



Embedding Orthomorphisms d -Algebra in Biorthomorphisms as Ordered Ideal

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Abstract — In the historical development of Riesz spaces, we can trace the history of ordered vector spaces to the International Mathematical Congress in Bologna in 1928. Studies related with f –Algebras for the Dedekind complete ordered vector space defined in Riesz spaces were initiated by Nakano and their current definition was made by Amemiya, Birkhoff and Pierce. The revival of f –algebras, which had a tendency to slow down for a period of time, emerged as a result of Pagter's doctoral thesis [9] and the examination of Alkansas lecture notes by Luxemburg. The concepts of homomorphism, isomorphism, orthomorphism and biorthomorphism in Riesz spaces are defined by Zaanen, Huijsmans, Boulabiar, Buskes and Triki. Algebraic structure of biorthomorphisms defined on Riesz space examined by [8]. f -Algebra on $Orth(X, X)$ were studied by [8] and [6]. [6] demonstrated that biorthomorphisms space have an f –algebraic structure with the help of the product defined as $(T_1 *_e T_2)(x, y) = T_1(x, T_2(e, y))$ for $e \in X^+$, $\forall x, y \in X$ and $T_1, T_2 \in Orth(X, X)$. [8] showed that if X are semiprime Dedekind complete f –algebras, $Orth(X)$ is an ordered ideal in biorthomorphisms. [6] developed an alternative proof for this situation. If X Archimedean Riesz space, $Orth(X)$ is an f –algebra according to compound operation with unit element. [11] showed that if X is a semiprime f –algebra, it is a d –algebra. In this study, we investigated embedding orthomorphism in biorthomorphisms when X is uniformly complete d –algebra.

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1. Introduction

In this section, we gave some definitions about Riesz space and Riesz algebra.

Definition 1. Let X is a set different from empty. If relation \leq defined on X satisfy the following properties, relation \leq is called the (partial) order relation, the pair (X, \leq) is called the (partial) ordered set. For $\forall x, y, z \in X$

- i. $x \leq x$ for $\forall x \in X$.
- ii. If $x \leq y$ and $y \leq x$, $x = y$.
- iii. If $x \leq y$ and $y \leq z$, $x \leq z$ [1].

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Definition 2. Let (X, \leq) is an ordered vector space. If every finite subset different from empty has a supremum, X is called a Riesz space (or a vector lattice). Supremum of $\{x, y\}$ is demonstrated with $x \vee y$ in classic Riesz space as notation [1].

Definition 3. Let X is an Riesz space. If $nx \leq y \Rightarrow x = 0$ is satisfy for $\forall n \in \mathbb{N}$ where $x, y \in X^+$, X is called Archimedean Riesz space [1].

Definition 4. A Riesz space is called Dedekind complete if every non-empty upper bound (bottom bounded) subset has a supremum (infimum) [1].

Definition 5. Let X is an Riesz space. If X is an associative algebra and $xy \in X^+$ for $\forall x, y \in X^+$, X is called Riesz algebra (or ordered lattice algebra) [1].

Definition 6. Let X is an Riesz algebra. If $x \cdot x = x^2 \in X^+$ for $\forall x \in X$, X is called positive square or X have the positive square property.

Definition 7. Let X is an Riesz algebra. If $cx \wedge y = xc \wedge y = 0$ is satisfy for $\forall x, y \in X, x \wedge y = 0$ and $\forall c \in X^+$, X is called f –algebra [4].

Definition 8. Let X is an Riesz algebra. If $xy = 0$ is satisfy for $\forall x, y \in X, x \wedge y = 0$, X is called almost f –algebra [5].

Definition 9. Let X is an Riesz algebra. If $cx \wedge cy = xc \wedge yc = 0$ is satisfy for $\forall x, y \in X, x \wedge y = 0$ and $\forall c \in X^+$, X is called d –algebra [12].

For more information about d –algebra, [3] can be given as reference.

Definition 10. Let X is an Riesz algebra. If $x = 0$ when $x^k = 0$ for $x \in X$ and $\exists k \in \mathbb{N}$, X is called semi prime [1].

Theorem 1. If X is a semiprime f –algebra, the following statements are are equivalent to each other.

- i. X is a f –algebra.
- ii. X is a d –algebra.
- iii. X is an almost f –algebra [11].

Definition 11. If the operator T transforms every ordered bounded subset of X to an ordered bounded subset of Y , where X and Y are ordered bounded vector spaces, transformation T is called an ordered bounded operator. The set of all ordered bounded operators is denoted by $L_b(X, Y)$ [1].

[13] can be examined about biliner operators.

Definition 12. Let X is an Riesz space. If $g \in S$ for $|g| \leq |f|$ where $S \subseteq X, g \in X$ and $f \in S$, S is called solid [1].

Definition 13. Let X is an Riesz space and $Y \subseteq X$. If Y is a solid linear subspace, Y is called ideal in X .

If A and B are ideals in Y , $A \cap B$ and $A + B$ are ideals in Y . $A \cap B$ is an ideal in Y because of $A \cap B \subseteq Y$ for $A \subseteq Y$ and $B \subseteq Y$. $A + B = \{A \cup B : A \cap B = \emptyset, A, B \in Y\}$ is an ideal in Y because of $A + B \subseteq A \cup B \subseteq Y$ [1].

Definition 14. Let X is an Riesz space and Y is an ideal of X . If any subset of Y has a supremum in X and this supremum is an element of Y , in other words, if $f \in Y$ is satisfied when $Z \subseteq Y$ and $f = \sup Z$, then the ideal Y is called band in X [14].

Definition 15. Let X is an Riesz space and $T: X \rightarrow X$ is a linear operator. If $T(B) \subseteq B$ for $\forall B \subseteq X$, in other words if operator T leaves all bands of X unchanged, T is called band preserving operator [1].

Definition 16. Let X and Y are two Riesz spaces and $T: X \rightarrow Y$ is a linear operator. For $\forall x, y \in X$, if operator T is satisfy $T(x \vee y) = Tx \vee Ty$, operator T is called Riesz homomorphism [1].

Definition 17. A band preserving ordered bounded operator is called an orthomorphism. So, let X is a Riesz space and $T: X \rightarrow X$ is a bounded operator. In X , $Tx \perp y$ is provided when $x \perp y$. Also, if the orthomorphism T is positive at the same time, T is called a positive orthomorphism. In other words, T is a positive orthomorphism if and only if $x \wedge y = 0$ then $Tx \wedge y = 0$ on X . The set of all orthomorphisms on X is denoted by $Orth(X)$ [1].

Definition 18. Let X is an Archimedean Riesz space. If bilinear transformation $T: X \times X \rightarrow X$ is an orthomorphism in each component of X , T is called biorthomorphism. In other words, if $T(x, \cdot), T(\cdot, x) \in Orth(X)$ for $\forall x \in X$, bilinear transformation $T: X \times X \rightarrow X$ is called biorthomorphism on X . The set of all biorthomorphism on X is denoted by $Orth(X, X)$.

Note. $\rho: Orth(X) \rightarrow Orth(X, X)$ defined with $\rho(T)(x, y) = T(xy) = T(x)y$ is one-to-one Riesz homomorphism for $\forall T \in Orth(X)$ and $(x, y) \in X \times X$ [8].

$$\rho: (Orth(X), X \times X) \rightarrow Orth(X, X)$$

$$(T, (x, y)) \rightarrow \rho(T)(x, y) = T(xy) = T(x)y.$$

Notation. $K(T) = \{x \in X : T(x, x) = 0\}$ for $\forall T \in Orth(X, X)$ [6].

Lemma 1. Let X is an Archimedean Riesz space and $T \in Orth(X, X)$.

$$K(T) = \{x \in X : T(x, y) = 0, \forall y \in X\}.$$

Specially, $K(T)$ is an ordered ideal in X [6].

Lemma 2. Let X is an Archimedean Riesz space and $T \in Orth(X, X)$. If $x \in X$, $T(x, x) = 0 \Leftrightarrow T(x, x) \in K(T)$ [6].

Proposition 1.

- i. If X is a semiprime f –algebra, $Orth(X)$ is Riesz space in $Orth(X, X)$.
- ii. If X is a semiprime Dedekind complete f –algebra, $Orth(X)$ is an ordered ideal in $Orth(X, X)$ [8].

Theorem 2. Let X is a semiprime Dedekind complete f –algebra. $\rho(Orth(X))$ is an ordered ideal in $Orth(X, X)$ [2].

2. Embedding Orthomorphisms in Biorthomorphisms

Let X is a semiprime f –algebra. Specially, $xT(y) = T(xy) = yT(x)$ is provided for $\forall x, y \in X$ and $T \in Orth(X)$ from $\rho: Orth(X) \rightarrow Orth(X)$, $\rho(\pi)(x, y) = \pi(xy) = \pi(x)y$ for $\forall \pi \in Orth(X)$ and $(x, y) \in X \times X$. If transformation $\hat{T}: X \times X \rightarrow X$ is satisfy $\hat{T}(x, y) = T(xy) = T(x)y$, \hat{T} is called biorthomorphism. Therefore, if $\rho(T)(x, y) = \hat{T}(x, y)$, $\rho(T) = \hat{T}$. Transformation $\rho: Orth(X) \rightarrow Orth(X, X)$ is an one-to-one Riesz homomorphism [8]. So, $Orth(X)$ as Riesz subspace is embedded in $Orth(X, X)$ under transformation ρ . Then, $T \in Orth(X)$ determined with $\hat{T} \in Orth(X, X)$.

Theorem 3. If X Riesz space is uniformly complete, $Orth(X, X)$ is uniformly complete [6].

2.1. Embedding Orthomorphisms in Biorthomorphisms when X is Uniformly Complete Semiprime f –Algebra

Let X is an uniformly complete semiprime f –algebra and $X^\odot = \{xy : x, y \in X\}$. Then the set X^\odot is a Riesz subspace of X with positive cone $\{xx = x^2 : x \in X\}$ [7].

Theorem 4. Let X is an uniformly complete semiprime f –algebra and transformation $T: X \times X \rightarrow X$. T is a (positive) biorthomorphism on $X \Leftrightarrow$ There is only one positive biorthomorphism $T^\odot: X^\odot \rightarrow X$ satisfying property $T(x, y) = T^\odot(xy)$ for $\forall x, y \in X$ [6].

Proof. \Rightarrow Let T is a biorthomorphism on X . Let positive biorthomorphism $T^\odot: X^\odot \rightarrow X$ is satisfy property $T(x, y) = T^\odot(xy)$ for $\forall x, y \in X$. Let us now show that there is only one biorthomorphism. Let T_1^\odot ve T_2^\odot are two biorthomorphisms satisfying $T(x, y) = T^\odot(xy)$ on X for $\forall x, y \in X$. $T_1^\odot(xy) = T_1(x, y) = T_1(xy)$ is provided for transformation $T_1^\odot: X^\odot \rightarrow X$. Similarly, $T_2^\odot(xy) = T_2(x, y) = T_2(xy)$ is provided for $T_2^\odot: X^\odot \rightarrow X$. If $T_1(xy) = T_2(xy)$, $T_1 = T_2$. Then $T_1^\odot = T_2^\odot$.

\Leftarrow If T is a positive biorthomorphism on X , T is called orthosymmetric Riesz bimorphism. Transformation $\odot: X \times X \rightarrow X$ defined with $\odot(x, y) = xy$ is orthosymmetric Riesz bimorphism, (X^\odot, \odot) is a Riesz space and square of X [7]. Therefore, there is an only Riesz homomorphism $T^\odot: X^\odot \rightarrow X$ defined with $T(x, y) = T^\odot(xy)$ for $\forall x, y \in X$. Moreover T^\odot is an orthomorphism. Indeed, let $|x| \wedge |v| = 0$ for $x \in X$ and $v \in X^\odot$. $T(x, v) = 0$ from T orthosymmetric. On the other hand, there are $y, z \in X$ satisfying property $v = yz$ from definition X^\odot for $v \in X^\odot$. From $\rho(\pi)(x, y) = \pi(xy) = \pi(x)y$ for $\forall \pi \in Orth(X)$ and $(x, y) \in X \times X$, then $xT(y, z) = T(xy, z) = yT(x, z) = T(x, yz) = T(x, v) = 0$ for $T \in Orth(X, X)$. Since X is semiprime,

$$|x| \wedge |T^\odot(v)| = |x| \wedge |T^\odot(yz)| = |x| \wedge |T(y, z)| = 0.$$

This demonstrated that operator T^\odot is a positive orthomorphism.

Conclusion 1. Let X is Dedekind complete semiprime f –algebra. $Orth(X)$ is an ordered ideal of $Orth(X, X)$ [6].

Proof. Since $Orth(X)$ is a Riesz subspace of $Orth(X, X)$, it is sufficient to prove the theorem to show that $Orth(X)$ is a solid in $Orth(X, X)$. For this let $T \in Orth(X, X)$. By definition of solid, let $0 \leq T \leq f$ for $f \in Orth(X)$. Here we have to show that $T \in Orth(X)$. Since X is uniformly complement, there is an only positive orthomorphism $T^\odot: X^\odot \rightarrow X$ satisfying $T(x, y) = T^\odot(xy)$ for $\forall x, y \in X$. From here,

$$0 \leq T^\odot(x^2) = T^\odot(xx) = T(x, x) \leq f(x^2)$$

is provided. In other words, $0 \leq T^\odot(v) \leq f(v)$ is provided for $\forall 0 \leq v = x^2 \in X^\odot$. The operator T^\odot has an extension to a positive operator that satisfies the property $0 \leq T^\odot \leq f$, which we can denote again with T^\odot . From here, $T^\odot \in Orth(X)$ and $\widehat{T^\odot} = T$ is obtained. Consequently, $T \in Orth(X)$ is found. This proves that $Orth(X)$ is a solid in $Orth(X, X)$ and therefore an ordered ideal.

2.2. Embedding Orthomorphism in Biorthomorphisms when X is Uniformly Complete Semiprime f –Algebra with Weak Ordered Unit

Theorem 5. If X is uniformly complete semiprime f –algebra with weak ordered unit, $Orth(X)$ is an ordered ideal in $Orth(X, X)$ [6].

Proof. $Orth(X, X)$ is an uniformly complement semiprime f –algebra. Let $e \in X$ is an positive weak ordered unit. $(Orth(X, X), *_e)$ is an semiprime f –algebra [6]. Let us show that $Orth(X)$ is a ring ideal in $(Orth(X, X), *_e)$. Let $f \in Orth(X)$ and $T \in Orth(X, X)$. From here, $(\hat{f} *_e T)(x, y) = \hat{f}(x, T(e, y)) = f(xT(e, y)) = xf(T(e, y)) = x(foT(e, .))(y) = fo\widehat{T(e, .)}(x, y)$ is provided from $xf(y) = f(xy) = yf(x)$ for $foT(e, .) \in Orth(X)$ and $\forall x, y \in X$. Then, $\hat{f} *_e T = fo\widehat{T(e, .)} \in Orth(X)$. This shows that $Orth(X)$ is a ring ideal in $Orth(X, X)$. On the other hand, since $Orth(X)$ is an uniformly complete f –algebra with unit element, $Orth(X)$ is square root closed. At the same time, ring ideal $Orth(X)$ is an ordered ideal. From this, it is concluded that $Orth(X)$ is an ordered ideal in $Orth(X, X)$.

2.3. Embedding Orthomorphism in Biorthomorphisms when X is Uniformly Complete d –Algebra

In this section, we defined the subspace $X^d = \{cx \wedge cy : x \wedge y = 0, c \in X^+, x, y \in X\}$ of X , which is an d –algebra. Following conclusion is obtained from Theorem 1.

Conclusion 2. Let X is an uniformly complete d –algebra. For $c \in X^+, x, y \in X$;

- i. If $y = x, X^d = X^\odot$.
- ii. If $y = x = c$ then set X^d is a Riesz subspace of space X with positive cone $\{cc = c^2 : c \in X\}$.

Proof.

- i. $X^d = \{cx \wedge cy : x \wedge y = 0, c \in X^+, x, y \in X\}$
 $= \{cx \wedge cx : x \wedge x = 0, c \in X^+, x \in X\}$ ($y = x$)
 $= \{cx : c \in X^+, x \in X\}$

From here, $X^d = X^\odot$ is provided.

- ii. $X^d = \{cx \wedge cy : x \wedge y = 0, c \in X^+, x, y \in X\}$
 $= \{cc \wedge cc : c \wedge c = 0, c \in X^+\}$ ($y = x = c$)

$$= \{c^2 : c \in X^+\}$$

From here, X^d is a Riesz subspace of X with positive cone $\{cc = c^2 : c \in X\}$.

Conclusion 3. Let X is uniformly complete d –algebra, $T : X \times X \rightarrow X$ is an transformation and T is an biorthomorphism on X . For $\forall x, y \in X$ and $c \in X^+$;

- i. If $y = x$, there is an only positive biorthomorphism $T^d : X^d \rightarrow X$ satisfying property $T(c, x) = T^d(cx \wedge cy)$.
- ii. If there is an only biorthomorphism $T^d : X^d \rightarrow X$ satisfying property $T(c, x) = T^d(cx \wedge cy)$ for $y = x$, T is a positive biorthomorphism on X .
- iii. If $y = x$ for $\forall x, y \in X$ and $c \in X^+$, $T^\odot = T^d$.

Proof.

i. Let $y = x$, T_1^d and T_2^d are two biorthomorphisms satisfying transformation $T^d : X^d \rightarrow X$.

$$T_1^d(cx \wedge cy) = T_1^d(cx \wedge cx) = T_1^d(cx) = T_1(c, x)$$

$$T_2^d(cx \wedge cy) = T_2^d(cx \wedge cx) = T_2^d(cx) = T_2(c, x)$$

$$T_1^d = T_2^d \text{ for } T_1^d(cx) = T_2^d(cx)$$

ii. Since X is semiprime f –algebra, X is d –algebra. In that case, proof is similar from Theorem 4. If T is a biorthomorphism on X , then T is an orthosymmetric Riesz bimorphism. In other words, if $x \wedge y = 0$ on X for $\forall c \in X^+$, $T(c, x) \wedge T(c, y) = 0$. If $y = x$, $T : X \times X \rightarrow X$ defined with $T(c, x) = cx$ is an orthosymmetric Riesz bimorphism. In addition, X^d for $y = x = c$ is Riesz space and X^d is square of X .

Therefore, there is an only Riesz homomorphism $T^d : X^d \rightarrow X$ defined with $T(c, x) = T^d(cx \wedge cy)$ for $\forall x, y \in X$ and $c \in X^+$. Moreover T^d is an orthomorphism. Indeed, let $|n| \wedge |v| = 0$ for $n \in X$ and $v \in X^d$. Since T is orthosymmetric, $T(n, v) = 0$. On the other hand, there are $x, y \in X$ and $c \in X^+$ satisfying property $v = cx \wedge cy$ from definition X^d for $v \in X^d$. From [7] and $\forall c \in X^+$ for $y = x$, $nT(c, x) = T(nc, x) = cT(n, x) = T(n, cx) = T(n, cx \wedge cy) = 0$ is provided for $T \in Orth(X, X)$. Since X is semiprime for $y = x$,

$$|n| \wedge |T^d(v)| = |n| \wedge |T^d(cx \wedge cy)| = |n| \wedge |T^d(cx)| = |n| \wedge |T(c, x)| = 0$$

is obtained. This showed that operator T^d is a positive orthomorphism.

iii. For $y = x$,

$$T^d(cx \wedge cy) = T^d(cx \wedge cx) = T^d(cx) = T(c, x)$$

$$T^\odot(cx) = T(c, x)$$

$T^d = T^\odot$ from $T^d(cx) = T^\odot(cx)$.

Conclusion 4. Let X is a Dedekind complete d –algebra. $Orth(X)$ is an ordered ideal of $Orth(X, X)$.

Proof. It suffices to show that $Orth(X)$ is a solid in $Orth(X, X)$ since $Orth(X)$ is a Riesz subspace of $Orth(X, X)$, as in the case of X being a Dedekind complete semiprime f –algebra [8]. Let $T \in Orth(X, X)$. By definition of solid, $0 \leq T \leq f$ for $f \in Orth(X)$. We have to show that $T \in Orth(X)$. Since X is uniformly complete d –algebra, there is an only positive orthomorphism $T^d : X^d \rightarrow X$ satisfying property $T(c, x) = T^d(cx \wedge cy)$ for $\forall x, y \in X$ and $\forall c \in X^+$. From here for $y = x = c$,

$$0 \leq T^d(cx \wedge cy) = T^d(cc \wedge cc) = T^d(c^2) = T(c, c) \leq T(c^2) \leq f(c^2)$$

is provided. In other words, $0 \leq T^d(v) \leq f(v)$ for $\forall 0 \leq v = c^2 \in X^d$. The operator T^d has the extension to a positive operator, again denoted by T^d , which satisfies the property $0 \leq T^d \leq f$. From here, $T^d \in Orth(X)$ and $\widehat{T^d} = T$ is obtained. Consequently, $T \in Orth(X)$. This proves that $Orth(X)$ is a solid in $Orth(X, X)$ and therefore an ordered ideal.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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