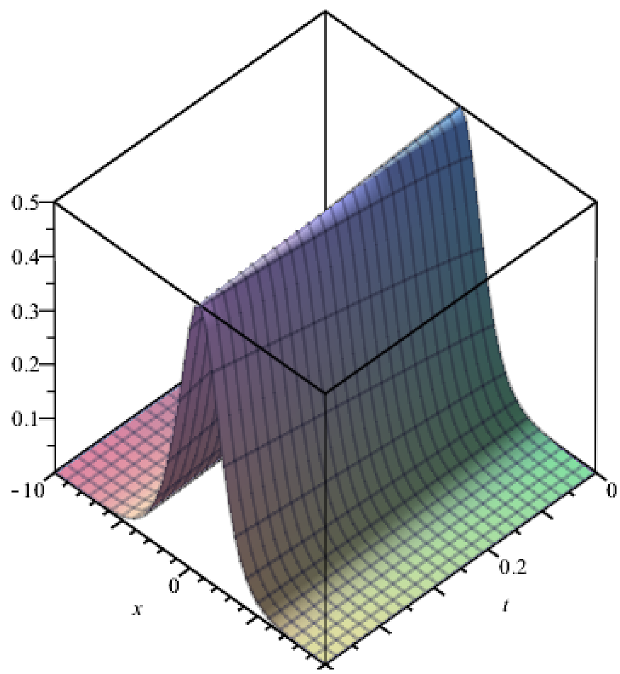
(a) $\alpha = 0.2$



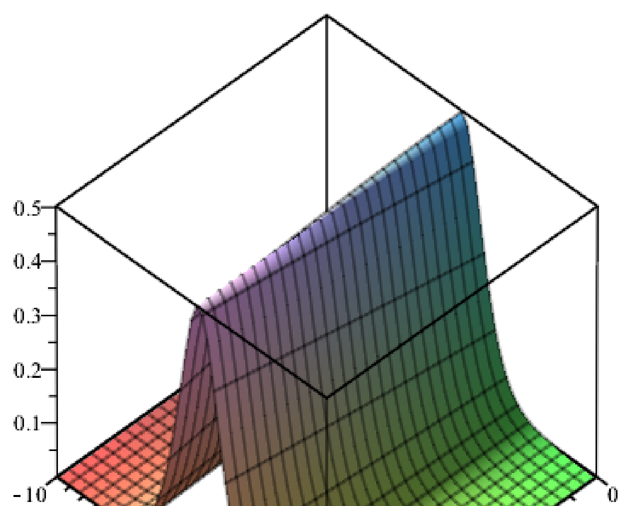
(b) $\alpha = 0.4$

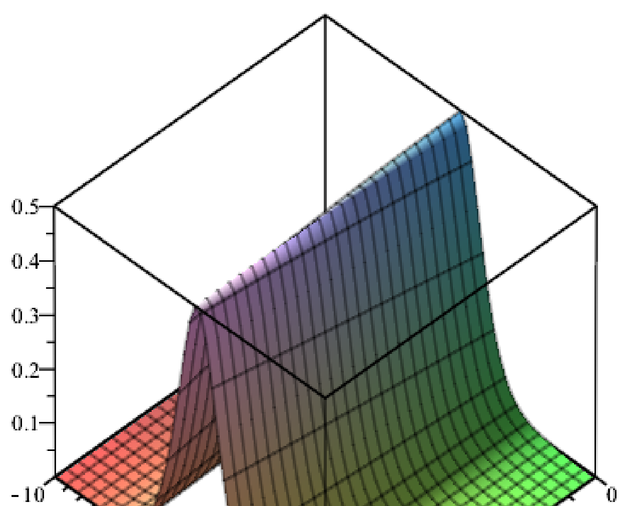


(c) $\alpha = 0.6$



(d) $\alpha = 0.8$





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