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Anti-pedals and Primitives of Curves in Minkowski Plane

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Abstract

The orthogonal projection of a fixed point on the tangent lines of a given curve yields a pedal curve of that curve. The aim of this study is to examine some special curves, such as pedal curves, which have singular points even for regular curves, in the Minkowski plane. For this, we investigate an anti-pedal and a primitive of curve, which is closely related to the pedal curve. The primitive of a curve is a curve that is provided by the inverse construction to make pedal. Using the envelope of a family of functions, we obtain the notion of primitive for the curves in the Minkowski plane. Then, we show that an anti-pedal of the original curve is equal to the inversion image of the pedal curve. Moreover, we analyze the relationships between primitive and anti-pedal of the curve using the inversion. We also present examples that provide our results.

Minkowski Düzleminde Eğrilerin Anti-Pedalları ve İlkelleri

Anahtar kelimeler
Düzlem Eğrileri; Pedal; Anti-pedal; İlkel; Minkowski Düzlem

Öz

Verilen bir eğrinin teğet doğruları üzerindeki sabit bir noktanın dik izdüşümü, o eğrinin bir pedal eğrisini oluşturur. Bu çalışmanın amacı, düzgün eğriler için bile tekil noktaları olan pedal eğriler gibi bazı özel eğrileri Minkowski düzleminde incelemektir. Bunun için, pedal eğrisi ile yakından ilişkili olan, eğrinin anti-pedalını ve ilkelini araştırdık. Bir eğrinin ilkeli, pedal yapmak için ters yapı tarafından sağlanan bir eğridir. Bir fonksiyon ailesinin örtüsünü kullanarak, Minkowski düzlemindeki eğriler için ilkel kavramını elde ettik. Daha sonra, orijinal eğrinin bir anti-pedalının, pedal eğrisinin inversiyon görüntüsüne eşit olduğunu gösterdik. Dahası, inversiyonu kullanarak eğrinin ilkeli ve anti-pedalı arasındaki ilişkileri analiz ettik. Ayrıca, sonuçlarımızı sağlayan örnekler sunduk.

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1. Introduction

Singularity theory is one of important topics to research because it arises in many problems in daily life. This theory is also used to link physics and mathematics. Many other sub-disciplines of mathematics, including differential geometry and algebra, utilize from it (Li and Sun 2019). The idea of combining differential geometry with singularity theory was proposed by Arnold (1990) and Thom (1956). Therefore, it can be said that they did pioneering work in this field. Many researchers later discussed the singularity of curves in accordance with their theories.

Pedal, anti-pedal and primitive curves, which form the basis of our study, are closely related to the singularity theory. Pedal curves are defined as the locus of the foot of the perpendicular from the given point to the tangent to given curve and primitive curves are defined as the envelope of the normal lines to its position vectors at their ends (Arnold 1989). There are numerous studies on pedal curves. One of the studies has been proposed by Nishimura (2008). He worked on pedal curves produced by dual curve germs that are non-singular. Another study on this subject has been introduced by Bakurova (2013). He examined pedal curves in Minkowski plane. After that, the pedaloids have been obtained as an analogous notion of evolutoids (Izumiya and Takeuchi 2019a). Also, using definition of the pedal curve, Izumiya and Takeuchi (2020) introduced the notion of the anti-pedal of a curve whose singularities also correspond to the inflection points of the original curve. Moreover, they gave the notion of primitivoids, which are relatives of the primitive. In another study, they examined pedal, anti-pedal and primitive for quadratic curves (Izumiya and Takeuchi 2019b).

In this study, we look at anti-pedals and primitives, both of which have singularities even for regular curves. Especially, we consider curves in the Minkowski plane. Our conclusions are Lorentzian analogue to the results of Izumiya and Takeuchi (2020). Then, we define the notions of anti-pedal, primitive in the Minkowski plane and examine the relationships between them.

2. Material and Method

It is well known that the Minkowski plane $\mathbb{R}_1^2$ is the plane $\mathbb{R}^2$ allowing the metric produced by the scalar product $<\mathbf{u},\mathbf{v}> = -u_1v_1 + u_2v_2$ where $\mathbf{u} = (u_1,u_2)$ and $\mathbf{v} = (v_1,v_2)$. The vectors in $\mathbb{R}_1^2$ are classified as follows by this product:

If $\langle\mathbf{u},\mathbf{u}\rangle > 0$ or $\mathbf{u} = 0$, then $\mathbf{u}$ is spacelike. If $\langle\mathbf{u},\mathbf{u}\rangle = 0$ or $\langle\mathbf{u},\mathbf{u}\rangle < 0$ for a non-zero vector, then $\mathbf{u}$ is lightlike or timelike, respectively (O'Neill 1983).

The norm of a vector $\mathbf{u} = (u_1,u_2) \in \mathbb{R}_1^2$ is given by $||\mathbf{u}|| = \sqrt{|\langle\mathbf{u},\mathbf{u}\rangle|}$ and the vector $\mathbf{u}^\perp$ is provided by $\mathbf{u}^\perp = (u_2,u_1)$, which is orthogonal to $\mathbf{u}$ (Izumiya *et al*. 2018). Furthermore, the signature of $\mathbf{u}$ is indicated by ε and so $\frac{\langle\mathbf{u},\mathbf{u}\rangle}{||\mathbf{u}||^2} = \varepsilon$.

Let $\boldsymbol{\gamma}: I \to \mathbb{R}_1^2$ be a regular curve, which is parametrized by an open interval I. For any $s \in I$, the curve is a spacelike curve, a timelike curve, a lightlike curve if $\langle\boldsymbol{\gamma}'(s),\boldsymbol{\gamma}'(s)\rangle > 0$, $\langle\boldsymbol{\gamma}'(s),\boldsymbol{\gamma}'(s)\rangle < 0$, $\langle\boldsymbol{\gamma}'(s),\boldsymbol{\gamma}'(s)\rangle = 0$, respectively. In addition, $\boldsymbol{\gamma}'(s)$ is velocity vector of $\boldsymbol{\gamma}$ and is written as $\boldsymbol{\gamma}'(s) = \frac{d\boldsymbol{\gamma}}{ds}(s)$. If a curve $\boldsymbol{\gamma}$ is timelike or spacelike, we call it a non-lightlike curve (Li and Sun 2019).

Assume that $\boldsymbol{\gamma}: I \to \mathbb{R}_1^2$ is a non-lightlike curve with an arc-length parameter s such that $||\boldsymbol{\gamma}'(s)|| = 1$. In this situation, $\mathbf{T}(s) = \boldsymbol{\gamma}'(s)$ is the unit tangent vector with

$$sgn\ \mathbf{T}(s) = \langle\mathbf{T}(s),\mathbf{T}(s)\rangle = \varepsilon \tag{1}$$

and $\mathbf{N}(s)$ is the unit normal vector with

$$sgn\ \mathbf{N}(s) = \langle\mathbf{N}(s),\mathbf{N}(s)\rangle = -\varepsilon. \tag{2}$$

Hence, we write the Frenet formula:

$$\frac{d}{ds}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\end{bmatrix} = \begin{bmatrix}0 & \kappa(s)\\ \kappa(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\end{bmatrix} \tag{3}$$

where $\kappa(s)$ is the curvature of $\boldsymbol{\gamma}$ (Li and Sun 2019).

Definition 2.1. For a fixed ϕ, the envelope of the family of lines defined by $F(s,\mathbf{x})$ consists of the points $\mathbf{x}$ in the plane where s exists with (Giblin and Warder 2014)

$$F(s,\mathbf{x}) = \frac{\partial F}{\partial s}(s,\mathbf{x}) = 0. \tag{4}$$

Definition 2.2. Let $\boldsymbol{\gamma}$ be a non-lightlike regular curve in Minkowski plane. A pedal of curve $\boldsymbol{\gamma}$ is given by (Aydın Şekerci and Izumiya 2021)

$$Pe_{\boldsymbol{\gamma}}(s) = -\varepsilon\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle\mathbf{N}(s). \tag{5}$$

3. Results and Discussion

Let $\boldsymbol{\gamma}: I \to \mathbb{R}_1^2/\{0\}$ be a non-lightlike curve with arc-parameter in Minkowski plane and there are no lightlike points. We define a family of functions

$$\begin{aligned} H: I \times (\mathbb{R}_1^2/\{0\}) &\to \mathbb{R} \\ (s,\mathbf{x}) &\mapsto H(s,\mathbf{x}) = \langle\mathbf{x}-\boldsymbol{\gamma}(s),\boldsymbol{\gamma}(s)\rangle. \end{aligned}$$

For fixed $s \in I$, $H(s,\mathbf{x}) = 0$ is an equation of the line through $\boldsymbol{\gamma}(s)$ and orthogonal to the position vector $\boldsymbol{\gamma}(s)$.
The envelope of family of lines is the primitive of curve γ. According to that, we obtain

$$\begin{aligned} \frac{\partial H}{\partial s}(s,\mathbf{x}) &= \langle-\boldsymbol{\gamma}'(s),\boldsymbol{\gamma}(s)\rangle + \langle\mathbf{x}-\boldsymbol{\gamma}(s),\boldsymbol{\gamma}'(s)\rangle \\ &= \langle\mathbf{x}-2\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle \end{aligned}$$

Any vector in $\mathbb{R}_1^2$ is represented by a linear combination as $\lambda\mathbf{T}(s) + \xi\mathbf{N}(s)$. Using this linear combination for the vector $\mathbf{x}-\boldsymbol{\gamma}(s)$ and substituting to $\frac{\partial H}{\partial s}(s,\mathbf{x}) = 0$, we obtain

$$\langle\lambda\mathbf{T}(s) + \xi\mathbf{N}(s) - \boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle = 0. \tag{6}$$

Then, we have the following equation for λ:

$$\lambda = \varepsilon\langle\mathbf{T}(s),\boldsymbol{\gamma}(s)\rangle. \tag{7}$$

Also, considering $H(s,\mathbf{x}) = 0$, we get

$$H(s,\mathbf{x}) = \langle\mathbf{x}-\boldsymbol{\gamma}(s),\boldsymbol{\gamma}(s)\rangle = 0. \tag{8}$$

Moreover, from $\mathbf{x}-\boldsymbol{\gamma}(s) = \lambda\mathbf{T}(s) + \xi\mathbf{N}(s)$, we have

$$\lambda\langle\mathbf{T}(s),\boldsymbol{\gamma}(s)\rangle + \xi\langle\mathbf{N}(s),\boldsymbol{\gamma}(s)\rangle = 0. \tag{9}$$

Using λ we write

$$\varepsilon\langle \mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle^2 + \xi\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle = 0. \quad (10)$$

Thus, we find

$$\xi = -\varepsilon\frac{\langle \mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle^2}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle} \quad (11)$$

Taking into account λ and ξ,

$$\mathbf{x} = \boldsymbol{\gamma}(s) + \varepsilon\langle \mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle \mathbf{T}(s) - \frac{\varepsilon\langle \mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle^2}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle}\mathbf{N}(s).$$

From $\boldsymbol{\gamma}(s) = \varepsilon\langle \mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle \mathbf{T}(s) - \varepsilon\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle \mathbf{N}(s)$, we write

$$\begin{aligned}\mathbf{x} &= \boldsymbol{\gamma}(s) + \varepsilon\langle \mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle \mathbf{T}(s) - \frac{\varepsilon\langle \mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle^2}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle}\mathbf{N}(s)\\ &= \boldsymbol{\gamma}(s) + \boldsymbol{\gamma}(s) + \varepsilon\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle \mathbf{N}(s)\\ &\quad - \frac{\varepsilon\langle \mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle^2}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle}\mathbf{N}(s)\\ &= 2\boldsymbol{\gamma}(s) + \varepsilon\left[\frac{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle^2}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle}\right]\mathbf{N}(s)\\ &\quad -\varepsilon\left[\frac{\langle \mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle^2}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle}\right]\mathbf{N}(s).\end{aligned}$$

Using the norm of $\boldsymbol{\gamma}(s)$, which is given as

$$\|\boldsymbol{\gamma}(s)\|^2 = sgn\,\boldsymbol{\gamma}(s)\begin{bmatrix}\varepsilon\langle \mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle^2\\ -\varepsilon\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle^2\end{bmatrix},$$

there exists

$$\mathbf{x} = 2\boldsymbol{\gamma}(s) - sgn\,\boldsymbol{\gamma}(s)\frac{\|\boldsymbol{\gamma}(s)\|^2}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle}\mathbf{N}(s).$$

Definition 3.1. The primitive $\mathrm{Pr}_{\boldsymbol{\gamma}}: I \to \mathbb{R}^2_1/\{0\}$ of Minkowski plane curve $\boldsymbol{\gamma}$ is given by

$$\mathrm{Pr}_{\boldsymbol{\gamma}}(s) = 2\boldsymbol{\gamma}(s) - sgn\,\boldsymbol{\gamma}(s)\frac{\|\boldsymbol{\gamma}(s)\|^2}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle}\mathbf{N}(s). \quad (12)$$

Now, let us define the anti-pedal curve in the Minkowski plane. The anti-pedal curve is defined by the pedal curve and inversion. It is known that the pedal is given as the envelope of a family of functions (Aydın Şekerci and Izumiya 2021):

$$\begin{aligned}G: I\times(\mathbb{R}^2_1/\{0\}) &\to \mathbb{R}\\ (s,\mathbf{x}) &\mapsto G(s,\mathbf{x}) = \langle \mathbf{x}-\boldsymbol{\gamma}(s), \mathbf{x}\rangle.\end{aligned}$$

We use the definition of inversion, which is given by

$$\begin{aligned}\Psi: \mathbb{R}^2_1/\{0\} &\to \mathbb{R}^2_1/\{0\}\\ \mathbf{x} &\mapsto \Psi(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|^2} = sgn\,\mathbf{x}\frac{\mathbf{x}}{\langle \mathbf{x},\mathbf{x}\rangle},\end{aligned}$$

to define an antipedal curve. Then we have

$$\Psi\left(g_s^{-1}(0)\right) = \{\mathbf{x}: \langle \mathbf{x}, \boldsymbol{\gamma}(s)\rangle = sgn\,\mathbf{x}\} \quad (13)$$

for $g_s(\mathbf{x}) = G(s,\mathbf{x})$. Here, $G(s,\mathbf{x}) = 0$ means that

$$sgn\,\mathbf{x}\langle\frac{\mathbf{x}}{\|\mathbf{x}\|^2}, \boldsymbol{\gamma}(s)\rangle = 1 \quad (14)$$

since $\langle \mathbf{x}-\boldsymbol{\gamma}(s), \mathbf{x}\rangle = 0$. Thus, we define a family of functions:

$$\begin{aligned}F: I\times(\mathbb{R}^2_1/\{0\}) &\to \mathbb{R}\\ (s,\mathbf{x}) &\mapsto F(s,\mathbf{x}) = \langle \mathbf{x}, \boldsymbol{\gamma}(s)\rangle - sgn\,\mathbf{x}.\end{aligned}$$

The envelope of the family of lines is the anti-pedal curve of γ. According to that, we obtain

$$\frac{\partial F}{\partial s}(s,\mathbf{x}) = \langle \mathbf{x}, \boldsymbol{\gamma}'(s)\rangle = 0. \quad (15)$$

Any vector in $\mathbb{R}^2_1$ is represented by a linear combination as $\lambda\mathbf{T}(s) + \xi\mathbf{N}(s)$. Using this linear combination for the vector $\mathbf{x}-\boldsymbol{\gamma}(s)$ and substituting to $\frac{\partial F}{\partial s}(s,\mathbf{x}) = 0$, we have the following equation for λ:

$$\lambda = 0\,. \quad (16)$$

Also, using λ in $F(s,\mathbf{x}) = 0$, we get

$$\xi\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle = sgn\,\mathbf{x}. \quad (17)$$

Then, we obtain

$$\xi = \frac{sgn\,\mathbf{x}}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle} \quad (18)$$

and so, $\mathbf{x}$ can be written as follows:

$$\mathbf{x} = \frac{sgn\,\mathbf{x}}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle}\mathbf{N}(s). \quad (19)$$

Moreover, the signature of $\mathbf{x}$ is equal to the signature of $\mathbf{N}(s)$. In that case, the following definition is expressed.

Definition 3.2. An anti-pedal $APe_{\boldsymbol{\gamma}}: I \to \mathbb{R}^2_1/\{0\}$ of the curve $\boldsymbol{\gamma}$ is given by

$$\mathrm{APe}_{\boldsymbol{\gamma}}(s) = \frac{-\varepsilon}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle}\mathbf{N}(s). \quad (20)$$

Proposition 3.3. For any unit speed non-lightlike regular curve $\boldsymbol{\gamma}: I \to \mathbb{R}^2_1/\{0\}$, there exists

$$\Psi \circ \mathrm{APe}_{\boldsymbol{\gamma}} = \mathrm{Pe}_{\boldsymbol{\gamma}} \tag{21}$$

where Ψ is an inversion, $\mathrm{APe}_{\boldsymbol{\gamma}}$ is an anti-pedal curve of $\boldsymbol{\gamma}$ and $\mathrm{Pe}_{\boldsymbol{\gamma}}$ is a pedal curve of $\boldsymbol{\gamma}$.

Proof. We obtain the equation with direct calculations using the anti-pedal of curve and inversion as follows:

$$\begin{aligned}(\Psi \circ \mathrm{APe}_{\boldsymbol{\gamma}})(\mathrm{s}) &= \Psi\left(\mathrm{APe}_{\boldsymbol{\gamma}}(\mathrm{s})\right) \\ &= \Psi\left(\frac{-\varepsilon}{\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle}\mathbf{N}(\mathrm{s})\right) \\ &= \frac{\dfrac{-\varepsilon}{\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle}\mathbf{N}(\mathrm{s})}{\left\| \dfrac{-\varepsilon}{\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle}\mathbf{N}(\mathrm{s})\right\|^2} \\ &= \frac{\dfrac{-\varepsilon}{\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle}\mathbf{N}(\mathrm{s})}{\dfrac{1}{\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle^2}} \\ &= -\varepsilon\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle\mathbf{N}(\mathrm{s}) \\ &= \mathrm{Pe}_{\boldsymbol{\gamma}}(\mathrm{s})\end{aligned}$$

Proposition 3.4. For any unit speed non-lightlike regular curve $\boldsymbol{\gamma}: \mathrm{I} \to \mathbb{R}^2_1/\{0\}$, there exists

$$\Psi \circ \mathrm{Pe}_{\boldsymbol{\gamma}} = \mathrm{APe}_{\boldsymbol{\gamma}} \tag{22}$$

where Ψ is an inversion, $\mathrm{APe}_{\boldsymbol{\gamma}}$ is an anti-pedal curve of $\boldsymbol{\gamma}$ and $\mathrm{Pe}_{\boldsymbol{\gamma}}$ is a pedal curve of $\boldsymbol{\gamma}$.

Proof. Similar to the proof of Proposition 3.3, we obtain the equation with direct calculations using the pedal of curve and inversion as follows:

$$\begin{aligned}\Psi \circ \mathrm{Pe}_{\boldsymbol{\gamma}}(\mathrm{s}) &= \Psi\left(\mathrm{Pe}_{\boldsymbol{\gamma}}(\mathrm{s})\right) \\ &= \Psi\left(-\varepsilon\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle\mathbf{N}(\mathrm{s})\right) \\ &= \frac{-\varepsilon\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle\mathbf{N}(\mathrm{s})}{\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle^2} \\ &= \frac{-\varepsilon\mathbf{N}(\mathrm{s})}{\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle} \\ &= \mathrm{APe}_{\boldsymbol{\gamma}}(\mathrm{s}).\end{aligned}$$

Theorem 3.5. Let $\boldsymbol{\gamma}$ be a unit speed non-lightlike curve in Minkowski plane. Assume that $\boldsymbol{\gamma}$ does not pass through the origin and there are no lightlike points. Then, the primitive and the anti-pedal of $\boldsymbol{\gamma}$ have the following relationship:

$$\mathrm{Pr}_{\boldsymbol{\gamma}}(\mathrm{s}) = -\varepsilon\, sgn\, \boldsymbol{\gamma}(\mathrm{s})\ \mathrm{APe}_{\Psi\circ\boldsymbol{\gamma}}(\mathrm{s}) \tag{23}$$

where ε is the signature of the tangent vector field of $\boldsymbol{\gamma}$, $sgn\, \boldsymbol{\gamma}(\mathrm{s})$ is the signature of $\boldsymbol{\gamma}$ and Ψ is an inversion.

Proof. Firstly, we find $\mathrm{APe}_{\Psi\circ\boldsymbol{\gamma}}(\mathrm{s})$. For this, the family of functions is $\mathrm{F}(\mathrm{s}, \mathbf{x}) = \langle \mathbf{x}, \boldsymbol{\gamma}(\mathrm{s})\rangle - sgn\, \mathbf{x}$. Since $\mathrm{F}(\mathrm{s}, \mathbf{x}) = 0$, we get

$$\langle \mathbf{x}, (\Psi \circ \boldsymbol{\gamma})(\mathrm{s})\rangle - sgn\, \mathbf{x} = 0,$$

$$\left\langle \mathbf{x}, \frac{\boldsymbol{\gamma}(\mathrm{s})}{\|\boldsymbol{\gamma}(\mathrm{s})\|^2}\right\rangle = sgn\, \mathbf{x}. \tag{24}$$

Any vector in $\mathbb{R}^2_1$ is represented by a linear combination as $\lambda\mathbf{T}(\mathrm{s}) + \xi\mathbf{N}(\mathrm{s})$. Using this linear combination for the vector $\mathbf{x}$,

$$\frac{\lambda}{\|\boldsymbol{\gamma}(\mathrm{s})\|^2}\langle \mathbf{T}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle + \frac{\xi}{\|\boldsymbol{\gamma}(\mathrm{s})\|^2}\langle \mathbf{N}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle = sgn\, \mathbf{x}. \tag{25}$$

Moreover, we obtain

$$\frac{\partial \mathrm{F}}{\partial s}(\mathrm{s}, \mathbf{x}) = \left\langle \mathbf{x}, \left(\frac{\boldsymbol{\gamma}(\mathrm{s})}{\|\boldsymbol{\gamma}(\mathrm{s})\|^2}\right)'\right\rangle = 0. \tag{26}$$

Since

$$\left(\frac{\boldsymbol{\gamma}(\mathrm{s})}{\|\boldsymbol{\gamma}(\mathrm{s})\|^2}\right)' = \frac{1}{\|\boldsymbol{\gamma}(\mathrm{s})\|^2}\mathbf{T}(\mathrm{s}) - \frac{2\, sgn\, \boldsymbol{\gamma}(\mathrm{s})\langle \mathbf{T}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle}{\|\boldsymbol{\gamma}(\mathrm{s})\|^4}\boldsymbol{\gamma}(\mathrm{s}),$$

Eq. (26) is written as

$$\begin{aligned}&\left\langle \lambda\mathbf{T}(\mathrm{s}) + \xi\mathbf{N}(\mathrm{s}), \frac{1}{\|\boldsymbol{\gamma}(\mathrm{s})\|^2}\mathbf{T}(\mathrm{s})\right\rangle \\ &- \left\langle \lambda\mathbf{T}(\mathrm{s}), \frac{2\, sgn\, \boldsymbol{\gamma}(\mathrm{s})\langle \mathbf{T}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle}{\|\boldsymbol{\gamma}(\mathrm{s})\|^4}\boldsymbol{\gamma}(\mathrm{s})\right\rangle \\ &- \left\langle \xi\mathbf{N}(\mathrm{s}), \frac{2\, sgn\, \boldsymbol{\gamma}(\mathrm{s})\langle \mathbf{T}(\mathrm{s}), \boldsymbol{\gamma}(\mathrm{s})\rangle}{\|\boldsymbol{\gamma}(\mathrm{s})\|^4}\boldsymbol{\gamma}(\mathrm{s})\right\rangle = 0.\end{aligned} \tag{27}$$

According to that, we have

$$\begin{aligned}&\lambda\frac{\varepsilon}{\|\boldsymbol{\gamma}(\mathrm{s})\|^2} - \frac{2\lambda\, sgn\, \boldsymbol{\gamma}(\mathrm{s})\langle \boldsymbol{\gamma}(\mathrm{s}), \mathbf{T}(\mathrm{s})\rangle^2}{\|\boldsymbol{\gamma}(\mathrm{s})\|^4} \\ &- \frac{2\xi\, sgn\, \boldsymbol{\gamma}(\mathrm{s})\langle \boldsymbol{\gamma}(\mathrm{s}), \mathbf{T}(\mathrm{s})\rangle\langle \boldsymbol{\gamma}(\mathrm{s}), \mathbf{N}(\mathrm{s})\rangle}{\|\boldsymbol{\gamma}(\mathrm{s})\|^4} = 0.\end{aligned} \tag{28}$$

If we multiply Eq.(25) by $\frac{2\, sgn\, \boldsymbol{\gamma}(s)\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^2}$ and add Eq.(28) side by side, then we can write as

$$\lambda = 2\varepsilon\, sgn\, \boldsymbol{\gamma}(s)\; sgn\, \mathbf{x}\, \langle\boldsymbol{\gamma}(s), \mathbf{T}(s)\rangle. \tag{29}$$

Using this λ, which we have obtained, we find ξ as follows:

$$\xi = \frac{sgn\, \boldsymbol{\gamma}(s)\; sgn\, \mathbf{x}}{\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle}[\langle\boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle - 2\varepsilon\langle\boldsymbol{\gamma}(s), \mathbf{T}(s)\rangle^2]. \tag{30}$$

Therefore, we get

$$\begin{aligned}
\mathbf{x} &= 2\varepsilon\, sgn\, \boldsymbol{\gamma}(s)\; sgn\, \mathbf{x}\, \langle\boldsymbol{\gamma}(s), \mathbf{T}(s)\rangle\mathbf{T}(s) \\
&\quad + \frac{sgn\, \boldsymbol{\gamma}(s)\; sgn\, \mathbf{x}}{\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle}[\langle\boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle \\
&\quad -2\varepsilon\, \langle\boldsymbol{\gamma}(s), \mathbf{T}(s)\rangle^2]\mathbf{N}(s) \\
&= 2\, sgn\, \boldsymbol{\gamma}(s)\; sgn\, \mathbf{x}[\boldsymbol{\gamma}(s) + \varepsilon\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle\mathbf{N}(s)] \\
&\quad + \frac{sgn\, \boldsymbol{\gamma}(s)\; sgn\, \mathbf{x}}{\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle}[\langle\boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle \\
&\quad -2\varepsilon\, \langle\boldsymbol{\gamma}(s), \mathbf{T}(s)\rangle^2]\mathbf{N}(s) \\
&= 2\, sgn\, \boldsymbol{\gamma}(s)\; sgn\, \mathbf{x}\, \boldsymbol{\gamma}(s) \\
&\quad + \frac{sgn\, \mathbf{x}\; sgn\, \boldsymbol{\gamma}(s)\langle\boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle}{\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle}\mathbf{N}(s) \\
&\quad - \frac{2 sgn\, \mathbf{x}}{\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle}[\varepsilon\, sgn\, \boldsymbol{\gamma}(s)\, \langle\boldsymbol{\gamma}(s), \mathbf{T}(s)\rangle^2 \\
&\quad -\varepsilon\, sgn\, \boldsymbol{\gamma}(s)\, \langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle^2]\mathbf{N}(s) \\
&= 2\, sgn\, \boldsymbol{\gamma}(s)\; sgn\, \mathbf{x}\, \boldsymbol{\gamma}(s) \\
&\quad + \frac{sgn \mathbf{x}\, \|\boldsymbol{\gamma}(s)\|^2}{\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle}\mathbf{N}(s) - \frac{2 sgn\, \mathbf{x}\|\boldsymbol{\gamma}(s)\|^2}{\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle}\mathbf{N}(s) \\
&= sgn\, \boldsymbol{\gamma}(s)\; sgn\, \mathbf{x}[2\boldsymbol{\gamma}(s) \\
&\quad \left. -sgn\, \boldsymbol{\gamma}(s)\frac{\|\boldsymbol{\gamma}(s)\|^2}{\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle}\mathbf{N}(s)\right] \\
&= sgn\, \boldsymbol{\gamma}(s)\; sgn\, \mathbf{x}\, \mathrm{Pr}_{\boldsymbol{\gamma}}(s)
\end{aligned}$$

Moreover, since $sgn\, \mathbf{x} = sgn\, \mathrm{Pr}_{\boldsymbol{\gamma}}(s)$, we get

$$\begin{aligned}
\langle \mathrm{Pr}_{\boldsymbol{\gamma}}(s), \mathrm{Pr}_{\boldsymbol{\gamma}}(s)\rangle &= \frac{\|\boldsymbol{\gamma}(s)\|^4}{\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle^2}\langle\mathbf{N}(s), \mathbf{N}(s)\rangle \\
&= -\varepsilon\frac{\|\boldsymbol{\gamma}(s)\|^4}{\langle\boldsymbol{\gamma}(s), \mathbf{N}(s)\rangle^2}.
\end{aligned}$$

This means that the signature of $\mathbf{N}(s)$ and the signature of $\mathrm{Pr}_{\boldsymbol{\gamma}}(s)$ are the same. According to that, we obtain

$$\mathrm{APe}_{\Psi\circ\boldsymbol{\gamma}}(s) = -\varepsilon\, sgn\, \boldsymbol{\gamma}(s)\, \mathrm{Pr}_{\boldsymbol{\gamma}}(s).$$

Theorem 3.6. Let $\boldsymbol{\gamma}$ be a unit speed non-lightlike curve in Minkowski plane. Assume that $\boldsymbol{\gamma}$ does not pass through the origin and there are no lightlike points. Then, the primitive and the anti-pedal of $\boldsymbol{\gamma}$ have the following relationship:

$$\mathrm{Pr}_{\Psi\circ\boldsymbol{\gamma}} = -\varepsilon\, sgn\, \boldsymbol{\gamma}\, \mathrm{APe}_{\boldsymbol{\gamma}} \tag{31}$$

Proof. Firstly, we find $\mathrm{Pr}_{\Psi\circ\boldsymbol{\gamma}}(s)$. For this, the family of functions is $\mathrm{H}(s,\mathbf{x}) = \langle\mathbf{x} - \boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle$. Since $\mathrm{H}(s,\mathbf{x}) = 0$, we get

$$\langle\mathbf{x} - (\Psi\circ\boldsymbol{\gamma})(s), (\Psi\circ\boldsymbol{\gamma})(s)\rangle = 0,$$

$$\langle\mathbf{x} - \frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}, \frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}\rangle = 0.$$

Any vector in $\mathbb{R}^2_1$ is represented by a linear combination as $\lambda\mathbf{T}(s) + \xi\mathbf{N}(s)$. Taking this linear combination for the vector $\mathbf{x} - (\Psi\circ\boldsymbol{\gamma})(s)$,

$$\langle\lambda\mathbf{T}(s) + \xi\mathbf{N}(s), \frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}\rangle = 0,$$

$$\frac{\lambda}{\|\boldsymbol{\gamma}(s)\|^2}\langle\mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle + \frac{\xi}{\|\boldsymbol{\gamma}(s)\|^2}\langle\mathbf{N}(s), \boldsymbol{\gamma}(s)\rangle = 0. \tag{32}$$

Moreover, we obtain

$$\frac{\partial\mathrm{H}}{\partial s}(s,\mathbf{x}) = \left\langle -\left(\frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}\right)', \frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}\right\rangle + \left\langle\mathbf{x} - \frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}, \left(\frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}\right)'\right\rangle = 0. \tag{33}$$

Since

$$\left(\frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}\right)' = \frac{1}{\|\boldsymbol{\gamma}(s)\|^2}\mathbf{T}(s) - \frac{2\, sgn\, \boldsymbol{\gamma}(s)\, \langle\mathbf{T}(s), \boldsymbol{\gamma}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^4}\boldsymbol{\gamma}(s),$$

Eq. (33) is written as

$$\left\langle\mathbf{x} - \frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2} - \frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}, \left(\frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}\right)'\right\rangle = 0.$$

According to that, using linear combination $\lambda\mathbf{T}(s)+\xi\mathbf{N}(s)$ for $\mathbf{x}-\frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}$, we have

$$\left\langle \lambda\mathbf{T}(s)+\xi\mathbf{N}(s)-\frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2},\frac{1}{\|\boldsymbol{\gamma}(s)\|^2}\mathbf{T}(s) - \frac{2sgn\,\boldsymbol{\gamma}(s)\,\langle\mathbf{T}(s),\boldsymbol{\gamma}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^4}\boldsymbol{\gamma}(s)\right\rangle=0.$$

Therefore, we obtain

$$\lambda\frac{\varepsilon}{\|\boldsymbol{\gamma}(s)\|^2}-\frac{2\lambda\,sgn\,\boldsymbol{\gamma}(s)\,\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle^2}{\|\boldsymbol{\gamma}(s)\|^4}$$

$$-\frac{2\xi\,sgn\,\boldsymbol{\gamma}(s)\,\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^4}-\frac{\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^4}$$

$$+\frac{2\,sgn\,\boldsymbol{\gamma}(s)\,\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle\langle\boldsymbol{\gamma}(s),\boldsymbol{\gamma}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^6}=0.$$

From that equation, we get

$$\lambda\frac{\varepsilon}{\|\boldsymbol{\gamma}(s)\|^2}-\frac{2\lambda\,sgn\,\boldsymbol{\gamma}(s)\,\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle^2}{\|\boldsymbol{\gamma}(s)\|^4} -\frac{2\xi\,sgn\,\boldsymbol{\gamma}(s)\,\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^4} =-\frac{\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^4} \quad (34)$$

If we multiply Eq.(32) by $\frac{2sgn\,\boldsymbol{\gamma}(s)\,\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^2}$ and add Eq.(34) side by side, then we have

$$\lambda=-\varepsilon\frac{\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^2}. \quad (35)$$

Using this λ, which we have obtained, we find ξ as follows:

$$\xi=\varepsilon\frac{\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle^2}{\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle\|\boldsymbol{\gamma}(s)\|^2}. \quad (36)$$

Therefore, we get

$$\begin{aligned}\mathbf{x}-\frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2} &=-\varepsilon\frac{\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^2}\mathbf{T}(s) +\varepsilon\frac{\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle^2}{\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle\|\boldsymbol{\gamma}(s)\|^2}\mathbf{N}(s)\\ &=-\frac{\boldsymbol{\gamma}(s)}{\|\boldsymbol{\gamma}(s)\|^2}-\varepsilon\frac{\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle}{\|\boldsymbol{\gamma}(s)\|^2}\mathbf{N}(s) +\varepsilon\frac{\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle^2}{\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle\|\boldsymbol{\gamma}(s)\|^2}\mathbf{N}(s)\end{aligned}$$

Thus, it can be written as

$$\begin{aligned}\mathbf{x} &=\frac{\varepsilon\langle\boldsymbol{\gamma}(s),\mathbf{T}(s)\rangle^2-\varepsilon\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle^2}{\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle\|\boldsymbol{\gamma}(s)\|^2}\mathbf{N}(s)\\ &=\frac{sgn\,\boldsymbol{\gamma}(s)}{\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle}\mathbf{N}(s).\end{aligned}$$

According to that, we obtain

$$\mathbf{x}=-\varepsilon\,sgn\,\boldsymbol{\gamma}(s)\frac{-\varepsilon}{\langle\boldsymbol{\gamma}(s),\mathbf{N}(s)\rangle}\mathbf{N}(s).$$

Therefore, the proof is completed.

4. Applications

Example 4.1. Let $\boldsymbol{\gamma}:\mathrm{I}\to\mathbb{R}^2_1/\{0\}$, $\mathrm{I}\subset\mathbb{R}$, be a curve which is expressed by $\boldsymbol{\gamma}(s)=(2\sinh s,\cosh s)$ and is shown in Figure 1.

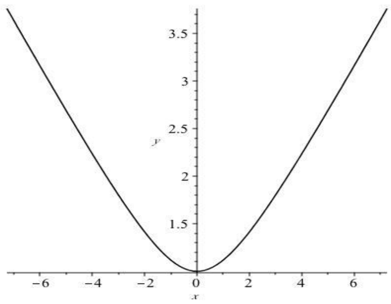

Figure 1. The curve $\boldsymbol{\gamma}(s)=(2\sinh s,\cosh s)$

Therefore, $\boldsymbol{\gamma}(s)$ is a regular curve, there are no lightlike points and $\boldsymbol{\gamma}'(s)=(2\cosh s,\sinh s)$. Then, we say that the curve is a timelike curve. The tangent vector field and the normal vector field of the curve $\boldsymbol{\gamma}(s)$ are given in the following forms:

$$\mathbf{T}(s)=\left(\frac{2\cosh s}{\sqrt{1+3\cosh^2 s}},\frac{\sinh s}{\sqrt{1+3\cosh^2 s}}\right)$$

$$\mathbf{N}(\mathrm{s})=\left(\frac{\sinh \mathrm{s}}{\sqrt{1+3\cosh^2 \mathrm{s}}},\frac{2\cosh \mathrm{s}}{\sqrt{1+3\cosh^2 \mathrm{s}}}\right)$$

Using (12), the primitive of $\boldsymbol{\gamma}$ is obtained as

$$\mathrm{Pr}_{\boldsymbol{\gamma}}(\mathrm{s})=\begin{pmatrix}\frac{\sinh \mathrm{s}}{2}(3\cosh^2\mathrm{s}+4),\\ \cosh \mathrm{s}\ (3\sinh^2\mathrm{s}+1)\end{pmatrix}$$

and is shown in Figure 2.

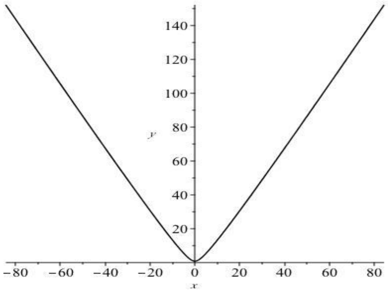

Figure 2. The primitive of curve $\boldsymbol{\gamma}(\mathrm{s})$

Also, from (20), the anti-pedal of $\boldsymbol{\gamma}(\mathrm{s})$ is given by

$$\mathrm{APe}_{\boldsymbol{\gamma}}(\mathrm{s})=\frac{1}{2}(\sinh \mathrm{s}, 2\cosh \mathrm{s})$$

and is shown in Figure 3.

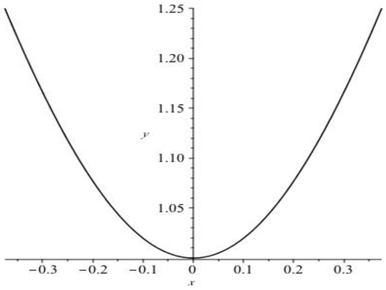

Figure 3. The anti-pedal of curve $\boldsymbol{\gamma}(\mathrm{s})$

Example 4.2. Let $\boldsymbol{\gamma}: \mathrm{I}\to\mathbb{R}_1^2/\{0\}$, $\mathrm{I}\subset\mathbb{R}$, be a curve which is expressed by $\boldsymbol{\gamma}(\mathrm{s})=(\cosh \mathrm{s}, \mathrm{s}+\sinh \mathrm{s})$ and is shown in Figure 4.

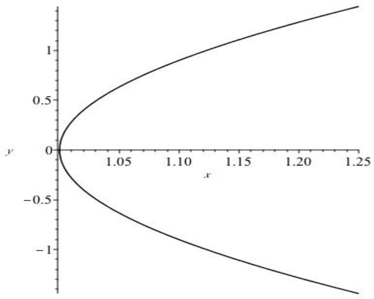

Figure 4. The curve $\boldsymbol{\gamma}(\mathrm{s})=(\cosh \mathrm{s}, \mathrm{s}+\sinh \mathrm{s})$

Therefore, $\boldsymbol{\gamma}(\mathrm{s})$ is a regular curve, there are no lightlike points and $\boldsymbol{\gamma}'(\mathrm{s})=(\sinh \mathrm{s}, 1+\cosh \mathrm{s})$. Then, we say that the curve is a spacelike curve.

Using (12), the primitive of $\boldsymbol{\gamma}$ is obtained as

$$\mathrm{Pr}_{\boldsymbol{\gamma}}(\mathrm{s})=\begin{pmatrix}2\cosh \mathrm{s}-\frac{(1+\cosh \mathrm{s})(\mathrm{s}^2-1+2\sinh \mathrm{s})}{\mathrm{s}\ \sinh \mathrm{s}-1-\cosh \mathrm{s}},\\ 2\sinh \mathrm{s}+2\mathrm{s}-\frac{(\sinh \mathrm{s})(\mathrm{s}^2-1+2\sinh \mathrm{s})}{\mathrm{s}\ \sinh \mathrm{s}-1-\cosh \mathrm{s}}\end{pmatrix}$$

and is shown in Figure 5.

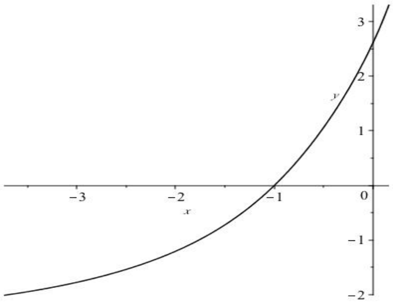

Figure 5. The primitive of curve $\boldsymbol{\gamma}(\mathrm{s})$

Also, from (20), the anti-pedal of $\boldsymbol{\gamma}(\mathrm{s})$ is given by

$$\mathrm{APe}_{\boldsymbol{\gamma}}(\mathrm{s})=\frac{-1}{\mathrm{s}\ \sinh \mathrm{s}-1-\cosh \mathrm{s}}(1+\cosh \mathrm{s}, \sinh \mathrm{s})$$

and is shown in Figure 6.

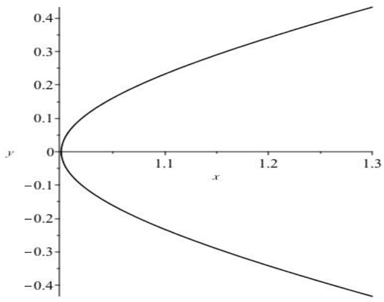

Figure 6. The anti-pedal of curve $\boldsymbol{\gamma}(\mathrm{s})$

5. Conclusions

In this study, the following conclusions are obtained about Minkowski plane curves:

(1)To obtain primitive and anti-pedal of the Minkowski plane curves, which does not pass through the origin and does not have lightlike point, the families of functions are defined and their envelopes are used. Thus, the primitive of curve is defined as

$$\mathrm{Pr}_{\boldsymbol{\gamma}}(\mathrm{s})=2\boldsymbol{\gamma}(\mathrm{s})-sgn\ \boldsymbol{\gamma}(\mathrm{s})\frac{\|\boldsymbol{\gamma}(\mathrm{s})\|^2}{\langle\mathbf{N}(\mathrm{s}),\boldsymbol{\gamma}(\mathrm{s})\rangle}\mathbf{N}(\mathrm{s}),$$

and the anti-pedal of curve is defined as

$$\mathrm{APe}_{\boldsymbol{\gamma}}(s) = \frac{-\varepsilon}{\langle \mathbf{N}(s), \boldsymbol{\gamma}(s) \rangle} \mathbf{N}(s).$$

(2) The pedal of the curve, which is non-lightlike Minkowski plane curve, does not pass through the origin and there is no lightlike point, is equivalent to the composition of an anti-pedal of the curve and inversion.

(3) The anti-pedal of the curve, which is non-lightlike Minkowski plane curve, does not pass through the origin and there is no lightlike point, is equivalent to the composition of a pedal of the curve and inversion.

(4) Assume that $\boldsymbol{\gamma}$ is a curve, which is a non-lightlike Minkowski plane curve, does not pass through the origin and there is no lightlike point, and Ψ is an inversion. Then, the primitive of the curve $\boldsymbol{\gamma}$ is written as the product of a constant and the anti-pedal of $\Psi \circ \boldsymbol{\gamma}$ where this constant is calculated by multiplying the signature of the normal vector field of the curve and the signature of the curve.

(5) Similar to the previous result , assume that $\boldsymbol{\gamma}$ is a curve, which is non-lightlike Minkowski plane curve, does not pass through the origin and there is no lightlike point, and Ψ is an inversion. Then, the primitive of $\Psi \circ \boldsymbol{\gamma}$ is written as the product of a constant and the anti-pedal of $\boldsymbol{\gamma}$ where this constant is calculated by multiplying the signature of the normal vector field of the curve and the signature of the curve.

References

Arnold, V.I., 1989, Dynamical systems VIII, Encyclopedia of mathematical sciences, 39, Springer, 88-92.

Arnold, V.I., 1990, Singularities of caustics and wave fronts, 62, Springer, 1-56.

Aydın Şekerci, G. and Izumiya, S., 2021. Evolutoids and pedaloids of Minkowski plane curves. *Bulletin of the Malaysian Mathematical Sciences Society*, **44**, 2813-2834.

Bakurova, V., 2013. On singularities of pedal curve in the Minkowski plane. *Proceedings of symposium on computer geometry SCG,* **22**, 5-10.

Giblin, P.J. and Warder J.P., 2014. Evolving evolutoids. *The American Mathematical Monthly*, **121**, 871-889.

Izumiya, S., Romero Fuster, M.C. and Takahashi, M., 2018. Evolutes of curves in the Lorentz-Minkowski plane. *Advanced Studies in Pure Mathematics*, **78**, 313-330.

Izumiya, S. and Takeuchi, N., 2019a. Evolutoids and pedaloids of plane curves. *Note di Matematica*, **39**, 13-23.

Izumiya, S. and Takeuchi, N., 2019b. Pedals and inversions of quadratic curves. *Arxiv: 1912.03114v1,* 1-18.

Izumiya, S. and Takeuchi, N., 2020. Primitivoids and inversions of plane curves. *Beitrage zur Algebra und Geometrie,* **61**, 317-334.

Li, Y. and Sun Q.Y., 2019. Evolutes of fronts in the Minkowski plane. *Mathematical Methods in the Applied Sciences,* **42**, 5416-5426.

Nishimura, T., 2008. Normal forms for singularities of pedal curves produced by non-singular dual curve germs in S^n. *Geometriae Dedicata*, **133**, 59-66.

O'Neill, B., 1983, Semi-Riemannian geometry with applications to relativity, 1, Academic Press Inc., 46-58.

Thom, R., 1956. Les singularities des applications differentiables. *Ann. Inst. Fourier*, **6**, 43-87.