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Neighborhoods of Certain Classes of Analytic Functions Defined By Miller-Ross Function

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Makalenin Alanı: Matematik

Makale Bilgileri	**Özet**
Geliş Tarihi 26.11.2021 **Kabul Tarihi** 28.12.2021 **Anahtar Kelimeler** Analitik Fonksiyon Yıldızıl ve Konveks Fonksiyonlar Miller-Ross Fonksiyonu Komşuluklar	Bu makalede, normalize edilmiş Miller-Ross yardımıyla tanımlanan negatif katsayılı açık U birim diskinde analitik fonksiyonların yeni bir alt sınıfını tanıtacağız. Bu makalenin amacı, tanıtılan bu alt sınıfa ait Miller-Ross fonksiyonu için katsayı eşitsizliklerini, indirgeme bağıntılarını ve komşuluk özelliklerini belirlemektir.

Article Info	**Abstract**
Received 26.11.2021 **Accepted** 28.12.2021 **Keywords** Analytic Function Starlike and Convex Functions Miller-Ross Function Neighborhoods	In this paper, we introduce a new subclass of analytic functions in the open unit disk U with negative coefficients defined by normalized of the Miller-Ross function. The object of the present paper is to determine coefficient inequalities, inclusion relations and neighborhoods properties for Miller-Ross function belonging to this subclass.

1. Introduction

Let A be a class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the open disk $\mathrm{U} = \{z : |z| < 1\}$. Denote by $\mathrm{A}(n)$ the class of functions consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{1.2}$$

which are analytic in U..

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We recall that the convolution (or Hadamard product) of two functions

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \quad \text{and} \quad g(z)=z+\sum_{n=2}^{\infty}b_n z^n$$

is given by

$$(f*g)(z):=z+\sum_{n=2}^{\infty}a_n b_n z^n=:(g*f)(z), \quad (z\in \mathrm{U}).$$

Note that $f*g\in \mathrm{A}$.

Next, following the earlier investigations by (Goodman, 1957), (Ruscheweyh, 1981), (Silverman, 1995), (Altıntaş & Owa, 1996; Altıntaş et al., 2000) and (Srivastava & Bulut, 2012) (see also Aktaş & Orhan, 2015; Çağlar & Orhan, 2017; Çağlar & Orhan, 2019; Çağlar et al., 2020; Darwish et al., 2015; Deniz & Orhan, 2010; Keerthi et al., 2008; Murugusundaramoorthy & Srivastava, 2004; Orhan, 2007), we define the $(n,\delta)-$neighborhood of a function $f\in \mathrm{A}(n)$ by

$$\mathrm{N}_{n,\delta}(f)=\left\{g\in \mathrm{A}(n): g(z)=z-\sum_{n=2}^{\infty}b_n z^n \text{ and } \sum_{n=2}^{\infty}n\left|a_n-b_n\right|\le\delta\right\}. \tag{1.3}$$

For $e(z)=z,$ we have

$$\mathrm{N}_{n,\delta}(e)=\left\{g\in \mathrm{A}(n): g(z)=z-\sum_{n=2}^{\infty}b_n z^n \text{ and } \sum_{n=2}^{\infty}n\left|b_n\right|\le\delta\right\}. \tag{1.4}$$

A function $f\in \mathrm{A}(n)$ is $\alpha-$starlike of complex order $\gamma,$ denoted by $f\in \mathrm{S}_n^*(\alpha,\gamma)$ if it satisfies the following condition

$$\Re\left\{1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>\alpha, \quad \left(\gamma\in\mathbb{C}\setminus\{0\},\ 0\le\alpha<1,\ z\in \mathrm{U}\right)$$

and a function $f\in \mathrm{A}(n)$ is $\alpha-$convex of complex order $\gamma,$ denoted by $f\in \mathrm{C}_n^*(\alpha,\gamma)$ if it satisfies the following condition

$$\Re\left\{1+\frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\}>\alpha, \quad \left(\gamma\in\mathbb{C}\setminus\{0\},\ 0\le\alpha<1,\ z\in \mathrm{U}\right).$$

The Miller-Ross (Miller & Ross, 1993) function $E_{\nu,c}(z),$ defined by

$$E_{\nu,c}(z)=z^{\nu}\sum_{n=0}^{\infty}\frac{c^n}{\Gamma(\nu+n+1)}z^{n(1+\alpha)}, \quad (\nu>-1,\ c\ge 0,\ z\in \mathrm{U}). \tag{1.5}$$

The Miller-Ross function $E_{\nu,c}(z)$ does not belong to the class A. Therefore, we consider the following normalization for the function $E_{\nu,c}(z)$:

$$\mathrm{E}_{\nu,c}(z)=\Gamma(1+\nu)z^{1-\nu}E_{\nu,c}(z)=\sum_{n=0}^{\infty}\frac{c^n\Gamma(1+\nu)}{\Gamma(\nu+n+1)}z^{n+1}, \quad (z\in \mathrm{U}). \tag{1.6}$$

In terms of Hadamard product and $\mathrm{E}_{\nu,c}(z)$ given by (1.6), a new operator $\varepsilon_{\nu,c}:\mathrm{A}\to\mathrm{A}$ can be defined as follows:

$$\varepsilon_{\nu,c} f(z) = (\varepsilon_{\nu,c} * f)(z) = z + \sum_{n=1}^{\infty} \frac{c^n \Gamma(1+\nu) a_{n+1}}{\Gamma(\nu+n+1)} z^{n+1}, \quad (z \in \mathrm{U}). \tag{1.7}$$

If $f \in \mathrm{A}(n)$ is given by (1.2), then we have

$$\varepsilon_{\nu,c} f(z) = z - \sum_{n=1}^{\infty} \frac{c^n \Gamma(1+\nu) a_{n+1}}{\Gamma(\nu+n+1)} z^{n+1}, \quad (z \in \mathrm{U}). \tag{1.8}$$

Finally, by using the differential operator defined by (1.8), we investigate the subclasses $\mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$ and $\mathrm{R}_{\nu,c}^{n}(\alpha,\gamma;\vartheta)$ of $\mathrm{A}(n)$ consisting of functions f as the followings:

However, throughout this paper, we restrict our attention to the case real-valued ν, c with $\nu > -1$ and $c \geq 0$.

Definition 1.1: The subclass $\mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$ of $\mathrm{A}(n)$ is defined as the class of functions f such that

$$\left| \frac{1}{\gamma} \left(\frac{z\left[\varepsilon_{\nu,c} f(z)\right]'}{\varepsilon_{\nu,c} f(z)} - 1 \right) \right| < \alpha, \quad (z \in \mathrm{U}), \tag{1.9}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$ and $0 \leq \alpha < 1$.

Definition 1.2: Let $\mathrm{R}_{\nu,c}^{n}(\alpha,\gamma;\vartheta)$ denote the subclass of $\mathrm{A}(n)$ consisting of f which satisfy the inequality

$$\left| \frac{1}{\gamma} \left[(1-\vartheta)\right] \frac{\varepsilon_{\nu,c} f(z)}{z} + \vartheta\left(\varepsilon_{\nu,c} f(z)\right)' - 1 \right| < \alpha, \tag{1.10}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$, $0 \leq \alpha < 1$ and $0 \leq \vartheta \leq 1$.

In this paper, we obtain the coefficient inequalities, inclusion relations and neighborhood properties of the subclasses $\mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$ and $\mathrm{R}_{\nu,c}^{n}(\alpha,\gamma;\vartheta)$.

2. Coefficient Inequalities For $\mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$ and $\mathrm{R}_{\nu,c}^{n}(\alpha,\gamma;\vartheta)$.

Theorem 2.1: Let $f \in \mathrm{A}(n)$. Then $f \in \mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{c^{n-1}\Gamma(1+\nu)}{\Gamma(\nu+n)} \left[n-1+\alpha|\gamma|\right] a_n \leq \alpha|\gamma| \quad (z \in \mathrm{U}) \tag{2.1}$$

for $\gamma \in \mathbb{C} \setminus \{0\}$ and $0 \leq \alpha < 1$.

Proof. Let $f \in \mathrm{A}(n)$. Then, by (1.9) we can write

$$\Re\left\{ \frac{z\left[\varepsilon_{\nu,c} f(z)\right]'}{\varepsilon_{\nu,c} f(z)} - 1 \right\} > -\alpha|\gamma|, \quad (z \in \mathrm{U}). \tag{2.2}$$

Using (1.2) and (1.8), we have,

$$\Re\left\{\frac{-\sum_{n=2}^{\infty}\frac{c^{n-1}\Gamma(1+\nu)}{\Gamma(\nu+n)}[n-1]a_n z^n}{z-\sum_{n=2}^{\infty}\frac{c^{n-1}\Gamma(1+\nu)}{\Gamma(\nu+n)}a_n z^n}\right\}>-\alpha|\gamma|, \quad (z\in \mathrm{U}). \tag{2.3}$$

Since (2.3) is true for all $z\in \mathrm{U}$, choose values of z on the real axis. Letting $z\to 1$, through the real values, the inequality (2.3) yields the desired inequality

$$\sum_{n=2}^{\infty}\frac{c^{n-1}\Gamma(1+\nu)}{\Gamma(\nu+n)}\left[n-1+\alpha|\gamma|\right]a_n \le \alpha|\gamma|.$$

Conversely, supposed that the inequality (2.1) holds true and $|z|=1$, then we obtain

$$\begin{aligned}\left|\frac{z\left[\varepsilon_{\nu,c}f(z)\right]'}{\varepsilon_{\nu,c}f(z)}-1\right| &\le \left|\frac{\sum_{n=2}^{\infty}\frac{c^{n-1}\Gamma(1+\nu)}{\Gamma(\nu+n)}[n-1]a_n z^n}{z-\sum_{n=2}^{\infty}\frac{c^{n-1}\Gamma(1+\nu)}{\Gamma(\nu+n)}a_n z^n}\right| \\ &\le \frac{\sum_{n=2}^{\infty}\frac{c^{n-1}\Gamma(1+\nu)}{\Gamma(\nu+n)}[n-1]a_n}{1-\sum_{n=2}^{\infty}\frac{c^{n-1}\Gamma(1+\nu)}{\Gamma(\nu+n)}a_n} \\ &\le \alpha|\gamma|.\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z)\in \mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$, which establishes the required result.

Theorem 2.2: Let $f\in \mathrm{A}(n)$. Then $f\in \mathrm{R}_{\nu,c}^{n}(\alpha,\gamma;\vartheta)$ if and only if

$$\sum_{n=2}^{\infty}\frac{c^{n-1}\Gamma(1+\nu)}{\Gamma(\nu+n)}\left[1+\vartheta(n-1)\right]a_n \le \alpha|\gamma| \tag{2.4}$$

for $\gamma\in \mathbb{C}\setminus\{0\}$, $0\le\alpha<1$ and $0\le\vartheta\le 1$.

Proof. We omit the proofs since it is similar to Theorem 2.1.

3. Inclusion Relations Involving $\mathrm{N}_{n,\delta}(e)$ of $\mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$ and $\mathrm{R}_{\nu,c}^{n}(\alpha,\gamma;\vartheta)$

Theorem 3.1: If

$$\delta=\frac{2\alpha|\gamma|(1+\nu)}{c(1+\alpha|\gamma|)}, \quad (|\gamma|<1), \tag{3.1}$$

then $\mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)\subset \mathrm{N}_{n,\delta}(e)$.

Proof. Let $f(z)\in \mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$. By Theorem 2.1, we have

$$\frac{c}{(1+\nu)}(1+\alpha|\gamma|)\sum_{n=2}^{\infty}a_n \leq \alpha|\gamma|,$$

which implies

$$\sum_{n=2}^{\infty}a_n \leq \frac{\alpha|\gamma|}{\frac{c}{(1+\nu)}(1+\alpha|\gamma|)}. \tag{3.2}$$

Using (2.1) and (3.2), we get

$$\begin{aligned}\frac{c}{(1+\nu)}\sum_{n=2}^{\infty}na_n &\leq \alpha|\gamma|+\frac{c}{(1+\nu)}(1-\alpha|\gamma|)\sum_{n=2}^{\infty}a_n \\ &\leq \frac{2\alpha|\gamma|}{(1+\alpha|\gamma|)}=\delta.\end{aligned}$$

That is,

$$\sum_{n=2}^{\infty}na_n \leq \frac{2\alpha|\gamma|}{\frac{c}{(1+\nu)}(1+\alpha|\gamma|)}=\delta.$$

Thus, by the definition given by (1.4), $f(z)\in \mathrm{N}_{n,\delta}(e)$, which completes the proof.

Theorem 3.2: If

$$\delta=\frac{2\alpha|\gamma|(1+\nu)}{c(1+\vartheta)}, \quad (|\gamma|<1), \tag{3.3}$$

then $\mathrm{R}_{\nu,c}^{n}(\alpha,\gamma;\vartheta)\subset \mathrm{N}_{n,\delta}(e)$.

Proof. For $f(z)\in \mathrm{R}_{\nu,c}^{n}(\alpha,\gamma;\vartheta)$ and making use of the condition (2.4), we obtain

$$\frac{c}{(1+\nu)}(1+\vartheta)\sum_{n=2}^{\infty}a_n \leq \alpha|\gamma|$$

so that

$$\sum_{n=2}^{\infty}a_n \leq \frac{\alpha|\gamma|}{\frac{c}{(1+\nu)}(1+\vartheta)}. \tag{3.4}$$

Thus, using (2.4) along with (3.4), we also get

$$\begin{aligned}\vartheta\frac{c}{(1+\nu)}\sum_{n=2}^{\infty}na_n &\leq \alpha|\gamma|+(\vartheta-1)\frac{c}{(1+\nu)}\sum_{n=2}^{\infty}a_n \\ &\leq \alpha|\gamma|+\frac{c(\vartheta-1)}{(1+\nu)}\frac{\alpha|\gamma|(1+\nu)}{c(1+\vartheta)} \\ &\leq \frac{2\vartheta\alpha|\gamma|}{(1+\vartheta)}=\delta.\end{aligned}$$

Hence,

$$\sum_{n=2}^{\infty} n a_n \leq \frac{2\alpha|\gamma|}{\frac{c}{(1+\nu)}(1+\vartheta)} = \delta$$

which in view of (1.4), completes the proof of theorem.

4. Neighborhood Properties For The Classes $\mathrm{M}_{\nu,c}^{n}(\alpha,\gamma,\eta)$ and $\mathrm{R}_{\nu,c}^{n}(\alpha,\gamma,\eta;\vartheta)$

Definition 4.1: For $0 \leq \eta < 1$ and $z \in \mathrm{U}$, A function $f(z) \in \mathrm{A}(n)$ is said to be in the class $\mathrm{M}_{\nu,c}^{n}(\alpha,\gamma,\eta)$ if there exists a function $g(z) \in \mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \eta. \tag{4.1}$$

Analogously, for $0 \leq \eta < 1$ and $z \in \mathrm{U}$, A function $f(z) \in \mathrm{A}(n)$ is said to be in the class $\mathrm{R}_{\nu,c}^{n}(\alpha,\gamma,\eta;\vartheta)$ if there exists a function $g(z) \in \mathrm{R}_{\nu,c}^{n}(\alpha,\gamma;\vartheta)$ such that the inequality (4.1) holds true.

Theorem 4.1: If $g(z) \in \mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$ and

$$\eta = 1 - \frac{\delta c\left(1+\alpha|\gamma|\right)}{2\left[c\left(1+\alpha|\gamma|\right) - \alpha|\gamma|(1+\nu)\right]} \tag{4.2}$$

then $\mathrm{N}_{n,\delta}(g) \subset \mathrm{M}_{\nu,c}^{n}(\alpha,\gamma,\eta)$.

Proof. Let $f(z) \in \mathrm{N}_{n,\delta}(g)$. Then,

$$\sum_{n=2}^{\infty} n\left|a_n - b_n\right| \leq \delta, \tag{4.3}$$

which yields the coefficient inequality,

$$\sum_{n=2}^{\infty} \left|a_n - b_n\right| \leq \frac{\delta}{2}, \quad (n \in \square).$$

Since $g(z) \in \mathrm{M}_{\nu,c}^{n}(\alpha,\gamma)$ by (3.2), we have

$$\sum_{n=2}^{\infty} b_n \leq \frac{\alpha|\gamma|}{\frac{c}{(1+\nu)}\left(1+\alpha|\gamma|\right)}, \tag{4.4}$$

and so

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}b_n} \\ &\leq \frac{\delta}{2}\frac{\frac{c}{(1+\nu)}(1+\alpha|\gamma|)}{\frac{c}{(1+\nu)}(1+\alpha|\gamma|)-\alpha|\gamma|} \\ &= 1-\eta.\end{aligned}$$

Thus, by the definition, $f(z)\in \mathrm{M}_{\nu,c}^{n}(\alpha,\gamma,\eta)$ for η given by (4.2), which establishes the desired result.

Theorem 4.2: If $g(z)\in \mathrm{R}_{\nu,c}^{n}(\alpha,\gamma;\vartheta)$ and

$$\eta=1-\frac{\delta c(1+\vartheta)}{2\left[c(1+\vartheta)-\kappa|\gamma|(1+\nu)\right]}, \tag{4.5}$$

then $\mathrm{N}_{n,\delta}(g)\subset \mathrm{R}_{\nu,c}^{n}(\alpha,\gamma,\eta;\vartheta)$.

Proof. We omit the proofs since it is similar to Theorem 4.1.

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