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The Determination of Eigenvalues and Eigenvectors of the Orbital Angular Momentum

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ABSTRACT

The theory of angular momentum performance has a significant position in the classical and quantum mechanical study of physical properties, such as studies into nuclear, atomic, and molecular processes, as well as other quantum problems, including spherical symmetry. In this analysis, angular momentum operators are described in multiple ways, based on the angular momentum operator's commutator, matrix, and geometric representation. The eigenvalue and eigenvector were also known for operators$\hat{J}_{\pm}$,$\hat{\vec{J}}^2$, $\hat{J}_x$,$\hat{J}_{y,}$ and $\hat{J}_z$within the $|j,m\rangle$ basis. Furthermore, in quantum mechanics, angular momentum is called a quantized variable, meaning that it comes in discrete quantities. In contrast to the macroscopic system case where a continuous variable is an angular momentum. In this study, the different factors of the angular momentum operator also attempted to focus on establishing it is eigenvalues and Eigen states. For the raising and lowering operators $(\hat{J}_{\pm})$ within the $|j,m\rangle$ of the eigenvalue and eigenvector within the basis have been discussed.

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1. Introduction

In both classical and quantum mechanics (QM), angular momentum has a significant role. Angular momentum is conserved in whole isolated systems (including linear momentum and energy): this fact decreases substantially the amount of required work in calculating rigid body rotations, planetary trajectories, also many more [1, 2].
Analogously, in QM, to comprehend the composition of atoms, angular momentum has a critical role, and several other quantum questions regarding rotational symmetry, highly depend on angular momentum [1, 3].
In quantum mechanics (QM), Angular momentum operators come in a variety of forms; the orbital angular momentum (commonly exhibited via$(\hat{L})$), the total angular momentum (generally exhibited via$(\hat{J})$), also spin angular momentum (indicated via$(\vec{S})$). Disorienting expression "Angular momentum" could hint at both the orbital angular momentum and the total angular momentum.
Angular momentum, like other measurable quantities, is represented in QM by an operator. Its vector operator is identical to the operator of the momentum. As we can quickly see, though, the angular momentum operator's three elements are always do not commute, contrary to the linear momentum operator. Various angular momentum operators exist in QM: entire angular momentum$(\vec{J})$), the orbital angular momentum$(\vec{L})$ then angular internal or spin angular momentum$(\vec{S})$. No classical equivalent has this last one (spin)[2]. The angular momentum vector $\vec{L}$ is classically described by the cross product of position $\vec{r}$ and momentum$(\vec{P})$ [1, 3]:

$$\vec{L} = (\vec{r} \times \vec{p}) = \left((yp_z - zp_y)\vec{\imath} + (zp_x - xp_z)\vec{J} + (xp_y - yp_x)\vec{k}\right) \tag{1}$$

$\vec{L}$ The orbital angular momentum operator could be achieved in quantum mechanics, where there is an operator for each measurable, by replacement ($\vec{r}$ and$\vec{p}$) with the equivalent operators within position description, ($\hat{\vec{R}}$ And$(\hat{\vec{p}} = -i\hbar\vec{\nabla})$:

$$\vec{L} = \vec{R} \times \vec{P} = \left(-i\hbar\hat{\vec{R}} \times \vec{\nabla}\right) \tag{2}$$

The Cartesian components of $\hat{\vec{L}}$ are $(\hat{L}_X, \hat{L}_{Y,}\hat{L}_Z)$ which are [1].:

$$\hat{L}_x = \hat{Y}\hat{P}_Z - \hat{Z}\hat{P}_y = -i\hbar\left(\hat{Y}\frac{\partial}{\partial Z} - \hat{Z}\frac{\partial}{\partial y}\right) \tag{3}$$

$$\hat{L}_y = \hat{L}_y\hat{Z}\hat{P}_x - \hat{X}\hat{P}_Z = -i\hbar\left(\hat{Z}\frac{\partial}{\partial x} - \hat{X}\frac{\partial}{\partial Z}\right) \tag{4}$$

$$\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x = -i\hbar\left(\hat{X}\frac{\partial}{\partial y} - \hat{Y}\frac{\partial}{\partial x}\right) \quad (5)$$

And the square of the angular momentum operator ($\hat{\vec{L}}$) is [4].:

$$\hat{\vec{L}}^2 = \hat{L}_x{}^2 + \hat{L}_y{}^2 + \hat{L}_z{}^2 \quad (6)$$

The angular momenta of the element are Hermitian, and$\hat{L}_x$,$\hat{L}_y$ and $\hat{L}_z$ do not commute (we can not evaluate them to arbitrary precision at the same time) with each other.:

$$[\ddot{L}_x, \ddot{L}_y] = i\hbar\ddot{L}_z, [\ddot{L}_y, \ddot{L}_Z] = i\hbar\ddot{L}_x, [\ddot{L}_{z,}\ddot{L}_x] = i\hbar\ddot{L}_y \quad (7)$$

The orbital angular momentum elements were also communally contradictory observables. The square's eigenvalues of the magnitude of the orbital angular momentum operator $\hat{L}^2$ are $\ell(\ell+1)\hbar^2$, while ℓ is quantum number orbital angular momentums which are natural numbers and ($\hbar$) is (reduced Planck constant). The $\hat{L}_z$ eigenvalues are $m\hbar$ wherever m = -ℓ, -$-\ell+1$, …, $-\ell$. Since $\hat{L}^2$ and $\hat{L}_z$ commute $([\hat{L}^2, \hat{L}_z] = 0)$, the quantum numbers of orbital angular momentum ℓ and m would be used to describe their corresponding eigenstates as$|l,m\rangle$. Furthermore, the orbital angular momentum ($\vec{L}$), fundamental fragments, as the way as electrons, additionally own spin for short (or intrinsic angular momentum spin $\vec{S}$), that is to say not according to motion in space position. The spin angular momentum and the orbitals algebras are uniform and the spin angular momentum operator's elements ($\hat{S}_z, \hat{S}_y$ *and* $\hat{S}_x$), fulfill commutation unique to the components of the orbital angular momentum operator's commutation affiliations $\hat{L}_Z, \hat{L}_y$ *and* $\hat{L}_x$ it means that [5, 6]:

$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$, $[\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x$, $[\hat{S}^2, \hat{S}_z] = 0$.

In addition {s(s+1)$\hbar^2$ } are the squared of the amount of the spin angular momentum operator S^2 eigenvalues, wherever s exhibits the spin angular momentum. The spin quantum number could be a natural number half-odd integer and a positive integer. In addition, the spin quantum value s for such electron is ($\frac{1}{2}$), while the values with ms, the spin quantum value for the z-dimension of spin, which is ($\pm\frac{1}{2}$). If the basis vectors are the eigenstates of the z elements of spin, also the operators are$\left\{\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z,\ \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y, and\ \hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x\right\}$, which could be illustrated via Pauli matrix [7, 8]:

$\hat{\sigma}_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \hat{\sigma}_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$ and $\hat{\sigma}_z = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

Since $(\hat{S}^2)$ and ($\hat{S}_z$) are commute, then quantum numbers (s) and ($m_{s)}$ can be utilized to indicate their eigenstates simultaneously like as $|s,m\rangle$ [1, 9]. If there is a quantum system that has two particles along with independent quantum orbital angular momentum numbers ℓ_1 and ℓ_2, the system's total quantum orbital angular momentum number would range from $\ell_1 + \ell_2$ down to $|\ell_1 - \ell_2$ | i.e. ℓ =$\ell_1 + \ell_2$, $\ell_1 + \ell_2$-1,…, $|\ell_1 - \ell_2$ |. The z-elements of the system's overall orbital angular momentum are equivalent to the sum of the z-components of the individual particles' orbital angular momenta, i.e. m $= m_1 + m_2$. The reasonable values of its overall quantum number of angular momentum (j) could be determined for a single particle of non-zero spin by adding its quantum number ℓ (orbital angular momentum) and its quantum number is (the angular momentum's spin). It means, j = s+ ℓ, s+ℓ - 1…, |s-l|. Comparably, the average quantum number of angular momentum of the process for two particles with overall angular momentum quantum numbers $j_1$and $j_2$is j $= j_1 + j_2, j_1 + j_2$– 1,…., $|j_1 - j_2$ | [10, 11].

In this work, we exhibited a review about the formalism nature of the general angular momentum through the Common formality of angular momentum which is explained clearly via the determination of the Eigenstates and eigenvalues of $\hat{\vec{J}}^2$and $\hat{J}_z$ as well as describing the eigenvalue and eigenvector for the raising and lowering operators ($\hat{J}_\pm$) within the $|j,m\rangle$ basis, and we will show different characteristics of the angular momentum operator, We will show different characteristics of the angular momentum operator. In this review, we tried to focus on specifying it is Eigen states and eigenvalues. Finally, we illustrate a review of angular momentums geometry and orbital angular momentum Eigenfunctions.

2. The Common Formality of Angular Momentum

To initiate a exceed common angular momentum operator ($\hat{\vec{J}}$) which is attributed via its three elements ($\hat{J}_x, \hat{J}_y$ *and* $\hat{J}_z$), which gratify the succeeding commutation association [9]:

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z, [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x, [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y \quad (8)$$

Or equivalently via

$$\hat{J} \times \hat{J} = i\hbar\hat{J} \quad (9)$$

Because $\hat{J}_x$,$\hat{J}_y$ and $\hat{J}_z$ do not reciprocally commute, they could not be diagonalized all at the same; notably, they do not have common eigenstates possessions. The angular momentum square is [4, 12]:

$$\hat{\vec{J}}^2 = \hat{J}_x{}^2 + \hat{J}_y{}^2 + \hat{J}_z{}^2 \quad (10)$$

It means the angular momentum's square is a scalar operator; and commutes with ($\hat{J}_x$), ($\hat{J}_y$) and ($\hat{J}_z$):

$$[\hat{\vec{J}}^2, \hat{J}_k] = 0 \quad (11)$$

Where k refers to (x, y, and z)

And since $[\hat{J}_x{}^2, \hat{J}_x] = 0, [\hat{J}_y, \hat{J}_x] = -i\hbar\hat{J}_z$, and $[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$. It may be noted that the operator $\hat{J}_x, \hat{J}_y, \hat{J}_z$, and $\hat{\vec{J}}^2$ altogether Hermitian; their eigenvalues are real [1]

2.1. Determination of the Eigenstates and Eigenvalues of $\hat{\hat{J}}^2$ And $\hat{J}_z$

Based on commutates of $\hat{\hat{J}}^2$with each of $(\hat{J}_x,\hat{J}_y\&\hat{J}_z)$, all components of $\hat{J}$ possibly diagonalized with ($\hat{\hat{J}}^2$).nevertheless, because the components $(\hat{J}_x,\hat{J}_y,\hat{J}_z)$ do not commutate frequently, only one of them might be specified to be diagonalized with $\hat{\hat{J}}^2$simultaneously. By signifying the joint eigenstates via $|\alpha,\beta\rangle$ also the eigenvalues of $(\hat{\hat{J}}^2) and\ (\hat{J}_z)$ via $(\hbar^2\alpha)$ &$(\hbar\beta)$, to await the joint eigenstates of $\hat{\hat{J}}^2$and $\hat{J}_z$and their relating eigenvalues, respectively[1, 7]

$$\hat{\hat{J}}^2|\alpha,\beta\rangle = \hbar^2\alpha|\alpha,\beta\rangle \quad (12)$$

$$\hat{J}_z|\alpha,\beta\rangle = \hbar\beta|\alpha,\beta\rangle \quad (13)$$

The constant $\hbar$ is presented so $(\alpha\&\beta)$ are dimensionless, taking into account that (ħ) is the dimensions of the angular momentum and the physical dimensions of ħ are [$\hbar$]=time×energy. To illustrate, it can be assumed that these eigenstates are orthonormal [4].

$$\langle\alpha',\beta'|\alpha,\beta\rangle = \delta_{\alpha'},\delta_{\beta'},\beta \quad (14)$$

Additionally, it might be introduced lowering and raising operators $(\hat{J}_-)$ and $(\hat{J}_+)$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \quad (15)$$

This result in

$$\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-), \hat{J}_y = \frac{1}{2}(\hat{J}_+,\hat{J}_-) \quad (16)$$

Consequently.

$$\hat{J}_x{}^2 = \frac{1}{4}\left(\hat{J}_+{}^2 + \hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+ + \hat{J}_-{}^2\right), \qquad \hat{J}_y{}^2 = -\frac{1}{4}\left(\hat{J}_+{}^2 - \hat{J}_+\hat{J}_- - \hat{J}_-\hat{J}_+ + \hat{J}_-{}^2\right) \quad (17)$$

Utilizing equation (8), it could be simply obtained in accordance commutation relations.

$$\left[\hat{\hat{J}}^2,\hat{J}_\pm\right] = 0, [\hat{J}_+,\hat{J}_-] = 2\hbar\hat{J}_z, [\hat{J}_z,\hat{J}_\pm] = \pm\hbar\hat{J}_\pm \quad (18)$$

Furthermore, $\hat{J}_+$ and $\hat{J}_-$ fulfill

$$\hat{J}_+\hat{J}_- = \hat{J}_x{}^2 + \hat{J}_y{}^2 + \hbar\hat{J}_z = \hat{\hat{J}}^2 - \hat{J}_z{}^2 + \hbar\hat{J}_z \quad (19)$$

$$\hat{J}_-\hat{J}_+ = \hat{J}_x{}^2 + \hat{J}_y{}^2 - \hbar\hat{J}_z = \hat{\hat{J}}^2 - \hat{J}_z{}^2 - \hbar\hat{J}_z \quad (20)$$

These relation results in

$$\hat{\hat{J}}^2 = \hat{J}_\pm\hat{J}_\mp + \hat{J}_z{}^2 \mp \hbar\hat{J}_z \quad (21)$$

Which intern yield

$$\hat{\hat{J}}^2 = \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) + \hat{J}_z{}^2 \quad (22)$$

Firstly, $\hat{J}_\pm$ operator on $|\alpha,\beta\rangle, because(\hat{J}_\pm)$ do not commute with $(\hat{J}_z)$, eigenstates of $\hat{J}_\pm$ are not the kets $(|\alpha,\beta\rangle)$, utilizing the relation (18), we have

$$\hat{J}_z(\hat{J}_\pm|\alpha,\beta\rangle) = (\hat{J}_\pm\hat{J}_z \pm \hbar\hat{J}_\pm)|\alpha,\beta\rangle = \hbar(\beta \pm 1)(\hat{J}_\pm|\alpha,\beta\rangle) \quad (23)$$

Hence the kets $\{\hat{J}_\pm|\alpha,\beta\rangle\}$is an eigenstate for $\hat{J}_z$with eigenvalues $\{\hbar(\beta \pm 1)\}$. Also $\hat{J}_z$with $\hat{J}_z{}^2$commute; $\{\hat{J}_\pm|\alpha,\beta\rangle\}$ have to be an eigenstate of$\hat{\hat{J}}^2$. Through producing use of the commutator $\left(\left[\hat{\hat{J}}^2,\hat{J}_\pm\right] = 0\right)$, the eigenvalues of $(\hat{\hat{J}}^2)$ while acting on $(\hat{J}_\pm|\alpha,\beta\rangle)$ may be determined. additionally, the state $\{(\hat{J}_\pm|\alpha,\beta\rangle)\}$ is as well as an eigenstate of $(\hat{J}^2)$ to eigenvalue $(\hbar^2\alpha)$:

$$\hat{\hat{J}}_\pm{}^2\ (\hat{J}_\pm|\alpha,\beta\rangle) = \hat{J}_\pm\hat{\hat{J}}^2|\alpha,\beta\rangle = \hbar^2\alpha(\hat{J}_\pm|\alpha,\beta\rangle) \quad (24)$$

From (23) and (24), when $\hat{J}_\pm$ come in $(|\alpha,\beta\rangle)$, it did not influence the first quantum number α, However, it lowers or raises β (the second quantum number) through one unit. It means that $\{\hat{J}_\pm|\alpha,\beta\rangle\}$ has a proportional relationship with $\{|\alpha,\beta \pm 1\rangle\}$:

$$\hat{J}_\pm|\alpha,\beta\rangle = C^\pm_{\alpha\beta}|\alpha,\beta \pm 1\rangle \quad (25)$$

Where $C^\pm_{\alpha\beta}$ is constant.

An upper limit for the quantum number β per unit exists for a certain eigenvalue (α) of $(\hat{\hat{J}}^2)$, Since the operator ($\hat{\hat{J}}^2 - \hat{J}_z{}^2$) is valid, because the matrix elements $\{\hat{\hat{J}}^2 - \hat{\hat{J}}_z{}^2 = \hat{J}_y{}^2 + \hat{J}_y{}^2\}$ are ≥ 0, it would be written as [1,18]:

$$\langle\alpha,\beta\left|\hat{\hat{J}}^2 - \hat{J}_z{}^2\right|\alpha,\beta\rangle = \hbar^2(\alpha - \beta^2) \geq 0, \rightarrow\ \alpha \geq \beta^2 \quad (26)$$

Since (β_{max}) is the upper limit of (β), it has to be a state $|\alpha,\beta_{max}\rangle$that could not be raised anymore:

$$\hat{J}_+|\alpha,\beta\rangle = 0 \quad (27)$$

Utilizing this equation along with $\{\hat{J}_-\hat{J}_+ = \hat{\hat{J}}^2 - \hat{J}_z{}^2 - \hbar\hat{J}_z\}$, it can be seen that

$\{(\hat{\hat{J}}^2 - \hat{J}_z{}^2\text{-}\hbar\hat{J}_z)|\alpha,\beta_{max}\rangle = \hbar^2(\alpha - \beta_{max}{}^2 - \beta_{min})|\alpha,\beta_{max}\rangle\}$ or $\{\hat{J}_-\hat{J}_+|\alpha,\beta_{max}\rangle = 0\}$, so that

$$\alpha = \beta_{max}(\beta_{max} + 1) \quad (28)$$

After (n) consecutive utilization of $(\hat{J}_-)$ on $(|\alpha,\beta_{max}\rangle)$, it should be competent to achieve a state $(|\alpha,\beta_{min}\rangle)$, what which could not be lowered anymore:

$$\hat{J}_-|\alpha,\beta_{min}\rangle = 0 \quad (29)$$

Employing $\{\hat{J}_+\hat{J}_- = \hat{\hat{J}}^2 - \hat{J}_z{}^2 + \hbar\hat{J}_z\}$, and through similarity along with equations (27) & (28), then

$$\beta_{max} = -\beta_{min} \quad (30)$$

Since (β_{min}) was attained by the utilization of ($\hat{J}_-$) upon ($|\alpha,\beta_{max}\rangle$), it indicated that

$$\beta_{max} = \beta_{min} + n \quad (31)$$

And because (β_{max}) may have an integer number or half-odd integer number, hinge on (n) being odd or even.it is suitable to define the notation j & m indicates that ($\beta_{\max}$) and (β), successively: $j = \beta_{max} = \frac{n}{2}, m = \beta$ thus the eigenvalue of $\hat{\hat{J}}^2$ is taken through $\alpha = j(j+1)$. Because$\beta_{min} = -\beta_{max}$, as well as (n) positive, the permitted values of m, are situated down among (-j) and (+j): j ≥ m ≥ -j. The gained outcomes will be recognized such as the ($\hat{\hat{J}}^2$) and (J_z) Eigenvalues. related to joint eigenvectors($|j,m\rangle$)are taken, respectively, through $\{\hbar^2 j(j+1)\}$ as well as($\hbar m$) [7, 13]:

$$\hat{\hat{J}}^2|j,m\rangle = \hbar^2 j(j+1)|\,j,m\rangle \;\&\; \hat{J}_z|j,m\rangle = \hbar m|\,j,m\rangle \quad (32)$$

While j = (1, 1/2, 1, 3/2, …). And m = (-j, - (j – 1) …... j – 1, j). Therefore, for each (j) there are (2j + 1) measures of (m). Such as, j = 1 then (m) accepts three prizes (-1, 0, 1); and if j = 5/2 then (m) makes the six measures (-5/2, -3/2, -1/2, 1/2, 3/2, 5/2). The values of (j) may be an integer number and a half-integer number. There is a discrete spectrum of angular momentum operators $\hat{\hat{J}}^2$ and $\hat{J}_z$. As the eigenstates are orthogonal, tending to various angular momentum. Because there are distinct angular momentum spectra, the orthogonally state is:$\langle j',m'|j,m\rangle = \delta_{j'},\delta_{m',m}$ [1, 14].

2.2. Describing The Eigenvalue and Eigenvector for The Raising and Lowering Operators ($\hat{J}_\pm$) within the $|j,m\rangle$ Basis

The state of the ($|j,m\rangle$) is not an eigenstate of($\hat{J}_\pm$), equation (25) can rewrite as:

$$\hat{J}_\pm|j,m\rangle = C^{\pm}_{jm}|j,m\pm 1\rangle \quad (33)$$

Derive of $C^{+}_{j\,m}$ can be given and then deduce $C^{-}_{j\,m}$.because ($|j,m\rangle$) is normalized, equation (33) can be utilized to acquire the following model of expressions [4]:

$$\left(\hat{J}_+|j,m\rangle\right)^\dagger\left(\hat{J}_+|j,m\rangle\right) = \left|C^{+}_{jm}\right|^2\langle j,m+1|j,m+1\rangle = \left|C^{+}_{j\,m}\right|^2 \quad (34)$$

$$\left|C^{+}_{j\,m}\right|^2 = \langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle \quad (35)$$

But because ($\hat{\hat{J}}^2 - \hat{J}_z{}^2 - \hbar\hat{J}_z$) is equal to $\hat{J}_-\hat{J}_+$ and assuming the random phase of ($C^{+}_{j\,m}$) being zero, so inferred that

$$C^{+}_{j\;m} = \sqrt{\left\langle j,m\middle|\hat{\hat{J}}^2 - \hat{J}_z{}^2 - \hbar\hat{J}_z\middle|j,m\right\rangle} = \hbar\sqrt{j(j+1) - m(m+1)} \quad (36)$$

By similarity with ($C^{+}_{j\,m}$), ($C^{-}_{j\,m}$) can be achieved:

$$C^{+}_{jm} = \hbar\sqrt{j(j+1) - m(m+1)} \quad (37)$$

So the eigenvalue formulas for $\hat{J}_+$and $\hat{J}_-$have taken by [1, 12].

$$\hat{J}_\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\;|j,m\pm 1\rangle \quad (38)$$

3. Matrix Picture of Angular Momentum

The section's formalism is common and exclusive of any individual description. There are several ways of portraying the operators of angular momentum and their eigenstates. The column vectors and square matrices will be illustrated according to eigenkets and operators in this section to consider the matrix picture of angular momentum. This is accomplished by the discrete basis of states and operators. Because ($\hat{\hat{J}}^2$) and ($\hat{J}_z$) commute, the general eigenstates ($|j,m\rangle$) could be adopted from the basis while basis is orthonormal, discrete (not continuous), and complete. For a specific measure of (j), the normalization stipulation for this basis is determined via $\{\langle j',m'|j,m\rangle = \delta_{j'},j\delta_{m',m}\}$, and the perfection is illustrated by [9].

$$\sum_{m}^{+j} = -\,j|j,m\rangle\langle j,m| = \hat{I} \quad (39)$$

Where ($\hat{I}$) represents unite matrix.$\hat{\hat{J}}^2$ and $\hat{J}_z$ operators are diagonal obtained based on their joint eigenstates

$$\left\langle j',m'\middle|\hat{\hat{J}}^2\middle|j,m\right\rangle = \hbar^2 j(j+1)\delta_{j'},j\delta_{m'm} \quad (40)$$

$$\left\langle j',m'\middle|\hat{J}_z\middle|j,m\right\rangle = \hbar m\delta_{j'},j\delta_{m'} \quad (41)$$

because the matrices illustration of ($\hat{\hat{J}}^2$) and ($\hat{J}_z$) in the ($|j,m\rangle$) eigenbasis are diagonal, further their diagonal elements adequate to $\{\hbar^2 j(j+1)\}$ and ($\hbar m$), consecutively.

Since $\hat{J}_z$ did not commute along with the operators $\hat{J}_\pm$, they are delineated in the ($|j,m\rangle$) basis via matrices that were nondiagonal [1, 15]:

$$\left\langle j',m'\middle|\hat{J}_\pm\middle|j,m\right\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,\delta_{j'},j\delta_{m',m\pm 1} \quad (42)$$

And from the:

$$\hat{J}_x|j,m\rangle = \tfrac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right)|j,m\rangle = \left[\!\left[\sqrt{(j-m)(j+m+1)}|j,m+1\rangle + \sqrt{(j+m)(j-m+1)}|j,m-1\rangle\right]\!\right]\frac{\hbar}{2} \quad (43)$$

However, $\hat{J}_y|j,m\rangle = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)|j,m\rangle$

$$= \left[\!\left[\sqrt{(j-m)(j+m+1)}|j,m+1\rangle - \sqrt{(j+m)(j-m+1)}|j,m-1\rangle\right]\!\right]\frac{\hbar}{2i} \quad (44)$$

The matrices for each $(\hat{J}_x)$ and $(\hat{J}_y)$ can be deduced by [15]:

$$\langle j',m'|\hat{J}_x|j,m\rangle = \frac{\hbar}{2}\left[\!\left[\sqrt{j(j+1)-m(m+1)} + \sqrt{j(j+1)-m(m-1)}\delta_{m',m-1}\right]\!\right]\delta_{j',j} \quad (45)$$

$$\langle j',m'|\hat{J}_y|j,m\rangle = \frac{\hbar}{2i}\left[\!\left[\sqrt{j(j+1)-m(m+1)}\delta_{m',m+1} - \sqrt{j(j+1)-m(m-1)}\delta_{m',m-1}\right]\!\right]\delta_{j',j} \quad (46)$$

To showing the matrix of angular momentum, it might be considered that j = 1
While for this case the forbidden prizes of (m) are (-1, 0, 1), however the link eigenstates of $(\hat{\hat{J}}^2)$ and $(\hat{J}_z)$ are $\{|1,1\rangle, |1,0\rangle, and\ |1,-1\rangle\}$.from equations (40) & (41) presentations of operator's matrix $(\hat{J}^2)$ and $(\hat{J}_z)$, can be deduced [18]:

$$\hat{\hat{J}}^2 = \begin{bmatrix} \langle 1,1|\hat{\hat{J}}^2|1,1\rangle & \langle 1,1|\hat{\hat{J}}^2|1,0\rangle & \langle 1,1|\hat{\hat{J}}^2|1,-1\rangle \\ \langle 1,0|\hat{\hat{J}}^2|1,1\rangle & \langle 1,0|\hat{\hat{J}}^2|1,0\rangle & \langle 1,0|\hat{\hat{J}}^2|1,-1\rangle \\ \langle 1,-1|\hat{\hat{J}}^2|1,1\rangle & \langle 1,-1|\hat{\hat{J}}^2|1,0\rangle & \langle 1,-1|\hat{\hat{J}}^2|1,-1\rangle \end{bmatrix} = 2\hbar^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (47)$$

$$\hat{J}^2 = \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (48)$$

Also, from formula (42), the matrices of $(\hat{J}_-)$ and $(\hat{J}_+)$ are taken via:

$$\hat{J}_+ = \hbar\sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \hat{J}_- = \hbar\sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (49)$$

For $(\hat{J}_x \& \hat{J}_{y,} and\ \hat{J}_z)$ the matrices in the $(|j,m\rangle$ basis consequences directly from the relation $(\hat{J}_x = (\hat{J}_- + \hat{J}_+)/2)$ as well as $(\hat{J}_y = i(\hat{J}_- - \hat{J}_+)/2)$ [3]:

$$\hat{J}_y = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\hat{J}_z = \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

4. Geometrical Description of Angular Momentum

The affiliation between momentum and the z-component is at issue here. This relation could be delineated geometrically in the following way. As exhibited in Figure 1. Total angular momentum $(\hat{\hat{J}})$ perhaps illustrated via a vector which length.is taken via $\left\{\sqrt{\langle\hat{\hat{J}}^2\rangle} = \hbar\sqrt{j(j+1)}\right\}$ and its component on z-axis is $(\langle\hat{J}_z\rangle = \hbar m)$ because $(\hat{J}_x)$ & $(\hat{J}_y)$ are individually indistinct. just their sum$\left\{\hat{J}_x^{\ 2} + \hat{J}_y^{\ 2} = \hat{J}^2 - \hat{J}_z^{\ 2}\right\}$, is well known inside the xy-plane [3, 7].

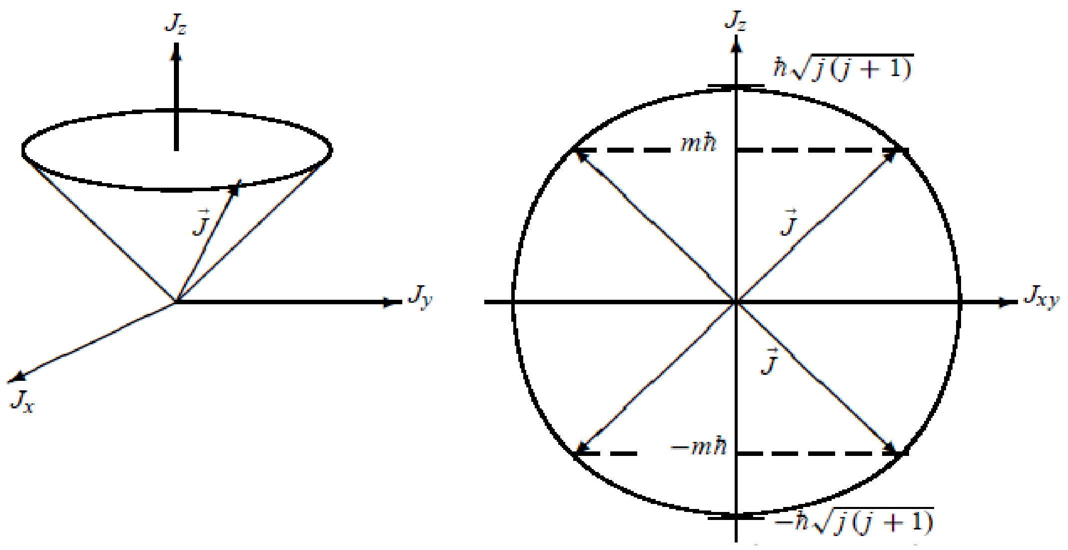

Figure 1. Geometrical illumination of angular momentum $(\hat{\hat{J}})$ [1].

The graphical illumination for the (j = 2) case is exhibited in Figure 2. In this situation when j=2, the angle θ receives just five values equivalent subsequently $\{m = -2, -1, 0, 1, 2\}$ they are taken via($\theta = -35.26^{\circ}, -65.91^{\circ}, 90^{\circ}, -65.91^{\circ}, 35.26^{\circ}$).

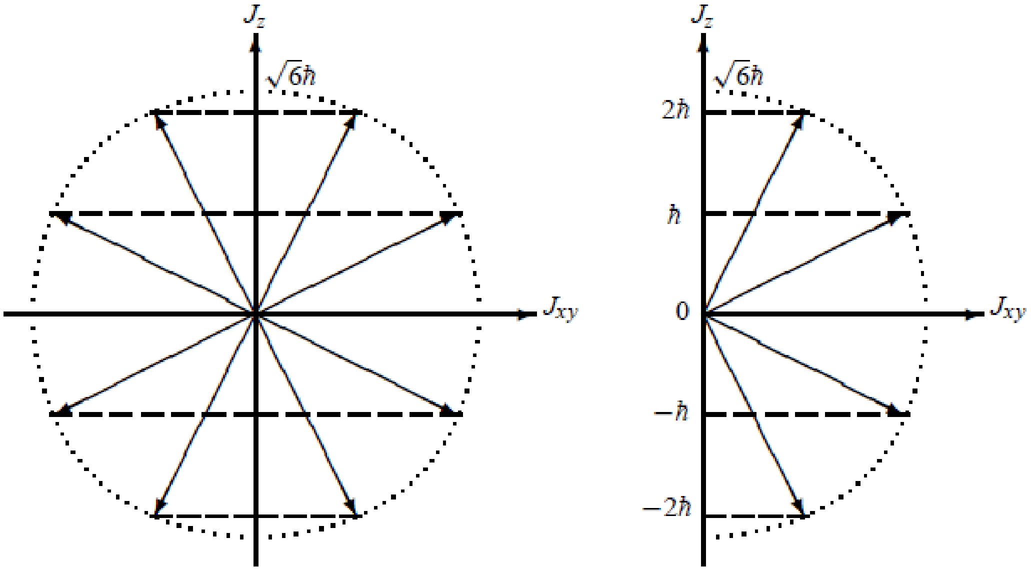

Figure 2.Graphical illumination of quantized angular momentum. While j =2 for the state$|2, m\rangle$ with *m = -2, -1, 0, 1, 2*. The radius of the circle is given by$\hbar\sqrt{\mathbf{2(2+1)}}$=$\hbar\sqrt{\mathbf{6}}$ [1, 9].

In classical, the vector ($\vec{J}$) should be thought of as integument a cone. Whose endpoint located on a circle with a radiance ($\sqrt{(j+1)j}\hbar$), revolving through cones surface of half-angle.

$$\theta = cos^{-1}\left(\frac{m}{\sqrt{j(j+1)}}\right) \quad (50)$$

In consequence of which its projection on the z-axis constantly ($m\hbar$). The angle θ is quantized when the quantum number value (*m*) is restrictive to *(-j, -j+1… j-1, j)*. The single reasonable value of θ contain values of $(2j+1)$;

$$\theta = cos^{-1}\left(\frac{-j}{\sqrt{(j+1)j}}\right), cos^{-1}\left(\frac{-j+1}{\sqrt{(j+1)j}}\right), \ldots, cos^{-1}\left(\frac{j-1}{\sqrt{(j+1)j}}\right), cos^{-1}\left(\frac{j}{\sqrt{(j+1)j}}\right) \quad (51)$$

Because on the cone's outer surface all ($\hat{\hat{J}}$) orientations are similar, the projection of ($\hat{\hat{J}}$) along each y and x-axes mean out to zero [12, 14]:

$$\langle \hat{J}_y \rangle = \langle \hat{J}_x \rangle = 0 \quad (52)$$

$while\langle \hat{J}_x \rangle$ be stands for $\{\langle j, m|\hat{J}_x|j, m\rangle\}$.

5. Orbital Angular Momentum Eigenfunctions

In an effort to achieve the eigenvalues and eigenfunctions of ($\hat{\hat{L}}^2$) and a component of (L), It's suitable to exhibit the operators in spherical coordinates (r, θ, ϕ).

According to spherical coordinates ($\hat{L}_z, \hat{L}_\pm, \hat{\hat{L}}^2$) could be expressed as [7, 16]:

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \quad (53)$$

$$\hat{L}_\pm = \hat{L}_X \pm i\hat{L}_y = \pm\hbar e^{\pm i\phi}\left[\frac{\partial}{\partial\theta} \pm i\frac{cos\theta}{sin\theta} i\frac{\partial}{\partial\phi}\right] \quad (54)$$

$$\hat{\hat{L}}^2 = -\hbar^2\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \quad (55)$$

Because the operators ($\hat{L}$) & ($\hat{\hat{L}}_x$) hinge just on the angle $\emptyset$ *and* θ, their eigenstate rely only on $\emptyset$ *and* θ. Their link eigenstates can be denoting by [15, 17]:

$$\langle \theta_\phi | 1, m\rangle = Y_{1m}(\theta, \phi) \quad (56)$$

When continuous functions of $\emptyset$ *and* θ are $Y_{1m}(\theta, \phi)$, the eigenvalue relation ($\hat{\hat{L}}^2 = |1, m\rangle = \hbar^2 1(1+1)|1, m\rangle$) and ($\hat{L}_z|1, m\rangle = \hbar m|1, m\rangle$) can be written as[10, 18]

$$\hat{\hat{L}}^2 Y_{1M}(\theta, \phi) = \hbar 1(1+1)Y_{1m} \quad (57)$$

$$\hat{L}_z Y_{1m}(\theta, \phi) = m\hbar Y_{1m}(\theta, \phi) \quad (58)$$

Because ($\hat{L}_z$) rely just upon $\emptyset$, as illustrated in equation (53), the previous two functions indicated that the eigenfunctions $\{Y_{1m}(\theta, \phi)\}$ are separable [12, 19]:

$$Y_{1m}(\theta, \emptyset) = \Theta_{1m}(\theta)\Phi_m(\phi) \quad (59)$$

Establish that

$$\hat{L}_\pm Y_{1m}(\theta, \phi) = \hbar\sqrt{1(1+1) - m(m\pm 1)}Y_{1m\pm 1}(\theta, \phi) \quad (60)$$

Conclusion

Orbital angular momentum has been one of the fundamental constants of motion in quite an isolated 3-dimensional model.Based on the obtained findings, we inferred that a:

- A commutator is very important and can be referred to as a physics trigger to establish relationships between observables with a short method, since a quantum mechanics commutator tells us that if the two 'observables' commutator is zero, such as $\hat{L}^2$ and $\hat{L}_z$ commute$[\hat{L}^2, \hat{L}_z]$ = 0, then they can be calculated at the same time, otherwise, an uncertain relationship remains between the two 'observables'. Such a $[\hat{L}_x,\hat{L}_y]$ = iħ$\hat{L}_z$, ($[\hat{L}_y,\hat{L}_z]$ = iħ$\hat{L}_x$, ($[\hat{L}_Z, \hat{L}_x]$ = iħ$\hat{L}_y$, on the other hand, the eigenvalue and eigenvector have also been calculated for operators $\hat{J}_{\pm}, \hat{J}^2, \hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$within the $|j,m\rangle$basis.
- The orbital angular momentum$\hat{L}^2$ eigenvalues are ℓ(ℓ+1)ħ^2, when (ℓ) is representse quantum number of the orbital angular momentum and the eigenvalues of $\hat{L}_z$ are mħ while m is the number of the magnetic quantum and m = -ℓ, -ℓ+1,, -ℓ. For quantized angular momentum, the overall angular momentum is a general form when both spin angular momentum and orbital angular momentum are coupled. Thus, the eigenvalues of $\hat{\hat{J}}^2$are j(j+1) and the eigenvalues of $\hat{J}_z$ are ħm, when the overall angular momentum quantum number j = ℓ± s. Only a discrete set of values can be used for all of them, so it can be inferred that angular momentum is quantized and quantized angular momentum values are represented as quantum numbers.
- Angular momentum did not change continuously, but rather in “quantum leaps” (sudden changes) between such permitted values.
- Quantization angular momentum principle is suitable to the macroscopic system; nevertheless, the discrete phases are too small to discern at the macroscale.
- The universal unit of angular momentum is shortened Planck's constant {quantum of action}, where it’s defined via two quantum numbers (magnetic and orbital).
- Except in the trivial case where all of the orthogonal components of angular momentum are (0), two orthogonal parameters of angular momentum cannot be known or evaluated at the same time.
- It is possible to know the length of both the angular momentum vector and one of its elements at the same time.

  Because of the isotropy of space, total angular momentum is still conserved, but orbital angular momentum is not (spin-orbit coupling may transfer angular momentum between orbital and spin degrees of freedom).

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