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Submersıons of Semi-Invarıant Submanifolds of A Kenmotsu Manifold

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Abstract
In this paper, we investigate the submersions of semi-invariant submanifolds of a Kenmotsu manifold onto almost contact manifold. We also obtain the decomposition theorems for such submersions and derive the relation between curvatures.

Keywords: Submersions, semi-invariant submanifolds, Kenmotsu manifold

1. INTRODUCTION

Bishop and O'Neill investigated manifolds with negatif curvature [3]. They studied warped product manifolds as a generalization of Riemannian product manifolds. In 1960's and 1970's, when almost contact manifolds were studied as an odd dimensional counterpart of almost complex manifolds, the warped product was used to make examples of almost contact manifolds. In 1972, Kenmotsu introduced the properties of the warped product of the complex space with the real line [6]. This manifold is called Kenmotsu manifold.
As a generalization of invariant and anti-invariant submanifolds, CR-submanifolds were introduced by Bejancu [1]. Later, this submanifolds are studied by some authours[2,8,12].
One way to compare two manifolds is to define smooth maps from one manifold to another. One of these maps is submersion, the rank of the map is equal to the dimension of the target manifold. An isometric submersion is called a Riemannian submersion. Riemannian submersion between Riemannian submanifolds was first introduced by O' Neill [9].
In 1981, Kobayashi [7] investigated submersion of CR-submanifold of a Kaehler manifold onto almost Hermitian manifold. After, Deshmekh et al. studied properties curvature of this submersions [4]. In 1989, Papaghuic [10] studied the submersion of semi-invariant submanifolds of a Sasakian manifold. Submersion of semi-invariant submanifolds of trans-Sasakian manifold were studied by Jamali et al [11]. Moreover, there are many papers about these subject in literatüre [5,13,14].
In this paper, we study submersions of semi-invariant submanifold of Kenmotsu manifold onto almost contact manifold. We have been shown that in submersion of a semi-invariant submanifold of a Kenmotsu manifold onto an almost contact metric manifold, an almost contact metric manifold is a Kenmotsu manifold. Moreover, we investigated decomposition theorems and curvature relation.

2. PRELIMINARIES

Let M be a *(2n+1)*-dimensional differentiable manifold endowed with a (φ, ξ, η, g), where φ is (1,1)-tensor field, ξ is a vector field, η is a 1-form, and g is a Riemannian metric such that

$$\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1 \tag{1}$$

$$g(\varphi x, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all $X, Y \epsilon \Gamma(TM)$.

The contact metric manifold M is called a Kenmotsu manifold if it satisfies the condition

$$(\overline{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{3}$$

for all $\mathrm{X}, \mathrm{Y} \epsilon \Gamma(\mathrm{TM})$ where $\overline{\nabla}$ is Levi-Civita connection on M.

Let M be an n-dimensional isometrically immersed submanifold of Kenmotsu manifold $\overline{M}$ and tangent to ξ and suppose $\overline{\nabla}$ (resp. ∇) be the covariant differentiation with respect to the Levi-Civita connection on $\overline{M}$ (resp. *M*). The Gauss and Weingarten formulae for *M* are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

And

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{5}$$

for $X, \mathrm{Y} \epsilon \Gamma(\mathrm{TM})$, $N \in \Gamma(T^{\perp}M)$, where h (resp. *A*) is the second fundamental form (resp. tensor) of *M* in $\overline{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover we have

$$g(h(X, Y), N) = g(A_N X, Y). \tag{6}$$

The curvature tensor R of the submanifold M is related to the curvature tensor $\bar{R}$ of $\overline{M}$ by the following Gauss formula

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) - g\big(h(Y, Z), h(X, W)\big) + g(h(X, Z), h(Y, W)) \tag{7}$$

Definition 2.1. An m-dimensional Riemannian submanifold M of a Kenmotsu manifold $\overline{M}$ is called a semi-invariant submanifold if ξ is tangent to M and it is endowed with a pair of orthogonal differentiable distributions (D,$D^{\perp}$) which satisfies

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$
(ii) The distribution $D_X : x \longrightarrow D \subset T_X M$ is invariant under ϕ i.e. $\phi D_X \subset D_X$ for each $x \in M$
(iii) The orthogonal complementary distribution $D^{\perp} : x \longrightarrow D^{\perp} \subset T_X M$ of the distribution D on M is totally real i.e. $\phi D^{\perp} \subset T_X^{\perp} M$ where $T_X M$, $T_X^{\perp} M$ are tangent space and the normal space of *M* at *x* respectively.

The projection of TM to D and $D^{\perp}$ are denoted by *h* and *v* respectively i.e., for any $X \in TM$ we have

$$X = hX + vX + \eta(X)\xi \tag{8}$$

The normal bundle to M has the decomposition

$$T^{\perp}M = \phi D^{\perp} \oplus \mu.$$

For any $U \in T^{\perp}M$, we put

$$U = pU + qU \tag{9}$$

where $pU \in \phi D^{\perp}$, $qU \in \mu$. Making use of the above equation, we may write

$$\phi U = \phi pU + \phi qU , \quad U \in \Gamma(T^{\perp}M) , \quad \phi pU \in \Gamma(D^{\perp}) , \quad \phi qU \in \Gamma(\mu).$$

On the otherhand, let $\pi : (M^n, g_M) \rightarrow (B^b, g_B)$ be a submersion between two Riemannian manifolds. Then π said to be Riemannian submersion if

i) π has maximal rank

ii) The diferantial π_* preserves the lengths of horizontal vector.

On the other hand, $\pi^{-1}(k)$ is an $(n-b)$-dimensional submanifold of M, for each $k \in M$ The submanifolds $\pi^{-1}(k)$ are called fibers. Moreover, vector fields tangent to fibers are called vertical and vector fields orthogonal to fibers are horizantal. A vector field X on M is called basic if X is horizontal and $\pi_* X_p = X'_{\pi_{*(q)}}$ for all $q \in M$. We determine that V and H define projections $ker\pi_*$ and $(ker\pi_*)^{\perp}$, respectively.

Lemma 2.1. Let X, Y be basic vector fields on M. Then

(i) $g(X, Y) = g'(X_*, Y_*) \circ \pi$,
(ii) The component $h([X, Y]) + \eta([X, Y])\xi$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, i.e., $\pi_*(h([X, Y]) + \eta([X, Y])\xi = [X_*, Y_*]$,
(iii) $[U, X] \in D^{\perp}$ for any $U \in D^{\perp}$,
(iv) $h(\nabla_X Y) + \eta(\nabla_X Y)\xi$ is a basic vector field corresponding to $\nabla^*_X Y_*$, where ∇^* denotes the Levi-Civita connection on M'.

For basic vector fields on M, we define the operator $\widetilde{\nabla}^*$ corresponding to ∇^* by setting

$\widetilde{\nabla}^*_X Y = h(\nabla_X Y) + \eta(\nabla_X Y)\xi$ for $X, Y \in (D \oplus \{\xi\})$. By (iv) of lemma 2.1., $\widetilde{\nabla}^*_X Y$ is a basic vector field and we have

$$\pi_*(\widetilde{\nabla}^*_X Y) = \nabla^*_{X_*} Y_* . \tag{10}$$

Define the tensor field C by

$$\nabla_X Y = \widetilde{\nabla}^*_X Y + C(X, Y), \quad X, Y \in \Gamma(D \oplus \{\xi\}) , \tag{11}$$

where $C(X, Y)$ is the vertical part of $\nabla_X Y$. It is known that C is skew-symmetric and satisfies

$$C(X, Y) = \frac{1}{2}\nu[X, Y] , \quad X, Y \in \Gamma(D \oplus \{\xi\}) .$$

The curvature tensors R, R^* of the connection ∇, ∇^* on M *and* M' respectively related by

$$R(X, Y, Z, W) = R^*(X_*, Y_*, Z_*, W_*) - g\big(C(Y, Z), C(X, W)\big) + g(C(X, Z), C(Y, W)) + 2g(C(X, Y), C(Z, W)) \tag{12}$$

where $X, Y, Z, W \in \Gamma(D \oplus \{\xi\})$, $\pi_* X = X_*$, $\pi_* Y = Y_*$, $\pi_* Z = Z_*$ and $\pi_* W = W_* \in \chi(M')$.

3. SUBMERSIONS OF SEMI INVARIANT SUBMANIFOLDS OF A KENMOTSU MANIFOLD

Definition 3.1. Let M be a semi-invariant submanifold of a Kenmotsu manifold $\bar{M}$ and M' be an almost contact metric manifold with the almost contact metric structure (ϕ', ξ', η', g'). Assume that there is a submersion $\pi : M \longrightarrow M'$ such that

(i) $D^{\perp} = ker\pi_*$, where $\pi_* : TM \longrightarrow TB$ is the tangent mapping to π,
(ii) $\pi_*: D_p \oplus\{\xi\} \longrightarrow T_{\pi(p)}B$ is an isometry for each $p \in M$ which satisfies $\pi_* \circ \phi = \phi' \circ \pi_*$; $\eta = \eta' \circ \pi_*$; $\pi_*(\xi_p) = \xi'_{\pi(p)}$, where $T_{\pi(p)}B$ denotes the tangent space of B at $\pi(p)$.

Proposition 3.2. Let $\pi:(M,g_M)\to(B,g_B)$ be a submersion of semi-invariant submanifold of a Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B. Then we have

$$(\widetilde{\nabla}^*_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X\ , \quad (13)$$

$$C(X,\phi Y) = \phi\, ph(X,Y)\ , \quad (14)$$

$$\phi C(X,Y) = ph(X,\phi Y)\ , \quad (15)$$

$$\phi qh(X,Y) = qh(X,\phi Y) \quad (16)$$

for any $X,Y\in\Gamma(TM)$.

Proof. For any $X,Y\in\Gamma(TM)$ and by using Gauss formula (4), decomposition equation (9) and (11) we have

$$\bar{\nabla}_X Y = \widetilde{\nabla}^*_X Y + C(X,Y) + ph(X,Y) + qh(X,Y). \quad (17)$$

Hence,we get

$$\phi\bar{\nabla}_X Y = \phi\widetilde{\nabla}^*_X Y + \phi C(X,Y) + \phi ph(X,Y) + \phi qh(X,Y). \quad (18)$$

Putting $Y=\phi Y$ in (17), it follows

$$\bar{\nabla}_X\phi Y = \phi\widetilde{\nabla}^*_X\phi Y + C(X,\phi Y) + ph(X,\phi Y) + qh(X,\phi Y). \quad (19)$$

On the other hand, using the definition of Kenmotsu manifold we find

$$\overline{(\nabla_X\phi)}Y = \bar{\nabla}_X\phi Y - \phi\bar{\nabla}_X Y = g(\phi X,Y)\xi - \eta(Y)\phi X. \quad (20)$$

Substituting (18) and (19) in (20) we get

$$\widetilde{\nabla}^*_X\phi Y + C(X,\phi Y) + ph(X,\phi Y) + qh(X,\phi Y) - \phi\widetilde{\nabla}^*_X Y - \phi C(X,Y)$$
$$-\phi ph(X,Y) - \phi qh(X,Y) = g(\phi X,Y)\xi - \eta(Y)\phi X.$$

Comparing components of $(D\oplus\{\xi\})$, $D^\perp$, $\phi D^\perp$ and q respectively on both sides in the above equation, we get the required results.

Theorem 3.3. Let $\pi:(M,g_M)\to(B,g_B)$ be a submersion of semi-invariant submanifold of a Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B. Then B is also a Kenmotsu manifold.

Proof. For any $X,Y\in\Gamma(TM)$, using (13), we get

$$(\widetilde{\nabla}^*_X\phi)Y = g_M(\phi X,Y)\xi - \eta(Y)\phi X\ .$$

Applying π_* to the above equation and using Lemma 2.1. , (10) and definition of submersion, we derive

$$\left(\widetilde{\nabla}^*_{X_*}\phi'\right)Y_* = g_B(\phi' X_*,Y_*)\xi' - \eta'(Y_*)\phi' X_*$$

The above equation shows that B is a Kenmotsu manifold.

Proposition 3.4. Let $\pi:(M,g_M)\to(B,g_B)$ be a submersion of semi-invariant submanifold of a Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B. Then

(i) $ph(\phi X,\phi Y) + ph(\phi X,Y) = 0$,
(ii) $ph(\phi X,\phi Y) = ph(X,Y)$,
(iii) $qh(\phi X,\phi Y) = -qh(X,Y)$,

(iv) $C(\phi X, \phi Y) = C(X, Y)$
for any $X, Y \in (D \oplus \{\xi\})$.
Proof.

(i) Interchanging X and Y in (15) we have

$\phi C(Y, X) = ph(Y, \phi X) = ph(\phi X, Y)$.
Then, we get
$ph(X, \phi Y) + ph(\phi X, Y) = \phi C(X, Y) + \phi C(Y, X) = \phi C(X, Y) - \phi C(X, Y) = 0$.

(ii) Putting $X = \phi X$ in (15), we get

$ph(\phi X, \phi Y) = \phi C(\phi X, Y) = -\phi C(Y, \phi X)$.

Using (14) in the above equation, we have
$ph(\phi X, \phi Y) = -\phi C(Y, \phi X) = -\phi\big(\phi ph(Y, X)\big)$.

Then, from (1) we conclude
$ph(\phi X, \phi Y) = ph(Y, X) - \eta\big(h(X, Y)\big)\xi = ph(Y, X)$.

(iii) Putting $X = \phi X$ in (16) and using again the same equation, we have

$qh(\phi X, \phi Y) = \phi qh(\phi X, Y) = \phi qh(Y, \phi X) = \phi^2 qh(Y, X) = -qh(X, Y)$.

(iv) Putting $X = \phi X$ in (14) and then using (15) yields

$C(\phi X, \phi Y) = \phi ph(\phi X, Y) = \phi ph(Y, \phi X) = \phi^2 C(Y, X)$.

Then from (1) we have
$C(\phi X, \phi Y) = C(Y, X)$.

4. CURVATURE RELATIONS

Proposition 4.1. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of semi-invariant submanifold of a Kenmotsu manifold $\overline{M}$ onto an almost contact metric manifold B. Then the φ-bisectional curvature of $\overline{M}$ and B are related by

$$\overline{B}(X, Y) = B'(X_*, Y_*) - 2\,\|ph\,(X, Y)\|^2 - 2\|ph(X, \phi Y\|^2 - 2g\big(ph(X, X), ph(Y, Y)\big)$$
$$+ 2\|qh(X, Y)\|^2 ,$$

where $X, Y \in (D \oplus \{\xi\})$.

Proof. We know
$\bar{B}(X, Y) = \bar{R}(X, \phi X, \phi Y, Y)$.
Put $Y = \phi X, Z = \phi Y, W = Y$ in equation (7) we get
$\bar{R}(X, \phi X, \phi Y, Y) = R(X, \phi X, \phi Y, Y) - g\big(h(X, Y), h(\phi X, \phi Y)\big) + g(h(X, \phi Y), h(\phi X, Y))$

Substituting $h = ph + qh$, in the above equation, we arrive at

$$\bar{R}(X,\phi X,\phi Y,Y)$$
$$= R(X,\phi X,\phi Y,Y) - g\big(qh(X,Y),qh(X,Y)\big) + g\big(ph(X,Y),ph(X,Y)\big)$$
$$- g\big(qh(X,\phi Y),qh(X,\phi Y)\big) + g\big(\phi ph(X,Y),\phi ph(X,Y)\big)$$

Then we have

$$\bar{R}(X,\phi X,\phi Y,Y) = R(X,\phi X,\phi Y,Y) - \|ph(X,Y)\|^2 + 2\|qh(X,Y)\|^2 - \|ph(X,\phi Y)\|^2 \tag{21}$$

Now by putting $Y = \phi X$, $Z = \phi Y$, $W = Y$ in (12) it follows

$$R(X,\phi X,\phi Y,Y) = R^*(X_*,\phi' X_*,\phi' Y_*,Y_*) - g\big(C(\phi X,\phi Y),C(X,Y)\big) - g\big(C(X,\phi Y),C(Y,\phi X)\big) - 2g(C(X,\phi X),C(Y,\phi Y)). \tag{22}$$

Applying φ to equation $\phi C(X,Y) = ph(X,\phi Y)$, we get $\phi^2 C(X,Y) = \phi ph(X,\phi Y)$. This gives
$-C(X,Y) + \eta\big(C(X,Y)\big)\xi = \phi ph(X,\phi Y)$
or
$C(X,Y) = -\phi ph(X,\phi Y)$.

Using the above relation in (22), we conclude

$$R(X,\phi X,\phi Y,Y) = R^*(X_*,\phi' X_*,\phi' Y_*,Y_*) - \|ph(X,Y)\|^2 - \|ph(X,\phi Y)\|^2 - 2g\big(ph(X,X),ph(Y,Y)\big).$$

Put this value of $R(X,\phi X,\phi Y,Y)$ in (21) we obtain

$$\bar{R}(X,\phi X,\phi Y,Y) = R^*(X_*,\phi' X_*,\phi' Y_*,Y_*) - \|ph(X,Y)\|^2 - \|ph(X,\phi Y)\|^2 - 2g\big(ph(X,X),ph(Y,Y)\big) - \|ph(X,Y)\|^2 + 2\|qh(X,Y)\|^2 - \|ph(X,\phi Y)\|^2 ,$$

which implies that

$$\overline{B}(X,Y) = B'(X_*,Y_*) - 2\,\|ph\,(X,Y)\|^2 - 2\|ph(X,\phi Y\|^2 - 2g\big(ph(X,X),ph(Y,Y)\big) + 2\|qh(X,Y)\|^2.$$

Corollary 4.2. Let $\pi: M \longrightarrow M'$ be a submersion of semi-invariant submanifold of a Kenmotsu manifold $\overline{M}$ onto an almost contact metric manifold. Then the ϕ-sectional curvature of $\overline{M}$ and M' are related by

$$\overline{H}(X) = H'(X_*) - 4\|ph(X,X)\|^2 + 2\|qh(X,X)\|^2,$$

where $X \in (D \oplus \{\xi\})$.
Proof. Putting $X = Y$ in the above expression of $\bar{B}(X,Y)$,we have

$$\bar{B}(X,X) = \overline{H}(X) = H'(X_*) - 2\,\|ph\,(X,X)\|^2 - 2\|ph(X,\phi X\|^2 - 2g\big(ph(X,X),ph(X,X)\big) + 2\|qh(X,X)\|^2.$$

Then we conclude

$$\bar{B}(X,X) = H'(X_*) - 4\|ph(X,X)\|^2 - 2\|ph(X,\phi X\|^2 + 2\|qh(X,X)\|^2.$$

Putting $Y = X$ in equation (15) we have

$$ph(X,\phi X) = \phi C(X,X) = 0.$$

Thus we get
$\bar{H}(X) = H'(X_*) - 4\|ph(X,X)\|^2 + 2\|qh(X,X)\|^2$.

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