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# Green's Function of Regular Sturm-Liouville Problem Having Eigenparameter in One Boundary Condition

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Abstract. In this paper we obtain Green's function for regular Sturm-Liouville problem having the eigenparameter in the quadratic boundary condition without smoothness conditions on the potential.

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## 1. Introduction

We consider the differential equation

$$-y''(t)+q(t)\,y(t)=\lambda y(t), \tag{1.1}$$

defined on the interval $[a,b]$ where $q\in L^1[a,b]$. We impose the following boundary conditions

$$\frac{y'(a)}{y(a)} = c\lambda^2+d\lambda+e,\ c\neq 0 \tag{1.2}$$

$$\sin\beta y'(b)-\cos\beta y(b) = 0,\ \beta\in[0,\pi) \tag{1.3}$$

where $c,d,e\in\mathbb{R}$.

This problem differs from the usual regular Sturm-Liouville problems in the sense that the eigenvalue parameter $\lambda$ is involved in one of the boundary conditions. The left-hand boundary condition contains quadratic $\lambda$-dependent function.

In this paper, we obtain Green's function of the problem (1.1)-(1.3) explicitly under the sole condition that $q$ is a member of $L^1[a,b]$. Constructions of Green's function is considered by some authors to give some expansion theorems and sampling representations for transforms associated with problem (1.1)-(1.3) in which $q(t)$ is assumed to be continuous [1, 6].

We suppose without loss of generality that $q$ has a mean value zero, i.e.,

$$\int_a^b q(t)\,dt=0.$$

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As an illustration of our results we obtain that for $a \leq x \leq y \leq b, \beta \neq 0$ and $q \in L^1[a,b]$

$$\begin{aligned} G(x,y,\lambda) &= -\lambda^{-1/2}\frac{\sin\lambda^{1/2}(x-a)\cos\lambda^{1/2}(b-y)}{\cos\lambda^{1/2}(b-a)} + \frac{\lambda^{-1}}{\cos\lambda^{1/2}(b-a)} \\ &\times\left\{\begin{array}{c} [\ \cot\beta - \frac{1}{2}\int_y^b q(t)\,dt\ ]\sin\lambda^{1/2}(x-a)\sin\lambda^{1/2}(b-y) \\ +\frac{1}{2}\left(\int_a^x q(t)\,dt\right)\cos\lambda^{1/2}(x-a)\cos\lambda^{1/2}(b-y) \\ -\cot\beta\tan\lambda^{1/2}(b-a)\sin\lambda^{1/2}(x-a)\cos\lambda^{1/2}(b-y) \end{array}\right\} \\ &+O\left(\lambda^{-1}\eta(\lambda)\right) \end{aligned}$$

where $\eta(\lambda) \to 0$ as $\lambda \to \infty$. Similar results hold for $a \leq y \leq x \leq b$ changing the role of $x$ and $y$.

## 2. The Method

As similar to the [3], we reduce (1.1) to the Riccati equation

$$v' = -\lambda + q - v^2. \tag{2.1}$$

It was shown in [7] that if $v(t,\lambda)$ is a complex-valued solution of (2.1) and

$$S(t,\lambda) := Re\{v(t,\lambda)\},$$

$$T(t,\lambda) := Im\{v(t,\lambda)\},$$

then any nontrivial real-valued solution, z, of (1.1) can be expressed as

$$z(t,\lambda) = c_1\exp\left(\int_a^t S(x,\lambda)dx\right)\cos\left\{c_2 + \int_a^t T(x,\lambda)dx\right\} \tag{2.2}$$

with

$$\begin{aligned} z'(t,\lambda) &= c_1 S(t,\lambda)\exp\left(\int_a^t S(x,\lambda)dx\right)\cos\left\{c_2 + \int_a^t T(x,\lambda)dx\right\} \\ &-c_1\exp\left(\int_a^t S(x,\lambda)dx\right)\sin\left\{c_2 + \int_a^t T(x,\lambda)dx\right\}T(t,\lambda). \end{aligned} \tag{2.3}$$

We suppose that there exist functions $A(t)$ and $\eta(\lambda)$ so that

$$\left|\int_t^b e^{2i\lambda^{1/2}x}q(x)dx\right| \leq A(t)\,\eta(\lambda), \qquad t \in [a,b]$$

where $A(t) := \int_t^b |q(x)|\,dx$ which is a decreasing function of $t$, $\eta(\lambda) \to 0$ as $\lambda \to \infty$ and $A(t) \in L^1[a,b]$. The existence of these functions are established in [2]. We restate $F(t,\lambda)$ for completeness.

We first define

$$F(t,\lambda) := \begin{cases} \left|\int_t^b e^{2i\lambda^{1/2}x}q(x)dx\right| / \int_t^b |q(x)|\,dx, & \text{if } \int_t^b |q(x)|\,dx \neq 0, \\ 0, & \text{if } \int_t^b |q(x)|\,dx = 0 \end{cases} \tag{2.4}$$

and we set $\eta(\lambda) := \sup_{a\leq t\leq b} F(t,\lambda)$ ( $0 \leq F(t,\lambda) \leq 1$). $\eta(\lambda)$ is well defined by (2.4) and goes to zero as $\lambda \to \infty$ [7].

We now approximate to a solution of (2.1) on $[a,b]$. For this reason, we set

$$v(t,\lambda) := i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(t,\lambda)$$

and choose $v_n$ so that

$$\begin{aligned} v_1' + 2i\lambda^{1/2}v_1 &= q, \\ v_2' + 2i\lambda^{1/2}v_2 &= -v_1^2 \end{aligned}$$

and for $n = 3, 4, \ldots.$

$$v_n' + 2i\lambda^{1/2}v_n = -\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right).$$

Solution of above equation for $n = 1, 2, 3, ...$ is

$$v_1(t,\lambda) = -e^{-2i\lambda^{1/2}t}\int_t^b e^{2i\lambda^{1/2}x}q(x)dx,$$

$$v_2(t,\lambda) = e^{-2i\lambda^{1/2}t}\int_t^b e^{2i\lambda^{1/2}x}v_1^2(x,\lambda)dx,$$

$$v_n(t,\lambda) = e^{-2i\lambda^{1/2}t}\int_t^b e^{2i\lambda^{1/2}x}\left( v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m \right)dx.$$

It is shown in [2] that the series $\sum_{n=1}^{\infty} v_n(t,\lambda)$ and $\sum_{n=1}^{\infty} v_n'(t,\lambda)$ are uniformly absolutely convergent for all $\lambda \geq \lambda_0$ and for all $t \in [a,b]$. The series $i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(t,\lambda)$ is a solution of (2.1) and

$$S(t,\lambda) = \mathrm{Re}\sum_{n=1}^{\infty} v_n(t,\lambda),$$

$$T(t,\lambda) = \lambda^{1/2} + Im\sum_{n=1}^{\infty} v_n(t,\lambda).$$

The asymptotic forms of $S(t,\lambda)$ and $T(t,\lambda)$ are obtained in [4] as follows

$$S(t,\lambda) = -\sin(2\lambda^{1/2}t + \zeta_t) + O(\eta^2(\lambda)) \tag{2.5}$$

and

$$T(t,\lambda) = \lambda^{1/2} - \cos(2\lambda^{1/2}t + \zeta_t) + O(\eta^2(\lambda)) \tag{2.6}$$

where

$$\sin\zeta_t = \int_t^b q(x)\cos 2\lambda^{1/2}x dx, \quad \cos\zeta_t = \int_t^b q(x)\sin 2\lambda^{1/2}x dx.$$

And following two equalities are obtained in [5]

$$\begin{aligned}\int_a^t S(x,\lambda)dx &= \frac{1}{2\lambda^{1/2}}\{ \cos(2\lambda^{1/2}t + \zeta_t) - \cos(2\lambda^{1/2}a + \zeta_a) \} \\ &\quad + O(\lambda^{-1/2}\eta^2(\lambda)),\end{aligned} \tag{2.7}$$

$$\begin{aligned}\int_a^t T(x,\lambda)dx &= \lambda^{1/2}(t-a) - \frac{1}{2\lambda^{1/2}}\{ \sin(2\lambda^{1/2}t + \zeta_t) - \sin(2\lambda^{1/2}a + \zeta_a) \\ &\quad + \int_a^t q(x)dx \} + O(\lambda^{-1/2}\eta^2(\lambda)).\end{aligned} \tag{2.8}$$

## 3. Approximations For The Eigenfunctions

In this section we obtain approximations for the solution of (1.1)-(1.3). We define two solutions, $\Psi(t,\lambda)$ and $\Phi(t,\lambda)$ of (1.1)-(1.3) with the initial conditions

$$\Psi(a,\lambda) = 1, \quad \Psi'(a,\lambda) = c\lambda^2 + d\lambda + e \tag{3.1}$$

and

$$\Phi(b,\lambda) = \sin\beta, \quad \Phi'(b,\lambda) = \cos\beta. \tag{3.2}$$

**Theorem 3.1.** *Let $\Psi(t,\lambda)$ and $\Phi(t,\lambda)$ be the solutions of (1.1) satisfying (3.1) and (3.2), respectively. Then*
*(i)*

$$\begin{aligned}\Psi(t,\lambda) &= \frac{1}{\cos\{ \cot^{-1}[ \frac{T(a,\lambda)}{-c\lambda^2-d\lambda-e+S(a,\lambda)} ] \}}\exp\left( \int_a^t S(x,\lambda)dx \right) \\ &\quad \times\cos\left\{ \cot^{-1}\left[ \frac{T(a,\lambda)}{-c\lambda^2 - d\lambda - e + S(a,\lambda)} \right] + \int_a^t T(x,\lambda)dx \right\},\end{aligned} \tag{3.3}$$

*(ii) a)* $\beta \neq 0$

$$\Phi(t,\lambda) = \frac{\sin\beta}{\cos\{\tan^{-1}[\frac{S(b,\lambda)-\cot\beta}{T(b,\lambda)}]\}}\exp(-\int_t^b S(x,\lambda)dx) \times\cos\{\tan^{-1}[\frac{S(b,\lambda)-\cot\beta}{T(b,\lambda)}]-\int_t^b T(x,\lambda)dx\}, \tag{3.4}$$

*b)* $\beta = 0$

$$\Phi(t,\lambda) = -\frac{1}{T(b,\lambda)}\exp(-\int_t^b S(x,\lambda)dx)\cos\{\frac{\pi}{2}-\int_t^b T(x,\lambda)\,dx\}.$$

*Proof.* (i) From (2.2), (2.3) and (3.1) we obtain

$$\Psi(a,\lambda) = c_1\cos c_2 = 1, \tag{3.5}$$

$$\Psi'(a,\lambda) = c_1 S(a,\lambda)\cos c_2 - c_1\sin c_2 T(a,\lambda) = c\lambda^2 + d\lambda + e. \tag{3.6}$$

From (3.5)

$$c_1 = \frac{1}{\cos c_2}.$$

Using $c_1$ in (3.6),

$$c_2 = \cot^{-1}[\frac{T(a,\lambda)}{-c\lambda^2 - d\lambda - e + S(a,\lambda)}]. \tag{3.7}$$

Hence

$$c_1 = \frac{1}{\cos\{\cot^{-1}[\frac{T(a,\lambda)}{-c\lambda^2-d\lambda-e+S(a,\lambda)}]\}}. \tag{3.8}$$

Substitution the values of $c_1$ and $c_2$ into (2.2) proves part (i).
(ii) *a)* For $\beta \neq 0$; from (2.2), (2.3) and (3.2) we obtain

$$\Phi(b,\lambda) = c_1\exp(\int_a^b S(x,\lambda)dx)\cos\{c_2+\int_a^b T(x,\lambda)dx\} = \sin\beta, \tag{3.9}$$

$$\begin{aligned}\Phi'(b,\lambda) &= c_1 S(b,\lambda)\exp(\int_a^b S(x,\lambda)dx)\cos\{c_2+\int_a^b T(x,\lambda)dx\} \\ &\quad -c_1\exp(\int_a^b S(x,\lambda)dx)\sin\{c_2+\int_a^b T(x,\lambda)dx\}T(b,\lambda) \\ &= \cos\beta. \end{aligned} \tag{3.10}$$

From (3.9)

$$c_1 = \frac{\sin\beta}{\exp(\int_a^b S(x,\lambda)dx)\cos\{c_2+\int_a^b T(x,\lambda)dx\}}.$$

Using $c_1$ in (3.10),

$$c_2 = \tan^{-1}\{\frac{S(b,\lambda)-\cot\beta}{T(b,\lambda)}\} - \int_a^b T(x,\lambda)dx. \tag{3.11}$$

Hence

$$c_1 = \frac{\sin\beta}{\exp(\int_a^b S(x,\lambda)dx)\cos\{\tan^{-1}[\frac{S(b,\lambda)-\cot\beta}{T(b,\lambda)}]\}}. \tag{3.12}$$

Substitution the values of $c_1$ and $c_2$ into (2.2) proves the result.

*b*) For $\beta = 0$; similarly, we obtain from (2.2), (2.3) and (3.2)

$$\begin{aligned}
\chi(b,\lambda) &= c_1 \exp\Big( \int_a^b S(x,\lambda)dx \Big) \cos\Big\{ c_2 + \int_a^b T(x,\lambda)dx \Big\} = 0 \\
\chi'(b,\lambda) &= c_1 S(b,\lambda) \exp\Big( \int_a^b S(x,\lambda)dx \Big) \cos\Big\{ c_2 + \int_a^b T(x,\lambda)dx \Big\} \\
&\quad - c_1 \exp\Big( \int_a^b S(x,\lambda)dx \Big) \sin\Big\{ c_2 + \int_a^b T(x,\lambda)dx \Big\} T(b,\lambda) \\
&= 1.
\end{aligned}$$

Hence from the last two equalities we have

$$\begin{aligned}
c_1 &= -\frac{1}{T(b,\lambda)\exp\big( \int_a^b S(x,\lambda)\,dx \big)}, \\
c_2 &= \frac{\pi}{2} - \int_a^b T(x,\lambda)\,dx.
\end{aligned}$$

Proof is completed by using the values of $c_1$ and $c_2$ into (2.2). □

**Theorem 3.2.** *As* $\lambda \longrightarrow \infty$

*(i)*

$$\Psi(t,\lambda) = c\lambda^{3/2} \sin \lambda^{1/2}(t-a) - \frac{c}{2}\lambda \Big[ \int_a^t q(x)dx \Big] \cos \lambda^{1/2}(t-a) + O(\lambda\eta(\lambda)), \tag{3.13}$$

*(ii) a)* $\beta \neq 0$

$$\begin{aligned}
\Phi(t,\lambda) &= \sin\beta \cos \lambda^{1/2}(b-t) + \lambda^{-1/2} \Big[ \frac{\sin\beta}{2} \int_t^b q(x)dx - \cos\beta \Big] \\
&\quad \times \sin \lambda^{1/2}(b-t) + O(\lambda^{-1/2}\eta(\lambda)).
\end{aligned} \tag{3.14}$$

*b)* $\beta = 0$

$$\Phi(t,\lambda) = -\lambda^{-1/2} \sin \lambda^{1/2}(b-t) + \frac{1}{2}\lambda^{-1} \Big[ \int_t^b q(x)dx \Big] \cos \lambda^{1/2}(b-t) + O\left(\lambda^{-1}\eta(\lambda)\right).$$

*Proof.* (i) We evaluate the terms in (3.3) as $\lambda \to \infty$. Firstly, using (2.5) and (2.6) together with the series expansion leads to

$$\begin{aligned}
\frac{T(a,\lambda)}{-c\lambda^2 - d\lambda - e + S(a,\lambda)} &= \frac{\lambda^{1/2} - \cos(2\lambda^{1/2}a + \zeta_a) + O(\eta^2(\lambda))}{-c\lambda^2 - d\lambda - e - \sin(2\lambda^{1/2}a + \zeta_a) + O(\eta^2(\lambda))} \\
&= \frac{\lambda^{1/2} - \cos(2\lambda^{1/2}a + \zeta_a) + O(\eta^2(\lambda))}{-c\lambda^2 \left[ \begin{array}{c} 1 + \frac{d}{c}\lambda^{-1} + \frac{e}{c}\lambda^{-2} + \frac{1}{c}\lambda^{-2} \\ \times \sin(2\lambda^{1/2}a + \zeta_a) + O\left(\lambda^{-2}\eta^2(\lambda)\right) \end{array} \right]} \\
&= \left[ \begin{array}{c} -\frac{1}{c}\lambda^{-3/2} + \frac{1}{c}\lambda^{-2}\cos(2\lambda^{1/2}a + \zeta_a) \\ +O\left(\lambda^{-2}\eta^2(\lambda)\right) \end{array} \right] \\
&\quad \times \left[ \begin{array}{c} 1 - \frac{d}{c}\lambda^{-1} - \frac{e}{c}\lambda^{-2} - \frac{1}{c}\lambda^{-2}\sin(2\lambda^{1/2}a + \zeta_a) \\ +O\left(\lambda^{-2}\eta^2(\lambda)\right) \end{array} \right] \\
&= -\frac{1}{c}\lambda^{-3/2} + O\left(\lambda^{-2}\eta(\lambda)\right).
\end{aligned}$$

From the last equality we obtain

$$\cot^{-1}\Big[ \frac{T(a,\lambda)}{-c\lambda^2 - d\lambda - e + S(a,\lambda)} \Big] = \frac{\pi}{2} + \frac{1}{c}\lambda^{-3/2} + O\left(\lambda^{-2}\eta(\lambda)\right),$$

and

$$\begin{aligned}
\cos\{\cot^{-1}[\frac{T(a,\lambda)}{-c\lambda^2-d\lambda-e+S(a,\lambda)}]\} &= -\sin[-\frac{1}{c}\lambda^{-3/2}+O(\lambda^{-2}\eta(\lambda))] \\
&= -\frac{1}{c}\lambda^{-3/2}+O(\lambda^{-2}\eta(\lambda)), \\
\sin\{\cot^{-1}[\frac{T(a,\lambda)}{-c\lambda^2-d\lambda-e+S(a,\lambda)}]\} &= \cos[-\frac{1}{c}\lambda^{-3/2}+O(\lambda^{-2}\eta(\lambda))] \\
&= 1-\frac{1}{2c^2}\lambda^{-3}+O(\lambda^{-7/2}\eta(\lambda)), \\
\frac{1}{\cos\{\cot^{-1}[\frac{T(a,\lambda)}{-c\lambda^2-d\lambda-e+S(a,\lambda)}]\}} &= \frac{1}{-\frac{1}{c}\lambda^{-3/2}+O(\lambda^{-2}\eta(\lambda))} \\
&= -c\lambda^{3/2}+O(\lambda\eta(\lambda)). \qquad (3.15)
\end{aligned}$$

Using $\int_a^t S(x,\lambda)dx$ given by (2.7) we get

$$\begin{aligned}
\exp(\int_a^t S(x,\lambda)dx) &= 1+\frac{1}{2\lambda^{1/2}}\{\cos(2\lambda^{1/2}t+\zeta_t)-\cos(2\lambda^{1/2}a+\zeta_a)\} \\
&\quad +O(\lambda^{-1/2}\eta^2(\lambda)) \qquad (3.16)
\end{aligned}$$

and from $\int_a^t T(x,\lambda)dx$ given by (2.8)

$$\begin{aligned}
\sin(\int_a^t T(x,\lambda)dx) &= \sin[\lambda^{1/2}(t-a)-\frac{1}{2\lambda^{1/2}}\int_a^t q(x)dx] \\
&\quad +O(\lambda^{-1/2}\eta(\lambda)), \\
\cos(\int_a^t T(x,\lambda)dx) &= \cos[\lambda^{1/2}(t-a)-\frac{1}{2\lambda^{1/2}}\int_a^t q(x)dx] \\
&\quad +O(\lambda^{-1/2}\eta(\lambda)).
\end{aligned}$$

Hence

$$\begin{aligned}
&\cos\{\cot^{-1}[\frac{T(a,\lambda)}{-c\lambda^2-d\lambda-e+S(a,\lambda)}+\int_a^t T(x,\lambda)dx\} \\
&= -\sin[\lambda^{1/2}(t-a)-\frac{1}{2\lambda^{1/2}}\int_a^t q(x)dx]+O(\lambda^{-1/2}\eta(\lambda)). \qquad (3.17)
\end{aligned}$$

Substituting the values of (3.15), (3.16) and (3.17) into (3.3) and using trigonometric expansions the proof is finished.

(ii) For $\beta \neq 0$; we evaluate the terms in (3.4) as $\lambda \to \infty$. Using (2.5) and (2.6), we obtain

$$\begin{aligned}
\frac{S(b,\lambda)-\cot\beta}{T(b,\lambda)} &= \frac{-\cot\beta+O\left(\eta^2(\lambda)\right)}{\lambda^{1/2}[1+O(\lambda^{-1/2}\eta^2(\lambda))]} \\
&= -\lambda^{-1/2}\cot\beta+O\left(\lambda^{-1/2}\eta^2(\lambda)\right)
\end{aligned}$$

and from

$$\tan^{-1}[\frac{S(b,\lambda)-\cot\beta}{T(b,\lambda)}] = -\lambda^{-1/2}\cot\beta+O\left(\lambda^{-1/2}\eta^2(\lambda)\right), \qquad (3.18)$$

$$\cos\{\tan^{-1}[\frac{S(b,\lambda)-\cot\beta}{T(b,\lambda)}]\} = 1-\frac{\lambda^{-1}\cot^2\beta}{2}+O(\lambda^{-1}\eta^2(\lambda)), \qquad (3.19)$$

$$\sin\{\tan^{-1}[\frac{S(b,\lambda)-\cot\beta}{T(b,\lambda)}]\} = -\lambda^{-1/2}\cot\beta+O\left(\lambda^{-1/2}\eta^2(\lambda)\right).$$

From (3.19), we find

$$\begin{aligned}\frac{\sin\beta}{\cos\{\,\tan^{-1}[\,\frac{S(b,\lambda)-\cot\beta}{T(b,\lambda)}\,]\,\}} &= \frac{\sin\beta}{1-\frac{\lambda^{-1}\cot^2\beta}{2}+O(\lambda^{-1}\eta^2(\lambda))} \\ &= \sin\beta\times[\,1+\frac{\lambda^{-1}\cot^2\beta}{2}+O(\lambda^{-1}\eta^2(\lambda))\,] \\ &= \sin\beta+\lambda^{-1}\frac{\cos^2\beta}{2\sin\beta}+O\left(\lambda^{-1}\eta^2(\lambda)\right) \qquad (3.20)\end{aligned}$$

Using $\int_a^t S(x,\lambda)dx$ given by (2.7) we get

$$\exp\left(-\int_t^b S(x,\lambda)dx\right) = 1+\frac{1}{2\lambda^{1/2}}\cos(2\lambda^{1/2}t+\zeta_t)+O\left(\lambda^{-1/2}\eta^2(\lambda)\right) \qquad (3.21)$$

and from $\int_a^t T(x,\lambda)dx$ given by (2.8)

$$\begin{aligned}\int_t^b T(x,\lambda)dx &= \int_a^b T(x,\lambda)dx-\int_a^t T(x,\lambda)dx \\ &= \lambda^{1/2}(b-t)+\frac{1}{2\lambda^{1/2}}\{\,\sin(2\lambda^{1/2}t+\zeta_t)-\int_t^b q(x)dx\,\} \\ &\quad +O(\lambda^{-1/2}\eta^2(\lambda)).\end{aligned}$$

From the last equality and (3.18) we see that

$$\begin{aligned}&\cos\{\,\tan^{-1}[\,\frac{S(b,\lambda)-\cot\beta}{T(b,\lambda)}\,]-\int_t^b T(x,\lambda)dx\,\} \\ =\;&\cos[\,\lambda^{1/2}(b-t)-\frac{1}{2\lambda^{1/2}}\int_t^b q(x)dx\,] \\ &-\lambda^{-1/2}\cot\beta\sin[\,\lambda^{1/2}(b-t)-\frac{1}{2\lambda^{1/2}}\int_t^b q(x)dx\,] \\ &+O(\lambda^{-1/2}\eta(\lambda)) \qquad (3.22)\end{aligned}$$

Finally, substituting the values of (3.20), (3.21) and (3.22) into (3.4) and using trigonometric expansions, we complete the prove. For $\beta = 0$; the prove is similar. □

## 4. Approximations For The Green's Function

In this section, we obtain asymptotic approximations for Green's function of (1.1)-(1.3). Let $W_t(\Psi,\Phi)$ be the Wronskian of $\Psi(t,\lambda)$ and $\Phi(t,\lambda)$. We define $w(\lambda)$ as follows

$$w(\lambda) := W_x(\Psi,\Phi) = \Psi(t,\lambda)\Phi'(t,\lambda)-\Psi'(t,\lambda)\Phi(t,\lambda). \qquad (4.1)$$

It is known that Green's function of problem (1.1)-(1.3) is

$$G(x,y,\lambda) = \begin{cases}\frac{\Psi(x,\lambda)\Phi(y,\lambda)}{w(\lambda)}, & a\le x\le y\le b, \\ \frac{\Phi(x,\lambda)\Psi(y,\lambda)}{w(\lambda)}, & a\le y\le x\le b.\end{cases} \qquad (4.2)$$

**Theorem 4.1.** *For $a\le x\le y\le b$, as $\lambda\to\infty$*

*(i)* $\beta \neq 0$

$$G(x,y,\lambda) = -\lambda^{-1/2}\frac{\sin\lambda^{1/2}(x-a)\cos\lambda^{1/2}(b-y)}{\cos\lambda^{1/2}(b-a)} + \frac{\lambda^{-1}}{\cos\lambda^{1/2}(b-a)} \times \left\{ \begin{array}{c} [\,\cot\beta - \frac{1}{2}\int_y^b q(t)\,dt\,]\sin\lambda^{1/2}(x-a)\sin\lambda^{1/2}(b-y) \\ +\frac{1}{2}\left(\int_a^x q(t)\,dt\right)\cos\lambda^{1/2}(x-a)\cos\lambda^{1/2}(b-y) \\ -\cot\beta\tan\lambda^{1/2}(b-a)\sin\lambda^{1/2}(x-a)\cos\lambda^{1/2}(b-y) \end{array} \right\} + O\left(\lambda^{-1}\eta(\lambda)\right),$$

*(ii)* $\beta = 0$

$$G(x,y,\lambda) = -\lambda^{-1/2}\frac{\sin\lambda^{1/2}(x-a)\sin\lambda^{1/2}(b-y)}{\sin\lambda^{1/2}(b-a)} + \frac{\lambda^{-1}}{\sin\lambda^{1/2}(b-a)} \times \left\{ \begin{array}{c} \frac{1}{2}\left(\int_y^b q(t)\,dt\right)\sin\lambda^{1/2}(x-a)\cos\lambda^{1/2}(b-y) \\ +\frac{1}{2}\left(\int_a^x q(t)\,dt\right)\cos\lambda^{1/2}(x-a)\sin\lambda^{1/2}(b-y) \end{array} \right\} + O\left(\lambda^{-1}\eta(\lambda)\right).$$

*Similar results are achieved for $a \le y \le x \le b$ by changing the role of $x$ and $y$.*

*Proof.* (i) For the Wronskian, $w(\lambda)$, we need $\Psi'(t,\lambda)$ and $\Phi'(t,\lambda)$ which are obtained from $z'(t,\lambda)$ given by (2.3). To obtain the derivation of $\Psi(t,\lambda)$, we substitute (3.7) and (3.8) into (2.3) and evaluate the terms as $\lambda \to \infty$. Hence

$$\Psi'(t,\lambda) = c\lambda^2\cos\lambda^{1/2}(t-a) + \frac{c}{2}\left(\int_a^t q(x)\,dx\right)\lambda^{3/2}\sin\lambda^{1/2}(t-a) + O\left(\lambda^{3/2}\eta(\lambda)\right). \tag{4.3}$$

At the same time, substituting (3.11) and (3.12) into (2.3) and evaluating the terms as $\lambda \to \infty$, we have

$$\Phi'(t,\lambda) = \lambda^{1/2}\sin\beta\sin\lambda^{1/2}(b-t) + \left[\cos\beta - \frac{\sin\beta}{2}\int_t^b q(x)\,dx\right]\cos\lambda^{1/2}(b-t) + O(\eta(\lambda)). \tag{4.4}$$

Using (3.13), (3.14), (4.3) and (4.4) in (4.1), we get

$$w(\lambda) = -c\lambda^2\sin\beta\cos\lambda^{1/2}(b-a) + c\lambda^{3/2}\cos\beta\sin\lambda^{1/2}(b-a) + O\left(\lambda^{3/2}\eta(\lambda)\right).$$

From which

$$\frac{1}{w(\lambda)} = \frac{1}{-c\lambda^2\sin\beta\cos\lambda^{1/2}(b-a) \times\left\{1-\lambda^{-1/2}\cot\beta\tan\lambda^{1/2}(b-a) + O\left(\lambda^{-1/2}\eta(\lambda)\right)\right\}} = -\frac{1}{c\sin\beta\cos\lambda^{1/2}(b-a)}\lambda^{-2} - \frac{\cos\beta}{c\sin^2\beta}\lambda^{-5/2}\frac{\sin\lambda^{1/2}(b-a)}{\cos^2\lambda^{1/2}(b-a)} + O\left(\lambda^{-5/2}\eta(\lambda)\right). \tag{4.5}$$

Theorem 4.1 (i) is proved by substituting (3.13), (3.14) and (4.5) into (4.2). The other part (ii) can be proved similarly. □

## References


[1] Annaby, M. H., Tharwat, M. M., *On sampling theory and eigenvalue problems with an eigenparameter in the boundary conditions*, Science University of Tokyo Journal of Mathematics, **42**(2006), 157–176. 1

[2] Coşkun, H., Harris, B. J., *Estimates for the periodic and semi-periodic eigenvalues of Hill's equation*, Proc. Roy. Soc. Edinburgh Sect. A, **130**(2000), 991–998. 2, 2

[3] Coşkun, H., *Asymptotic approximations of eigenvalues and eigenfunctions for regular Sturm-Liouville problems*, Rocky Mountain J. Math, **36**(2006), 867–883. 2

[4] Coşkun, H., Başkaya, E., *Asymptotics of the eigenvalues of regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition for integrable potential*, Math. Scand., **107**(2010), 209–223. 2

[5] Coşkun, H., Kabataş, A., *Asymptotic approximations of eigenfunctions for regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition for integrable potential*, Math. Scand., **113**(2013), 143–160. 2

[6] Fulton, C. T., *Two point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edin. A., **77**(1977), 293–308. 1

[7] Harris, B. J., *The form of the spectral functions associated with Sturm-Liouville problems with continuous spectrum*, Mathematika, **44**(1997), 149–161. 2, 2