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# Dynamics and Bifurcation of $x_{n+1}=\frac{\alpha+\beta x_{n-1}}{A+Bx_n+Cx_{n-1}}$

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**Abstract**
The main goal of this paper is to study the bifurcation of a second order rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_{n-1}}{A+Bx_n+Cx_{n-1}},\quad n=0,1,2,\ldots$$

with positive parameters $\alpha,\beta,A,B,C$ and non-negative initial conditions $\{x_{-k},x_{-k+1},\ldots,x_0\}$. We study the dynamic behavior and the direction of the bifurcation of the period-two cycle. Numerical discussion with figures are given to support our results.

**Keywords:** Fixed point, period-doubling bifurcation.
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## 1. Introduction

In this paper we studies the second order rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_{n-1}}{A+Bx_n+Cx_{n-1}},\quad n=0,1,2,\ldots, \tag{1.1}$$

with positive parameters $\alpha,\beta,A,B$ and $C$ and non-negative initial conditions $\{x_{-k},x_{-k+1},\ldots,x_0\}$. We focus on the dynamic behavior of the positive fixed points and the type of bifurcation exists where the change of stability occurs.

Equation (1.1) was studied by Lin-Xia Hu, Wan-Tong Li, Hong-Wu Xu in [4]. Boundedness, invariant intervals, semicycles and global stability of the positive fixed point was investigated. Also it was studied by Ladas in [5] and [1].

Recently, bifurcation and dynamics of higher order nonlinear difference equations have been studied in [8, 7, 6, 3].

Changing of variables convert the second-order rational difference equation with five positive parameters

$$x_{n+1}=\frac{\alpha+\beta x_{n-1}}{A+Bx_n+Cx_{n-1}},\quad n=0,1,2,\ldots$$

into

$$y_{n+1}=\frac{p+qy_{n-1}}{1+y_n+ry_{n-1}},\quad n=0,1,2,\ldots,$$

with three positive parameters $p$ , $q$ and $r$.
In this paper, regarding $p$ as a parameter, we investigate the existence of Period-Doubling bifurcation and use the normal form theory for discrete dynamical system to determine the direction of bifurcation of period-two cycle. Then, we give numerical discussion with figures to support our results.

## 2. Dynamics of $y_{n+1} = \frac{p+qy_{n-1}}{1+y_n+ry_{n-1}}$

In this section we study the stability of the positive fixed points of

$$y_{n+1} = \frac{p+qy_{n-1}}{1+y_n+ry_{n-1}}. \tag{2.1}$$

Note that the discrete difference equation (2.1) has the unique positive fixed point

$$\bar{y} = \frac{q-1+\sqrt{(1-q)^2+4p(1+r)}}{2(1+r)}.$$

In order to convert equation (2.1) to a second dimensional system with three positive parameters $p, q$, and $r$, let $u_n = x_{n-1}$ and $w_n = x_n$. We have the following system

$$u_{n+1} = w_n,$$

$$w_{n+1} = \frac{p+qu_n}{1+w_n+ru_n}, n = 0,1,2,\dots. \tag{2.2}$$

System (2.2) has the unique positive fixed point $(u^*,w^*)^T = (\bar{y},\bar{y})^T$.

The Jacobian matrix associated with system (2.2) at the positive fixed point is

$$JF(u,w)\mid_{(\bar{y},\bar{y})} = \begin{pmatrix} 0 & 1 \\ \frac{q+q\bar{y}-rp}{(1+\bar{y}+r\bar{y})^2} & -\frac{p+q\bar{y}}{(1+\bar{y}+r\bar{y})^2} \end{pmatrix}.$$

Note that

$$\det(JF(\bar{y},\bar{y})) = -\frac{q+q\bar{y}-rp}{(1+\bar{y}+r\bar{y})^2} = -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$$

and

$$tr(JF(\bar{y},\bar{y})) = -\frac{p+q\bar{y}}{(1+\bar{y}+r\bar{y})^2} = -\frac{\bar{y}}{1+\bar{y}+r\bar{y}}$$

where det and $tr$ denote the determinant and trace of the Jacobian matrix J, respectively.
We will use the following lemmas.

**Lemma 2.1.** *[2] Consider the map $f : G \subset \mathbb{R}^2 \to \mathbb{R}^2$ ba a $C^1$ map, where $G$ is an open subset of $\mathbb{R}^2$, $\bar{x}$ is a fixed point of f, $A = Jf(\bar{x})$ and $\rho(A)$ is the spectral norm of $A$ where $\rho(A) = max_i\{\mid \lambda_i \mid, \lambda_i$ are the eigenvalues of $A\}$. Then the following statement hold true:*

1. *If $\rho(A) < 1$, then $\bar{x}$ is asymptotically stable.*
2. *If $\rho(A) > 1$, then $\bar{x}$ is unstable.*
3. *If $\rho(A) = 1$, then $\bar{x}$ may or may not be stable.*

**Lemma 2.2.** *[2] Consider the map*

$$x \to f(x), \quad x \in \mathbb{R}^2,$$

*with $\bar{x}$ as a fixed point of f and $A = Jf(\bar{x})$. Then $\rho(A) < 1$ if and only if*

$$\mid trA \mid -1 < \det A < 1$$

*where $trA$ and $\det A$ denote trace and determinant of the matrix $A$ respectively.*

**Theorem 2.3.** *[9] The equilibrium point $\bar{y}$ of (2.1) is locally asymptotically stable if one of the following holds*

1. $q \leq 1$
2. $q > 1$ *and* $(r-1)(q-1)^2+4pr^2>0$.

**Proof:** We want to show that

$$\left|\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}\right| < 1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}} < 2.$$

That is equivalent to

$$\frac{\bar{y}}{1+\bar{y}+r\bar{y}}+\left|\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}\right| < 1 \quad and \quad \frac{\bar{y}}{1+\bar{y}+r\bar{y}} > -1.$$

The first inequality is equivalent to

$$|q-r\bar{y}| < 1+r\bar{y}. \tag{2.3}$$

If $q-r\bar{y}<0$, then(2.3) becomes $r\bar{y}-q<1+r\bar{y}$ and this is obvious .
If $q-r\bar{y}\geq 0$, then(2.3) becomes $q-r\bar{y}<1+r\bar{y}$,
or

$$q-1<2r\bar{y}. \tag{2.4}$$

If $q \leq 1$ , then (2.4) holds. If $q>1$, then

$$r\bar{y} > r\frac{\sqrt{(q-1)^2+4p(1+r)}}{r+1} > r\sqrt{(q-1)^2+4p(1+r)}$$

and if $(r-1)(q-1)^2+4pr^2>0$, multiply both sides by $r+1$ we can get

$$(r^2-1)(q-1)^2+4pr^2(1+r)>0.$$

Rearrange the terms of the previous inequality, we get

$$r^2((q-1)^2+4p(1+r)) > (q-1)^2.$$

Take the square of both sides, we obtain

$$r\sqrt{(q-1)^2+4p(1+r)} > (q-1).$$

Now, add $r(q-1)$ for both sides, we have

$$r(q-1+\sqrt{(q-1)^2+4p(1+r)} > (r+1)(q-1).$$

That is equivalent to

$$2r(r+1)\bar{y} > (r+1)(q-1),$$

or

$$2r\bar{y} > q-1.$$

This shows in this case inequality (2.4) holds and hence

$$\frac{\bar{y}}{1+\bar{y}+r\bar{y}}+\left|\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}\right| < 1.$$

Note that the second inequality $1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}}<2$ is always true.
So in both cases the equilibrium point $\bar{y}$ is locally asymptotically stable.

## 3. Existence of Period-Doubling Bifurcation

In this section we will study the bifurcation of (2.1). we will use the following theorem.

**Lemma 3.1.** *[2] Consider the map*

$$x \to f(x,\alpha), x \in \mathbb{R}^2, \alpha \in \mathbb{R}. \tag{3.1}$$

*Let $A = Jf(x^*,\alpha^*)$ where $(x^*,\alpha^*)$ is a fixed point of $f(x,\alpha)$. Then the following hold*

1. *If $\det A = -trA - 1$, then the eigenvalues of $A$ are $\lambda_1 = -\det A$ and $\lambda_2 = -1$.*
2. *If $\det A = trA - 1$, then $\lambda_1 = 1$ and $\lambda_2 = \det A$.*
3. *If $|\, trA \,| - 1 < \det A$ and $\det A = 1$, then $A$ has complex eigenvalues $\lambda_{1,2} = e^{\pm i\theta}$ where $\theta = cos^{-1}(\frac{trA}{2})$.*

**Corollary 3.2.** *For the one-parameter of two-dimensional map*

$$x \to f(x,\alpha), x \in \mathbb{R}^2, \alpha \in \mathbb{R}, \tag{3.2}$$

*with the fixed point $(x^*,\alpha^*)$ and $A = Jf(x^*,\alpha^*)$, then the following hold*

1. *If $\det A = -trA - 1$, then the system (3.2) undergoes a period-doubling bifurcation.*
2. *If $\det A = trA - 1$, then then the system (3.2) undergoes a saddle-node bifurcation.*
3. *If $|\, trA \,| - 1 < \det A$ and $\det A = 1$, then the system (3.2) undergoes a Neimark-Sacker bifurcation.*

Using the previous corollary, system (2.2) can not undergoes a saddle-node or Neimark-Sacker bifurcation.

**Theorem 3.3.** *The fixed point $(\bar{y},\bar{y})^T$ of the system (2.2) undergoes a period-doubling (flip) bifurcation when $p = \frac{(1-r)(q-1)^2}{4r^2}$ if $q > 1$ and $r < 1$.*

**Proof:** Assume that $q > 1$ and $r < 1$. Corollary (3.2) implies that period-doubling bifurcation occurs if $\det(JF(\bar{y},\bar{y})^T) = -tr(JF(\bar{y},\bar{y})) - 1$.
That is equivalent to

$$-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{\bar{y}}{1+\bar{y}+r\bar{y}} - 1,$$

$$-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{\bar{y}-(1+\bar{y}+r\bar{y})}{1+\bar{y}+r\bar{y}} - 1,$$

or

$$-(q-r\bar{y}) = \bar{y}-(1+\bar{y}+r\bar{y}).$$

That is equivalent to

$$2r\bar{y} = q-1,$$

or

$$2\bar{y} = \frac{q-1}{r}.$$

Substitute the value of $\bar{y}$, we obtain

$$\frac{q-1+\sqrt{(1-q)^2+4p(1+r)}}{1+r} = \frac{q-1}{r},$$

or

$$q-1+\sqrt{(1-q)^2+4p(1+r)} = q-1+\frac{q-1}{r}.$$

Take the square of both sides, we get

$$(1-q)^2+4p(1+r) = (\frac{q-1}{r})^2,$$

multiply both sides by $r^2$

$$r^2[(1-q)^2+4p(1+r)] = (q-1)^2,$$

or

$$(r^2-1)(q-1)^2+4pr^2(1+r) = 0.$$

Since $r > 0$, $r+1 \neq 0$, so we can divide into $1+r$. We obtain

$$(r-1)(q-1)^2+4pr^2 = 0,$$

$$p = \frac{(1-r)(q-1)^2}{4r^2}.$$

## 4. Direction of The Period-Doubling (Flip) Bifurcation

In this section we will use the normal form theory for discrete dynamical system to find the direction of the period-doubling bifurcation of system (2.2) which exists at $p = \frac{(1-r)(q-1)^2}{4r^2}$. Firstly, we shift the fixed point $(\bar{y},\bar{y})^T$ to the origin. Let

$$x_n = u_n - \bar{y}, \;\; z_n = w_n - \bar{y}.$$

System (2.2) corresponds to

$$x_{n+1} = z_n,$$

$$z_{n+1} = \frac{p+q(x_n+\bar{y})}{1+(z_n+\bar{y})+r(x_n+\bar{y})} - \bar{y}, \tag{4.1}$$

or

$$Y_{n+1} = AY_n + G(Y_n), \tag{4.2}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & -\frac{\bar{y}}{1+\bar{y}+r\bar{y}} \end{pmatrix}, Y_n = \begin{pmatrix} x_n \\ z_n \end{pmatrix},$$

and

$$G(Y) = \frac{1}{2}B(Y,Y)+\frac{1}{6}C(Y,Y,Y)+O(\| Y \|^3),$$

$$B(Y,Y) = \begin{pmatrix} 0 \\ B_2(Y,Y) \end{pmatrix} \text{ and } C(Y,Y,Y) = \begin{pmatrix} 0 \\ C_2(Y,Y,Y) \end{pmatrix},$$

where

$$B_2(\phi,\psi) = -\frac{2r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}\phi_1\psi_1 + \frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}[\phi_1\psi_2+\phi_2\psi_1] + 2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}\phi_2\psi_2,$$

and

$$\begin{aligned} C_2(\phi,\psi,\eta) =& 6\frac{r^2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3}\phi_1\psi_1\eta_1 + \frac{4qr-6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3}[\phi_1\psi_1\eta_2+\phi_2\psi_0\eta_1+\phi_1\psi_2\eta_1] \\ &+ \frac{2q-6r\bar{y}}{(1+\bar{y}+r\bar{y})^3}[\phi_1\psi_2\eta_2+\phi_2\psi_1\eta_2+\phi_1\psi_2\eta_2] - 6\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^3}\phi_2\psi_2\eta_2. \end{aligned}$$

Let $q$ and $p^*$ be the eigenvectors of $A$ and $A^T$ corresponding to the eigenvalue $\lambda = -1$, respectively. We have $Aq = -q$ and $A^T p^* = -p^*$, where

$$q \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ and } p^* \sim \begin{pmatrix} -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 \end{pmatrix}.$$

Normalize $p^*$ and $q$,

$$< p^*, q >= \sum_{i=1}^{2} p_i^* q_i = -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - 1.$$

Take

$$p = \xi * \begin{pmatrix} -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 \end{pmatrix}, \text{ where } \xi = \frac{1}{-1-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}} = -\frac{1+\bar{y}+r\bar{y}}{q+1+\bar{y}}.$$

The critical eigenspace $T^c$ corresponding to the eigenvalue $\lambda$ is a one-dimensional map, and is spanned by the eigenvector $q$. Let $T^{su}$ denote a one-dimensional linear eigenspace of $A$ corresponding to the other eigenvalue than $\lambda$. Note that the matrix $A - \lambda I$ which is equivalent to the matrix $A + T$ has common invariant spaces with the matrix $A$, we conclude that $y \in T^{su}$ if and only if $< p, y >= 0$. Any vector $x \in \mathbb{R}^2$ can be decomposed as

$$x = uq + y,$$

where $uq \in T^c, y \in T^{su}$, and

$$u =< p, x >,$$

$$y = x- < p, x > q. \tag{4.3}$$

In the coordinates $(u, y)$, the map (4.2) can be written as

$$\tilde{u} = \lambda u+ < p, F(uq+y) >,$$

$$\tilde{y} = Ay + F(uq+y)- < p, F(uq+y) > q. \tag{4.4}$$

Using Taylor expansions, (4.4) can be written as

$$\tilde{u} = \lambda u + \frac{1}{2}\sigma u^2 + u < b, y > + \frac{1}{6}\delta u^3 + \ldots,$$

$$\tilde{y} = Ay + \frac{1}{2} a u^2 + \ldots, \tag{4.5}$$

where $u \in \mathbb{R}^1, y \in \mathbb{R}^2, \sigma, \delta \in \mathbb{R}^1, a, b \in \mathbb{R}^2$ and $< b, y >= \sum_{i=1}^{2} b_i y_i$ is the standard scaler product $< b, y >$ can be expressed as

$$< b, y >=< p, B(q, y) > .$$

The center manifold of (4.5) has the representation

$$y = V(u) = \frac{1}{2} w_2 u^2 + O(u^3),$$

where $w_2 \in T^{su} \subset \mathbb{R}^2$, so that $< p, w >= 0$. The vector $w_2$ satisfies

$$(A - I)w_2 + a = 0.$$

Note that the matrix $A-I$ is invertible in $\mathbb{R}^2$ because $\lambda = 1$ is not an eigenvalue of A. Thus, we have

$$w_2 = -(A-I)^{-1}a,$$

and the restriction of (4.5) to the center manifold takes the form

$$\tilde{u} = -u + \frac{1}{2}\sigma u^2 + \frac{1}{6}(\delta - 3 < p, B(q,(A-I)^{-1}a) >)u^3 + O(u^4),$$

where

$$\sigma =< p, B(q,q) >, \delta =< p, C(q,q,q) >, \text{ and } a = B(q,q)- < p, B(q,q) > q.$$

Using the identity $(A-I)^{-1}q = -\frac{1}{2}q$, the restricted map can be written as

$$\tilde{u} = -u + a(0)u^2 + b(0)u^3 + O(u^4), \tag{4.6}$$

where

$$a(0) = \frac{1}{2} < p, B(q,q) >,$$

and

$$b(0) = \frac{1}{6} < p, C(q,q,q) > -\frac{1}{4}(< p, B(q,q) >)^2 - \frac{1}{2} < p, B(q,(A-I)^{-1}B(q,q)) > .$$

$$B(q,q) = \begin{pmatrix} 0 \\ -2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}} - 2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2} \end{pmatrix},$$

$$< p, B(q,q) >= -\frac{1+\bar{y}+r\bar{y}}{q+1+\bar{y}}[-2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}} - 2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}],$$

$$C(q,q,q) = \begin{pmatrix} 0 \\ 6\frac{r^2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3} - 3\frac{4qr-6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3} + 3\frac{2q-6r\bar{y}}{(1+\bar{y}+r\bar{y})^3} + 6\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^3} \end{pmatrix},$$

$$< p, C(q,q,q) >= -\frac{1+\bar{y}+r\bar{y}}{q+1+\bar{y}}[6\frac{r^2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3} - 3\frac{4qr-6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3} + 3\frac{2q-6r\bar{y}}{(1+\bar{y}+r\bar{y})^3} + 6\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^3}],$$

$$(A-I)^{-1} = \begin{pmatrix} -1 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & -1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}} \end{pmatrix}^{-1} = \frac{1+\bar{y}+r\bar{y}}{2\bar{y}} \begin{pmatrix} -1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}} & -1 \\ -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & -1 \end{pmatrix},$$

$$(A-I)^{-1}B(q,q) = \frac{1+\bar{y}+r\bar{y}}{2\bar{y}} \begin{pmatrix} -2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}} - 2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2} \\ -2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}} - 2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2} \end{pmatrix},$$

$$B(q,(A-I)^{-1}B(q,q)) = \frac{1+\bar{y}+r\bar{y}}{2\bar{y}} \begin{pmatrix} 0 \\ S \end{pmatrix}$$

where

$$S = [\frac{2r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2} + \frac{2\bar{y}}{(1+\bar{y}+r\bar{y})^2}][-2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2} + 2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2} - 2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}],$$

$$\begin{aligned} < p, B(q,(A-I)^{-1}B(q,q)) >=& 2\frac{r^2(q-r\bar{y})^2}{\bar{y}(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2} - 2\frac{\bar{y}}{(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2} \\ &+ 2\frac{r(q-r\bar{y})(2r\bar{y}-q)}{\bar{y}(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2} + 2\frac{2r\bar{y}-q}{(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2}. \end{aligned}$$

The map (4.6) can be transformed to the normal form

$$\tilde{\xi} = -\xi + c(0)\xi^3 + O(\xi^4),$$

where

$$c(0) = a^2(0) + b(0).$$

Thus, the critical normal form coefficient $c(0)$ allows us to predict the direction of bifurcation of period-two cycle. $c(0)$ is given by the following invariant formula:

$$c(0) = \frac{1}{6} < p, C(q,q,q) > -\frac{1}{2} < p, B(q,(A-I)^{-1}B(q,q)) > .$$

If $c(0) > 0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $p = \frac{(1-r)(1-q)^2}{4r^2}$.

## 5. Numerical Discussion

In this section we give numerical examples which support our results in the previous sections. Figure that we get using Matlab will be attached with example to illustrate the bifurcation.

**Example 5.1.** *Consider equation (2.1). In this example we fix the parameters q, r and consider p as bifurcation parameter. Take* $q = 1.1$, $r = 0.09$ *and* $0 < p \leq 2$. *Equation (2.1) becomes*

$$y_{n+1} = \frac{p + 1.1y_{n-1}}{1 + y_n + 0.09y_{n-1}}, n = 0, 1, 2, \ldots \tag{5.1}$$

*The planer form corresponding to equation (5.1) is*

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{p+1.1y_1(n)}{1+y_2(n)+0.09y_1(n)}. \end{pmatrix} \tag{5.2}$$

*Positive equilibrium point of system (5.2) is* $(\bar{y}, \bar{y})$, *where* $\bar{y} = \frac{0.1+\sqrt{0.01+4.36p}}{2.18}$. *Theorem (3.3) determined the bifurcation point at* $(r-1)(1-q)^2 + 4pr^2 = 0$. *So, the fixed point undergoes a period-doubling bifurcation at* $p = 0.2808642$.

$$q = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad and \quad p = \begin{pmatrix} 0.39539749 \\ -0.60460251 \end{pmatrix},$$

$$B(q,q) = \begin{pmatrix} 0 \\ 0.71303782 \end{pmatrix},$$

$$< p, B(q,q) >= -0.43110446,$$

$$C(q,q,q) = \begin{pmatrix} 0 \\ -0.4797597 \end{pmatrix},$$

$$< p, C(q,q,q) >= 0.2900639,$$

$$(A-I)^{-1} = \begin{pmatrix} -1 & 1 \\ 0.65397924 & -1.34602076 \end{pmatrix},$$

$$B(q,(A-I)^{-1}B(q,q)) = \begin{pmatrix} 0 \\ 0.8212105 \end{pmatrix},$$

$$< p, B(q,(A-I)^{-1}B(q,q)) >= -0.49947486,$$

$$c(0) = 0.20139345 > 0.$$

*So, this verify that a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point* $p = 0.2808642$. *See figure (5.1).*

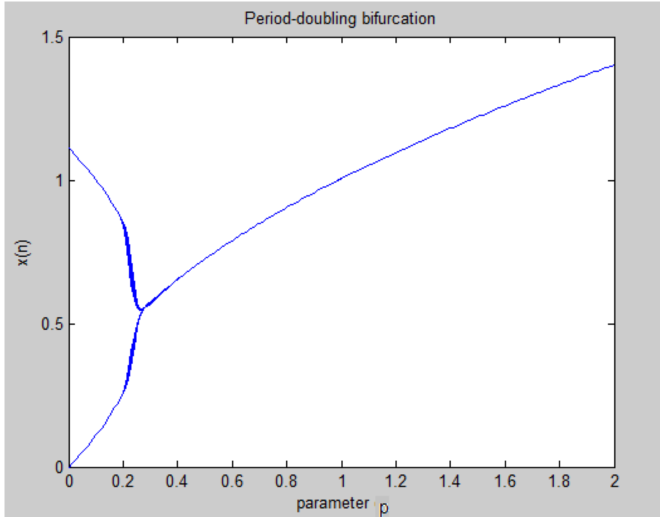


**Figure 5.1.** Period-doubling bifurcation of $y_{n+1} = \frac{p+1.1y_{n-1}}{1+y_i+0.09y_{i-1}}$, $p$ is a parameter.

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## Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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