ON A NEW SUBCLASS OF P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

Ömer DURMAZPINAR
Department of Mathematics, Faculty of Science, Atatürk University, 25240 Erzurum, Turkey

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Abstract

We introduce a new subclass $P_p^*(\alpha,\beta,\xi,\Omega,m)$ *of analytic and* $p-$*valent functions with negative coefficients. Coefficient theorem, distortion theorem and closure theorem of functions belonging to the class* $P_p^*(\alpha,\beta,\xi,\Omega,m)$ *are determined. Also we obtain radius of convexity for* $P_p^*(\alpha,\beta,\xi,\Omega,m)$. *Integral operators of functions belonging to the class* $P_p^*(\alpha,\beta,\xi,\Omega,m)$ *are studied here. Furthermore the extreme points of* $P_p^*(\alpha,\beta,\xi,\Omega,m)$ *are also determined.*

Keywords: *Distortion theorem, exreme points, analytic function, radius of convexity, Salagean operator.*

NEGATİF KATSAYILI p-VALENT FONKSİYONLARIN BİR YENİ ALTSINIFI HAKKINDA

Özet

Bu makalede negative katsayılı $p-$*valent analitik fonksiyonların* $P_p^*(\alpha,\beta,\xi,\Omega,m)$ *ile gösterilen yeni bir sınıfı tanıtıldı.* $P_p^*(\alpha,\beta,\xi,\Omega,m)$ *sınıfına ait fonksiyonlar için katsayı teoremi, distorsiyon teoremi ve kapanış teoremi belirlendi. Ayrıca* $P_p^*(\alpha,\beta,\xi,\Omega,m)$ *sınıfı için konvekslik yarıçapı elde edildi. Bundan başka* $P_p^*(\alpha,\beta,\xi,\Omega,m)$ *sınıfına ait* *fonksiyonların integral operatörleri çalışıldı. Bunlara ilave olarak* $P_p^*(\alpha,\beta,\xi,\Omega,m)$ *sınıfının extreme noktaları belirlendi.*

Anahtar kelimeler: *Distorsiyon teoremi, extreme noktaları, analitik fonksiyon, konvekslik yarıçapı, Salagean operator.*

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1. Introduction and Definitions

We recall some basic facts together with terminology and notation that will be needed.

Let *A* be class of functions $f(z)$ of the form $f(z)=z+\sum_{k=2}^{\infty}a_k z^k$ which are analytic in the open unit disk $U=\{z:|z|<1\}$. For $f(z)$ belong to *A*, Sălăgean [7] has introduced the following operator called the Sălăgean operator:

$$D^0 f(z)=f(z), \quad D^1 f(z)=Df(z)=zf'(z)$$
$$D^n f(z)=D(D^{n-1}f(z)) \quad (n\in\mathbb{N}=\{1,2,3,\ldots\}).$$

We note that

$$D^n f(z)=z+\sum_{k=2}^{\infty}k^n a_k z^k, \quad n\in\mathbb{N}_0=\{0\}\cup\mathbb{N}.$$

Let S_p ($p\in\mathbb{N}$) denote the class of functions of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}$$

that are holomorphic and p-valent in the unit disk $|z|<1$.

Also let T_p denote the subclass of S_p consisting of functions that can be expressed in the form

$$f(z)=z^{p}-\sum_{n=1}^{\infty}\left|a_{n+p}\right|z^{n+p} \qquad (1.1)$$

We can write the following equalities for the functions $f(z)$ belonging to the class T_p

$$D^{0}f(z)=f(z),$$

$$D^{1}f(z)=Df(z)=zf'(z)=pz^{p}-\sum_{n=1}^{\infty}(n+p)\left|a_{n+p}\right|z^{n+p},$$

$$D^{2}f(z)=D(Df(z))=p^{2}z^{p}-\sum_{n=1}^{\infty}(n+p)^{2}\left|a_{n+p}\right|z^{n+p},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$D^{\Omega}f(z)=D(D^{\Omega-1}f(z))=p^{\Omega}z^{p}-\sum_{n=1}^{\infty}(n+p)^{\Omega}\left|a_{n+p}\right|z^{n+p}.$$

A function $f(z)\in T_p$ in $P_p^{*}(\alpha,\beta,\xi,\Omega,m)$ if and only if

$$\left|\frac{(D^{\Omega}f(z))^{(m)}z^{m-p}-\dfrac{p^{\Omega}p!}{(p-m)!}}{2\xi\left[(D^{\Omega}f(z))^{(m)}z^{m-p}-\alpha\right]-\left[(D^{\Omega}f(z))^{(m)}z^{m-p}-\dfrac{p^{\Omega}p!}{(p-m)!}\right]}\right|<\beta,$$

$$(p,m\in\mathbb{N},\ \Omega\in\mathbb{N}_0,\ p\geq m),\ |z|<1,\ \text{for } 0\leq\alpha<\frac{p}{2\xi},\ 0<\beta\leq 1,\ \frac{1}{2}<\xi\leq 1.$$

Particularly, the symbol $(D^{\Omega}f(z))^{(m)}$ was named as the $m-$ th order derivative operator.

Such type of investigation was carried out by Aouf [1] for $P_p^{*}(\alpha,\beta)$. We note that $P_1^{*}(\alpha)\equiv P_1^{*}(0,\alpha,1,0,1)$ is precisely the class of functions in U studied by

Caplinger [2]. The class $P_1^*(\alpha,1,\beta,0,1)\equiv P_1^*(\alpha,\beta)$ is the class of analytic functions investigated by Juneja-Mogra [4]. Gupta-Jain [3] investigated the family of analytic univalent functions that have the form $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ and satisfy the condition

$$\left|\frac{f'(z)-1}{f'(z)+(1-2\alpha)}\right|<\beta,\quad (0\le\alpha<1,\quad 0<\beta\le 1)$$

Kulkarni [5] has studied above mentioned properties for the functions having Taylor series expansion of the form $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$.

A function $f\in T_p$ is in $P_p^*(\alpha,\beta,\xi)$ if and only if

$$\left|\frac{f'(z)z^{1-p}-p}{2\xi(f'(z)z^{1-p}-\alpha)-(f'(z)z^{1-p}-p)}\right|<\beta, |z|<1,\ \text{for}\ 0\le\alpha<\frac{p}{2\xi},\ 0<\beta\le 1,\ \frac{1}{2}<\xi\le 1.$$

The class $P_p^*(\alpha,\beta,\xi)$ investigated by Kulkarni *et al*. [6].

A function $f\in T_p$ is in $P_p^*(\alpha,\beta,\xi,\Omega)$ if and only if

$$\left|\frac{(D^{\Omega}f(z))'z^{1-p}-p^{\Omega+1}}{2\xi((D^{\Omega}f(z))'z^{1-p}-\alpha)-((D^{\Omega}f(z))'z^{1-p}-p^{\Omega+1})}\right|<\beta,$$

$$\Omega\in\mathbb{N}_0,\ |z|<1,\ \text{for}\ 0\le\alpha<\frac{p}{2\xi},\ 0<\beta\le 1,\ \frac{1}{2}<\xi\le 1.$$

The class $P_p^*(\alpha,\beta,\xi,\Omega)$ studied by Orhan *et al*. [8].

In this paper sharp results concerning coefficients, distortion theorem, closure theorem and the radius of convexity for the class $P_p^*(\alpha,\beta,\xi,\Omega,m)$ are determined. Furthermore, we give integral operators of functions belonging to the class $P_p^*(\alpha,\beta,\xi,\Omega,m)$.

We note that $P_p^*(\alpha,\beta,\xi,\Omega,1)\equiv P_p^*(\alpha,\beta,\xi,\Omega)$. Therefore our class $P_p^*(\alpha,\beta,\xi,\Omega,m)$ is the generalization of $P_p^*(\alpha,\beta,\xi,\Omega,m)$ by Orhan *et al*. [8].

2. Coefficient Theorem

We begin by proving some sharp coefficient inequalities contained in following theorem.

Theorem 1. A function $f(z)=z^p-\sum_{n=1}^{\infty}\left|a_{n+p}\right|z^{n+p}$ is in $P_p^*(\alpha,\beta,\xi,\Omega,m)$ if and only if

$$\sum_{n=1}^{\infty}\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|\le 2\beta\xi\left(\frac{p!\,p^{\Omega}}{(p-m)!}-\alpha\right).$$

The result is sharp, the extremal function being

$$f(z)=z^p-\frac{2\beta\xi\left(\frac{p!\,p^{\Omega}}{(p-m)!}-\alpha\right)}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}z^{n+p}. \tag{2.1}$$

Proof. Let $|z|=1$. Then

$$\left|(D^{\Omega}f(z))^{(m)}z^{m-p}-\frac{p!\,p^{\Omega}}{(p-m)!}\right|-\beta\left|2\xi\left[(D^{\Omega}f(z))^{(m)}z^{m-p}-\alpha\right]-\left[(D^{\Omega}f(z))^{(m)}z^{m-p}-\frac{p!\,p^{\Omega}}{(p-m)!}\right]\right|$$

$$=\left|-\sum_{n=1}^{\infty}\frac{(n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|z^{n}\right|-\beta\left|2\xi\left(\frac{p!\,p^{\Omega}}{(p-m)!}-\alpha\right)-\sum_{n=1}^{\infty}\frac{(2\xi-1)(n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|z^{n}\right|$$

$$\le\sum_{n=1}^{\infty}\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|-2\beta\xi\left(\frac{p!\,p^{\Omega}}{(p-m)!}-\alpha\right)\le 0$$

by hypothesis. Hence, by maximum modulus theorem $f(z)\in P_p^*(\alpha,\beta,\xi,\Omega,m)$ For the converse we suppose that

$$\left|\frac{(D^{\Omega}f(z))^{(m)}z^{m-p}-\dfrac{p!\,p^{\Omega}}{(p-m)!}}{2\xi\left[(D^{\Omega}f(z))^{(m)}z^{m-p}-\alpha\right]-\left[(D^{\Omega}f(z))^{(m)}z^{m-p}-\dfrac{p!\,p^{\Omega}}{(p-m)!}\right]}\right|$$

$$=\left|\frac{\sum_{n=1}^{\infty}\dfrac{(n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|z^{n}}{2\xi(\dfrac{p^{\Omega}p!}{(p-m)!}-\alpha)-\sum_{n=1}^{\infty}\dfrac{(2\xi-1)(n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|z^{n}}\right|<\beta.$$

Since $\left|\mathrm{Re}(z)\right|\leq\left|z\right|$ for all z we have

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}\dfrac{(n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|z^{n}}{2\xi(\dfrac{p^{\Omega}p!}{(p-m)!}-\alpha)-\sum_{n=1}^{\infty}\dfrac{(2\xi-1)(n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|z^{n}}\right\}<\beta.$$

We select the values of z on the real axis so that $(D^{\Omega}f(z))^{(m)}z^{m-p}$ is real. Simplifying the denominator in the above expression and letting $z\to 1$ through real values, we obtain

$$\sum_{n=1}^{\infty}\frac{(n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|z^{n}\leq 2\beta\xi(\frac{p^{\Omega}p!}{(p-m)!}-\alpha)-\sum_{n=1}^{\infty}\frac{\beta(2\xi-1)(n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|z^{n}$$

and it result in the required condition.

The result is sharp for the function (2.1).

Corollary 1. Let the function $f(z)$ defined by (1.1) be in the class $P_{p}^{*}(\alpha,\beta,\xi,\Omega,m)$. Then

$$a_{n+p} \le \frac{2\beta\xi\left[\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right]}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}, \qquad n=1,2,3,....$$

3. Distortion Theorem

Let us start with the following theorem.

Theorem 2. If $f(z)\in P_p^*(\alpha,\beta,\xi,\Omega,m)$, then for $|z|=r$,

$$r^p-\frac{2\beta\xi\left[\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right]}{\frac{\left[1+(2\xi-1)\beta\right](1+p)^{\Omega}(1+p)!}{(1+p-m)!}}r^{p+1}\le |f(z)|\le r^p+\frac{2\beta\xi\left[\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right]}{\frac{\left[1+(2\xi-1)\beta\right](1+p)^{\Omega}(1+p)!}{(1+p-m)!}}r^{p+1} \tag{3.1}$$

and

$$pr^{p-1}-\frac{2\beta\xi\left[\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right]}{\frac{\left[1+(2\xi-1)\beta\right](1+p)^{\Omega}(1+p)!}{(1+p-m)!}}r^{p}\le |f'(z)|\le pr^{p-1}+\frac{2\beta\xi\left[\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right]}{\frac{\left[1+(2\xi-1)\beta\right](1+p)^{\Omega}(1+p)!}{(1+p-m)!}}r^{p} \tag{3.2}$$

Proof. In view of Theorem 1, we have

$$\sum_{n=1}^{\infty}|a_{n+p}|\le \frac{2\beta\xi\left[\frac{p^{\Omega}p!}{(p-m)!}\right]}{\frac{\left[1+(2\xi-1)\beta\right](1+p)^{\Omega}(1+p)!}{(1+p-m)!}}.$$

Hence

$$\left|f(z)\right| \le r^p + \sum_{n=1}^{\infty}\left|a_{n+p}\right| r^{n+p} \le r^p + \frac{2\beta\xi\left[\dfrac{p^{\Omega}p!}{(p-m)!}\right]}{\dfrac{\left[1+(2\xi-1)\beta\right](1+p)^{\Omega}(1+p)!}{(1+p-m)!}} r^{1+p}$$

and

$$\left|f(z)\right| \ge r^p - \sum_{n=1}^{\infty}\left|a_{n+p}\right| r^{n+p} \ge r^p - \frac{2\beta\xi\left[\dfrac{p^{\Omega}p!}{(p-m)!}\right]}{\dfrac{\left[1+(2\xi-1)\beta\right](1+p)^{\Omega}(1+p)!}{(1+p-m)!}} r^{1+p}.$$

In the same way we have

$$\left|f'(z)\right| \le pr^{p-1} + \sum_{n=1}^{\infty}(n+p)\left|a_{n+p}\right| r^{n+p-1} \le pr^{p-1} + \frac{2\beta\xi\left[\dfrac{p^{\Omega}p!}{(p-m)!}\right]}{\dfrac{\left[1+(2\xi-1)\beta\right](1+p)^{\Omega-1}(1+p)!}{(1+p-m)!}} r^{p}$$

and

$$\left|f'(z)\right| \ge pr^{p-1} - \sum_{n=1}^{\infty}(n+p)\left|a_{n+p}\right| r^{n+p-1} \ge pr^{p-1} - \frac{2\beta\xi\left[\dfrac{p^{\Omega}p!}{(p-m)!}\right]}{\dfrac{\left[1+(2\xi-1)\beta\right](1+p)^{\Omega-1}(1+p)!}{(1+p-m)!}} r^{p}.$$

This completes the proof of the theorem.

The above bounds are sharp. Equalities are attended for the following function:

$$f(z) = z^p - \frac{2\beta\xi\left[\dfrac{p^{\Omega}p!}{(p-m)!}-\alpha\right]}{\dfrac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}} z^{p+1} \qquad z=\pm r. \tag{3.3}$$

Theorem 3. Let $f(z)\in P_p^*(\alpha,\beta,\xi,\Omega,m)$. Then the disk $|z|<1$ is mapped on a domain that contain the disk

$$|w|<\frac{\frac{(1+p)^{\Omega}(1+p)!}{(1+p-m)!}+\beta\left\{\frac{(2\xi-1)(1+p)^{\Omega}(1+p)!}{(1+p-m)!}-2\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right)\right\}}{\frac{\left[1+2\xi-1)\beta\right](1+p)^{\Omega}(1+p)!}{(1+p-m)!}}.$$

The result is sharp with extremal function (3.3).

Proof. The result follows upon letting $r\rightarrow 1$ in (3.1).

Theorem 4. $f(z)\in P_p^*(\alpha,\beta,\xi,\Omega,m)$, then $f(z)$ is convex in the disk $|z|<r=r(\alpha,\beta,\xi,\Omega,m)$, where

$$r(\alpha,\beta,\xi,\Omega,m)=\inf_{n\in\mathbb{N}}\left\{\frac{\frac{p^2\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}{2\beta\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right)(n+p)^2}\right\}^{\frac{1}{n}},\quad n=1,2,3,...$$

the result is sharp, the extremal function being of the form (2.1).

Proof. It is enough to show that

$$\left|(1+\frac{zf''(z)}{f'(z)}-p\right|\le p \qquad \text{for} \qquad |z|<1.$$

First we note that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|=\left|\frac{zf''(z)+(1-p)f'(z)}{f'(z)}\right|\le\frac{\sum_{n=1}^{\infty}n(n+p)|a_{n+p}||z|^n}{p-\sum_{n=1}^{\infty}(n+p)|a_{n+p}||z|^n}.$$

Thus, the result follows if

$$\sum_{n=1}^{\infty} n(n+p)\left|a_{n+p}\right|\left|z\right|^n \le p\left\{p-\sum_{n=1}^{\infty}(n+p)\left|a_{n+p}\right|\left|z\right|^n\right\},$$

or, equivalently,

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^2\left|a_{n+p}\right|\left|z\right|^n \le 1.$$

But in view of Theorem 1, we have

$$\sum_{n=1}^{\infty}\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right| \le 2\beta\xi\left(\frac{p!\,p^{\Omega}}{(p-m)!}-\alpha\right).$$

Thus f is convex if

$$\left(\frac{n+p}{p}\right)^2\left|z\right|^n \le \frac{\dfrac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}{2\beta\xi\left(\dfrac{p^{\Omega}p!}{(p-m)!}-\alpha\right)}, \quad n=1,2,3,...$$

$$\left|z\right| \le \left\{\frac{\dfrac{p^2\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}{2\beta\xi\left(\dfrac{p^{\Omega}p!}{(p-m)!}-\alpha\right)(n+p)^2}\right\}^{\frac{1}{n}}, \quad n=1,2,3,...$$

which completes the proof of our theorem.

4. Closure Theorem

We shall prove the following result for the closure of functions in the class $P_p^*(\alpha,\beta,\xi,\Omega,m)$

Theorem 5. If $f(z)=z^p-\sum_{n=1}^{\infty}\left|a_{n+p}\right|z^{n+p}$ and $g(z)=z^p-\sum_{n=1}^{\infty}\left|b_{n+p}\right|z^{n+p}$ are in the $P_p^*(\alpha,\beta,\xi,\Omega,m)$, then $h(z)=z^p-\frac{1}{2}\sum_{n=1}^{\infty}\left|a_{n+p}+b_{n+p}\right|z^{n+p}$ is also in $P_p^*(\alpha,\beta,\xi,\Omega,m)$.

Proof. f and g both being members of $P_p^*(\alpha,\beta,\xi,\Omega,m)$, we have in accordance with Theorem 1

$$\sum_{n=1}^{\infty}\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}\right|\leq 2\beta\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right) \tag{4.1}$$

and

$$\sum_{n=1}^{\infty}\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|b_{n+p}\right|\leq 2\beta\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right) \tag{4.2}$$

To show that h is a member of $P_p^*(\alpha,\beta,\xi,\Omega,m)$ it is enough to show that

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|a_{n+p}+b_{n+p}\right|\leq 2\beta\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right)$$

This is exactly an immediate consequence of (4.1) and (4.2).

5. Integral Operators

In this section, we prove the following.

Theorem 6. Let the function $f(z)$ defined (1.1) be in the class $P_p^*(\alpha,\beta,\xi,\Omega,m)$ and let c be real number such that $c>-p$. Then the function $F(z)$ defined by

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \tag{5.1}$$

also belongs to the class $P_p^*(\alpha,\beta,\xi,\Omega,m)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z)=z^{p}-\sum_{n=1}^{\infty}b_{n+p}z^{n+p},$$

where

$$b_{n+p}=\left(\frac{c+p}{c+p+n}\right)a_{n+p}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left|b_{n+p}\right|$$

$$=\sum_{n=1}^{\infty}\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}\left(\frac{c+p}{c+p+n}\right)a_{n+p}$$

$$\leq\sum_{n=1}^{\infty}\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}a_{n+p}\leq 2\beta\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right).$$

Since $f(z)\in P_{p}^{*}(\alpha,\beta,\xi,\Omega,m).$ Hence by Theorem 1, $F(z)\in P_{p}^{*}(\alpha,\beta,\xi,\Omega,m).$

6. Extreme points for $P_{p}^{*}(\alpha,\beta,\xi,\Omega,m)$

We shall now determine te extreme points of $P_{p}^{*}(\alpha,\beta,\xi,\Omega,m).$

Theorem 7. Let $f_{p}(z)=z^{p}$ and

$$f_{n+p}(z)=z^{p}-\frac{2\beta\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right)}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}z^{n+p},\qquad n=1,2,3,...$$

Then $f(z)\in P_p^*(\alpha,\beta,\xi,\Omega,m)$ if and only if it can be expressed in the form

$f(z)=\sum_{n=0}^{\infty}\lambda_{n+p}f_{n+p}(z)$ where $\lambda_{n+p}\geq 0$ and $\sum_{n=0}^{\infty}\lambda_{n+p}=1$.

Proof. Suppose that $f(z)=\sum_{n=0}^{\infty}\lambda_{n+p}f_{n+p}(z)=\lambda_p f_p(z)+\sum_{n=1}^{\infty}\lambda_{n+p}f_{n+p}(z)$

$$=\lambda_p z^p+\sum_{n=1}^{\infty}\lambda_{n+p}\left\{z^p-\frac{2\beta\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right)}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}z^{n+p}\right\}$$

$$=\lambda_p z^p+\sum_{n=1}^{\infty}\lambda_{n+p}z^p-\sum_{n=1}^{\infty}\lambda_{n+p}\frac{2\beta\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right)}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}z^{n+p}$$

$$=\left(\sum_{n=0}^{\infty}\lambda_{n+p}\right)z^p-\sum_{n=1}^{\infty}\lambda_{n+p}\frac{2\beta\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right)}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}z^{n+p}$$

$$=z^p-\sum_{n=1}^{\infty}\lambda_{n+p}\frac{2\beta\xi\left(\frac{p^{\Omega}p!}{(p-m)!}-\alpha\right)}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}z^{n+p}.$$

Thus

$$\sum_{n=1}^{\infty}\lambda_{n+p}\left(\frac{2\beta\xi(\frac{p^{\Omega}p!}{(p-m)!}-\alpha)}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}\right)\left(\frac{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}{2\beta\xi(\frac{p^{\Omega}p!}{(p-m)!}-\alpha)}\right)$$

$$\sum_{n=1}^{\infty}\lambda_{n+p}=\sum_{n=0}^{\infty}\lambda_{n+p}-\lambda_{p}=1-\lambda_{p}\leq 1,$$

so by Theorem 1, $f(z)\in P_p^{*}(\alpha,\beta,\xi,\Omega,m)$.

Conversely, suppose $f(z)\in P_p^{*}(\alpha,\beta,\xi,\Omega,m)$. Since

$$a_{n+p}\leq\frac{2\beta\xi(\frac{p^{\Omega}p!}{(p-m)!}-\alpha)}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}\qquad(n=1,2,3,...),$$

we may set

$$\lambda_{n+p}=\frac{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}{2\beta\xi(\frac{p^{\Omega}p!}{(p-m)!}-\alpha)}a_{n+p}$$

and

$$\lambda_{p}=1-\sum_{n=1}^{\infty}\lambda_{n+p}.$$

Then

$$f(z)=z^{p}-\sum_{n=1}^{\infty}a_{n+p}z^{n+p}=z^{p}-\sum_{n=1}^{\infty}\lambda_{n+p}\frac{2\beta\xi(\frac{p^{\Omega}p!}{(p-m)!}-\alpha)}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}}z^{n+p}$$

$$= z^p - \sum_{n=1}^{\infty} \lambda_{n+p}(z^p - f_{n+p}(z)) = z^p - \sum_{n=1}^{\infty} \lambda_{n+p} z^p + \sum_{n=1}^{\infty} \lambda_{n+p} f_{n+p}(z)$$

$$= (1 - \sum_{n=1}^{\infty} \lambda_{n+p}) z^p + \sum_{n=1}^{\infty} \lambda_{n+p} f_{n+p}(z)$$

$$= \lambda_p z^p + \sum_{n=1}^{\infty} \lambda_{n+p} f_{n+p}(z)$$

$$= \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z) = \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_{n+p} f_{n+p}(z).$$

This status is completes proof of theorem.

Corollary 2. The extreme points of $P_p^*(\alpha,\beta,\xi,\Omega,m)$ are given by $f_p(z) = z^p$ and

$$f_{n+p}(z) = z^p - \frac{2\beta\xi\left(\frac{p^{\Omega} p!}{(p-m)!} - \alpha\right)}{\frac{\left[1+(2\xi-1)\beta\right](n+p)^{\Omega}(n+p)!}{(n+p-m)!}} z^{n+p}, \quad n = 1,2,3,...$$

Remark. If we take $m = 1$ and $\Omega = 0$ in the class $P_p^*(\alpha,\beta,\xi,\Omega,m)$ then we have the results by Kulkarni *et al*. [6].

Remark. If we take $m = 1$ in the class $P_p^*(\alpha,\beta,\xi,\Omega,m)$ then we have the results by Orhan *et al.* [8].

7. References

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