Actions and semi-direct products in categories of groups with action

Tamar Datuashvili*1, Tunçar Şahan2

1A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvii Str., Tbilisi 0177, Georgia
2Department of Mathematics, Aksaray University, Aksaray, Turkey

Abstract

Derived actions in the category of groups with action on itself Gr* are defined and described. This category plays a crucial role in the solution of two problems of Loday stated in the literature. A full subcategory of reduced groups with action rGr* of Gr* is introduced, which is not a category of interest but has some properties, which can be applied in the investigation of action representability in this category; these properties are similar to those, which were used in the construction of universal strict general actors in the category of interest. Semi-direct product constructions are given in Gr* and rGr* and it is proved that an action is a derived action in Gr* (resp. rGr*) if and only if the corresponding semi-direct product is an object of Gr* (resp. rGr*). The results obtained in this paper will be applied in the forthcoming paper on the representability of actions in the category rGr*.

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1. Introduction

Action theories were developed in many algebraic categories like the categories of groups, associative algebras, (pre)crossed modules, non-associative algebras, in particular Lie, Leibniz, alternative algebras and others and, in more general settings of semi-abelian category [1] and category of interest [12, 13]. There were two different approaches to the definition of action, which turned out to be equivalent. In this paper we give a new example of a category, where action theory can be developed. It is a category of groups with action on itself introduced in [5–7], where it played a main role in the solution of two problems of Loday stated in [10, 11]. This category is neither a category of interest, nor a modified category of interest [2]. It is a category of groups with operations, but doesn’t satisfy all conditions stated in [14]. The category Gr* is a category of Ω-groups in the sense of Kurosh [9]. Actions are defined in Gr* as derived actions from split extensions in this category as it is in the category of interest or in any semi-abelian category. We describe derived action conditions in this category and construct a semi-direct product

*Corresponding Author.
Email addresses: tamar.datu@gmail.com (T. Datuashvili), tuncarsahan@gmail.com (T. Şahan)
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B \ltimes A$, where $A, B \in \mathbf{Gr^\bullet}$ and $B$ has a derived action on $A$. We prove that an action of $B$ on $A$ is a derived action if and only if $B \ltimes A \in \mathbf{Gr^\bullet}$ (Theorem 3.2). Then we define a full subcategory $\mathbf{rGr^\bullet}$ in $\mathbf{Gr^\bullet}$, give examples of its objects including a construction of free objects and describe derived actions in $\mathbf{rGr^\bullet}$. Our interest is to investigate the existence of a universal acting object on an object $A \in \mathbf{Gr^\bullet}$ applying the results obtained in [3,4] for categories of interest. Since the category $\mathbf{Gr^\bullet}$ is far from being category of interest we found its subcategory $\mathbf{rGr^\bullet}$, which is not a category of interest, but has interesting properties which are close to those ones used in the construction of a universal strict general actor for any object of a category of interest in [3,4]. We prove necessary and sufficient condition for the action of $B$ on $A$, $A, B \in \mathbf{rGr^\bullet}$, to be a derived action in terms of the semi-direct product $B \ltimes A$, like we have in $\mathbf{Gr^\bullet}$ (Theorem 4.4). Applying the results of this paper, in [8], we prove that under certain conditions on the object $A \in \mathbf{rGr^\bullet}$, it has representable actions in the sense of [1], i.e. a universal acting object, which represents all actions on $A$.

2. Preliminary definitions and results

Let $G$ be a group which acts on itself from the right side, i.e. we have a map $\varepsilon: G \times G \to G$ with

$$\varepsilon(g, g' + g'') = \varepsilon(\varepsilon(g, g'), g''),$$
$$\varepsilon(g, 0) = g$$
$$\varepsilon(g' + g'', g) = \varepsilon(g' + g) + \varepsilon(g'' + g)$$

for $g, g', g'' \in G$. Denote $\varepsilon(g, h) = g^h$, for $g, h \in G$.

We denote the group operation additively, nevertheless the group is not commutative in general. From the third condition on $\varepsilon$ it follows that $0^h = 0$, for any $h \in G$.

If $(G', \varepsilon')$ is another group with action then a homomorphism $(G, \varepsilon) \to (G', \varepsilon')$ is a group homomorphism $\varphi: G \to G'$, for which the diagram

$$\begin{array}{ccc}
G \times G & \overset{\varepsilon}{\longrightarrow} & G \\
\varphi \times \varphi \downarrow & & \downarrow \varphi \\
G' \times G' & \overset{\varepsilon'}{\longrightarrow} & G'
\end{array}$$

commutes. In other words, we have

$$\varphi \left( g^h \right) = \varphi(g)^{\varphi(h)} \quad (2.1)$$

for all $g, h \in G$.

Note that action defined above is a split derived action in the sense of [12,13].

According to Kurosh [9] an $\Omega$-group is a group with a system of $n$-ary algebraic operations $\Omega_{n\geq0}$, which satisfy the condition

$$000 \cdots 0_\omega = 0, \quad (2.2)$$

where 0 is the identity element of $G$, and 0 on the left side occurs $n$ times if $\omega$ is an $n$-ary operation. In special cases $\Omega$-groups give groups, rings, associative and non-associative algebras like Lie and Leibniz algebras etc. and groups with action on itself as well. In the latter case $\Omega$ consists of one binary operation which is an action or $\Omega$ consists of only unary operations, which are elements of $G$, and this operation is an action again. In both cases condition (2.2) is satisfied. Denote the category of groups with action on itself by $\mathbf{Gr^\bullet}$; here the action is considered as a binary operation and morphisms between the objects in $\mathbf{Gr^\bullet}$ are group homomorphisms satisfying condition (2.1).
Example 2.1. [5] Every group with trivial action on itself or with an action by conjugation is an object of $\text{Gr}^\ast$. There are two pairs of adjoint functors between the category of groups and the category $\text{Gr}^\ast$ [5].

Example 2.2. [6] For any set $X$ there exists a free group with action $F(X)$ with the basis $X$ in $\text{Gr}^\ast$; one can see the construction in [6].

Example 2.3. Let $\mathbb{Z}^\ast$ be an abelian group of integers which acts on itself in the following way:

$$x^y = (-1)^y x$$

for any $x, y \in \mathbb{Z}$. It is easy to check that $\mathbb{Z}^\ast \in \text{Gr}^\ast$.

Let $G \in \text{Gr}^\ast$.

Definition 2.4. [5] A non-empty subset $A$ of $G$ is called an ideal of $G$ if it satisfies the following conditions

1) $A$ is a normal subgroup of $G$ as a group;
2) $a^g \in A$, for any $a \in A$ and $g \in G$;
3) $-g + g^\ast \in A$, for any $a \in A$ and $g \in G$.

Note that the condition 3 in this definition is equivalent to the condition, that $g^a - g \in A$, since $(-g)^a = -g^a$, for any $a \in A$ and $g \in G$. This definition is equivalent to the definition of an ideal given in [9] for $\Omega$-groups in the case where $\Omega$ consists of one binary operation of action, one can see the proof in [5].

3. Actions and semi-direct products in $\text{Gr}^\ast$

Let $A, B \in \text{Gr}^\ast$. An action of $B$ on $A$ by definition is a triple of mappings $\beta = (\beta_+, \beta_-, \beta_\circ): B \times A \to A$, where $\ast$ is a binary operation of action, $\ast^\circ$ is its dual operation in $\text{Gr}^\ast$, i.e. $\beta_+(b, a) = b \cdot a$, $\beta_-(b, a) = a \ast b = a^b$ and $\beta_\circ(b, a) = a^\circ b = b^a$.

In the category of interest or category of groups with operations there is a condition $0 \ast g = g \ast 0 = 0$, for any binary operation $\ast \in \Omega \setminus \{+\}$, any object $G$ in this category and any element $g \in G$. In the category $\text{Gr}^\ast$ we have $0^g = 0$, for any $G \in \text{Gr}^\ast$ and any $g \in G$, but $g^0 \neq 0$ in general. Therefore we modify the definition of derived action due to split extensions [12–14], known for the category of groups with operations or category of interest, for the category $\text{Gr}^\ast$. Note that the definition of derived action from the split extension agrees with the definition of action in a semi-abelian category [1].

Let $A, B \in \text{Gr}^\ast$. An extension of $B$ by $A$ is a sequence

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0 \quad (3.1)$$

in which $p$ is surjective and $i$ is the kernel of $p$. We say that an extension is split if there is a morphism $j: B \to E$, such that $pj = 1_B$. We will identify $i(a)$ with $a$.

A split extension induces a triple of actions of $B$ on $A$ corresponding to the operation of addition, action and its dual operation in $\text{Gr}^\ast$. From the split extension (3.1) for any $b \in B$ and $a \in A$ we define

$$b \cdot a = j(b) + a - j(b) \quad (3.2)$$

$$b^a = j(b)^a - j(b) \quad (3.3)$$

$$a^b = a^i(b) \quad (3.4)$$

Actions defined by (3.2)-(3.4) will be called derived actions of $B$ on $A$ as it is in the case of groups with operations or category of interest. Note that (3.3) differs from what we have in the noted known cases, since as we have mentioned above $b^0 \neq 0$ in $B$.

Proposition 3.1. Let $A, B, \in \text{Gr}^\ast$. Derived actions of $B$ on $A$ satisfy the following conditions:
(a) well-known group action conditions for the dot left action:

\[
\begin{align*}
    b \cdot (a_1 + a_2) &= b \cdot a_1 + b \cdot a_2 \\
    (b_1 + b_2) \cdot a &= b_1 \cdot (b_2 \cdot a) \\
    0 \cdot a &= a
\end{align*}
\]

where \( a, a_1, a_2 \in A \) and \( b, b_1, b_2 \in B \).

(b) \( 0_A^a = 0_A, 0_B^a = 0_A, 0_B^a = 0_A, a^{0_B} = a \) where \( 0_A \) and \( 0_B \) denote the zero elements of \( A \) and \( B \) respectively. For any \( a, a' \in A \) and \( b, b' \in B \),

\[
\begin{align*}
    (1_A) \quad (a + a')^b &= a^b + (a')^b, \\
    (2_A) \quad (b + b')^a &= b^a + b \cdot ((b')^a),
\end{align*}
\]

(3.2) follows from the action properties (3.4) and the definition of the derived action corresponding to the action operation and its dual (3.3) and (3.4).

(1A) Let \( a, a' \in A \) and \( b \in B \); then

\[
\begin{align*}
    (a + a')^b &= (a + a')^{j(b)} \\
    &= a^{j(b)} + (a')^{j(b)} \\
    &= a^b + (a')^b
\end{align*}
\]

(2A) Let \( a \in A \) and \( b, b' \in B \); then

\[
\begin{align*}
    (b + b')^a &= (j(b + b'))^a - j(b + b') \\
    &= j(b)^a + j(b')^a - j(b') - j(b) \\
    &= j(b)^a - j(b) + j(b) + j(b')^a - j(b') - j(b) \\
    &= b^a + b \cdot ((b')^a)
\end{align*}
\]

(3A) Let \( a, a' \in A \) and \( b \in B \); then

\[
\begin{align*}
    (b \cdot a)^{a'} + b^{a'} &= (j(b) + a - j(b))^{a'} + j(b)^{a'} - j(b) \\
    &= j(b)^{a'} + a^{a'} - j(b)^{a'} + j(b)^{a'} - j(b) \\
    &= j(b)^{a'} - j(b) + j(b) + a^{a'} - j(b) \\
    &= b^{a'} + b \cdot (a^{a'})
\end{align*}
\]

Note that as it will be shown in the proof of Theorem 3.2, all properties noted in (b) except \( a^{0_B} = a \) follow from (1A), (2A) and (1B). Nevertheless we preferred for explicitness to state in the theorem these properties separately.
(4A) Let \( a \in A \) and \( b, b' \in B \); then
\[
(b \cdot a)' = (j(b) + a - j(b))b'
\]
\[
= j(b)j(b') + a^j(b') - j(b)j(b')
\]
\[
= j\left(b'^a\right) + a^j(b') - j\left(b'^a\right)
\]
\[
= b'^a \cdot a^b
\]

(1B) Let \( a, a' \in A \) and \( b \in B \); then
\[
b'^{(a+a')} = j(b)^{(a+a')} - j(b)
\]
\[
= (j(b)^a)^{a'} - j(b)^a + j(b)^a' - j(b)
\]
\[
= (j(b)^a - j(b))^a' + j(b)^a' - j(b)
\]
\[
= (b^a)^{a'} + b^a
\]

(2B) Let \( a \in A \) and \( b, b' \in B \); then
\[
(a^b + b') = a^{j(b) + j(b')}
\]
\[
= (a^b)^{b'}
\]

(3B) Let \( a, a' \in A \) and \( b \in B \); then
\[
\left(a^{(b-a')}\right)^b = \left(a^{j(b) + a' - j(b)}\right)^b
\]
\[
= a^{j(b) + a'}
\]
\[
= \left(a^{j(b)}\right)^{a'}
\]
\[
= \left(a^b\right)^{a'}
\]

(4B) Let \( a, a' \in A \) and \( b, b' \in B \); then
\[
(b^{(b-a')})^b = \left(j(b)^{j(b') + a - j(b')}ight) - j(b)
\]
\[
= \left(\left(j(b)^{j(b')}\right)^a - j(b)^{j(b')}\right) - j(b)
\]
\[
= \left(j\left(b'ight)^a\right) - j\left(b'ight)
\]
\[
= \left(b'^a\right)^a - j\left(b'^a\right)
\]

Given a triple of actions of \( B \) on \( A \) in \( \text{Gr}^* \), we can define operations on the product \( B \times A \) in the following way:
\[
(b, a) + (b', a') = (b + b', a + b \cdot a')
\]
\[
(b, a)(b', a') = (b', a'\cdot b + (b'^a)b')
\]
for any \((b, a), (b', a') \in B \times A\). This kind of universal algebra will be called semi-direct product and denoted by \( B \times A \).

**Theorem 3.2.** Let \( A, B \in \text{Gr}^* \), if \( \beta = (\beta_+, \beta_*, \beta_v) \) is a triple of actions of \( B \) on \( A \), then the following conditions are equivalent:

1. \( \beta \) is a triple of derived actions of \( B \) on \( A \).
We have

(2) $\beta_+$ satisfies group action conditions, $\beta$ satisfies conditions $(1_A)- (4_A), (1_B)- (4_B)$ and the condition $a^0_1 = a$, for any $a \in A$.

(3) The semi-direct product $B \ltimes A$ is an object in $\text{Gr}^\ast$.

**Proof.** (1)$\Rightarrow$(2): by Proposition 3.1.

(2)$\Rightarrow$(3): First of all we will show that from $(1_A), (2_A)$ and $(1_B)$ follow the conditions of (b) except the one $a^0_{0B} = a$. From $(1_A)$ we have

$$0^b_A = 0_A + 0^b_A = 0_A^b + 0^b_A$$

and then $0^b_A = 0_A$.

From $(2_A)$ we have

$$0^a_B = 0_B + 0^a_B = 0^a_B + 0_B \cdot (0^b_B) = 0^a_B + 0^a_B$$

and then $0^a_B = 0_A$. Note that we will not use this property in the proof of $(2)\Rightarrow(3)$.

From $(1_B)$ we have

$$0^{b^0_A} = 0^{a^0_A + 0_A} = (0^a_A + 0_A)$$

since $a^0_A = a$ for any $a \in A$, we obtain

$$0^{b^0_A} = 0^{b^0_A} + 0^{b^0_A}$$

and then $0^{b^0_A} = 0_A$.

Now we shall prove that the semi-direct product $B \ltimes A \in \text{Gr}^\ast$. Obviously, $B \ltimes A$ is a group as it is in the case of groups. We have to show the following equalities for any $(b, a), (b', a'), (b'', a'') \in B \ltimes A$:

(a) $(b, a)(b', a') + (b'', a'') = (b, a)(b', a') + (b'', a'')$

(b) $(b, a) + (b', a') = (b, a) + (b', a')$

(c) $(b, a)(0_B, 0_A) = (b, a)$.

First we prove the equality in (a).

(a) We have

$$(b, a)(b', a') + (b'', a'') = (b, a)(b + b', a' + b'')$$

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(a) $(b, a)(b', a') + (b'', a'') = (b, a)(b', a') + (b'', a'')$

(b) $(b, a) + (b', a') = (b, a) + (b', a')$

(c) $(b, a)(0_B, 0_A) = (b, a)$.

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(a) $(b, a)(b', a') + (b'', a'') = (b, a)(b', a') + (b'', a'')$

(b) $(b, a) + (b', a') = (b, a) + (b', a')$

(c) $(b, a)(0_B, 0_A) = (b, a)$.

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(a) $(b, a)(b', a') + (b'', a'') = (b, a)(b', a') + (b'', a'')$

(b) $(b, a) + (b', a') = (b, a) + (b', a')$

(c) $(b, a)(0_B, 0_A) = (b, a)$.

First we prove the equality in (a).

(a) We have

$$(b, a)(b', a') + (b'', a'') = (b, a)(b + b', a' + b'')$$

Note that we will not use this property in the proof of $(2)\Rightarrow(3)$.
On the other hand
\[
(b, a)^{(b', a')}^{(b'', a'')} = \left( b^{b'}, (a^{a'})^{b''} + (b^{a'})^{b''} \right)^{(b'', a'')}
\]
\[
= \left( b^{b'}, \left( \left( a^{a'} \right)^{b''} + \left( b^{a'} \right)^{b''} \right) \right)^{b''} + \left( b^{b'} \right)^{b''}
\]
\[
= \left( b^{b'}, \left( \left( a^{a'} \right)^{b''} \right) + \left( \left( a^{a'} \right)^{b''} \right) \right)^{b''} + \left( b^{b'} \right)^{b''}
\]
\[
= \left( b^{b'} + b^{b' a''}, \left( a^{a''} \right)^{b''} \right) + \left( b^{b'} \right)^{b''} + \left( b^{b'} a^{a''} \right)^{b''}
\]
\[
= \left( b^{b'} + b^{b' a''}, \left( a^{a''} \right)^{b''} \right) + \left( b^{b'} \right)^{b''} + \left( b^{b'} a^{a''} \right)^{b''}
\]

From which we conclude that condition (a) holds in $B \ltimes A$.

Now we check condition (b).

\[
((b, a) + (b', a'))^{(b'', a'')} = (b + b', a + b \cdot a')^{(b'', a'')}
\]
\[
= \left( b + b', (a + b' a'')^{b''} + (b + b' a'')^{b''} \right)
\]
\[
= \left( b^{b''} + b^{b' a''}, \left( a^{a''} \right)^{b''} \right) + \left( b^{b'} \right)^{b''} + \left( b^{b'} a^{a''} \right)^{b''}
\]
\[
= \left( b^{b''} + b^{b' a''}, \left( a^{a''} \right)^{b''} \right) + \left( b^{b'} \right)^{b''} + \left( b^{b'} a^{a''} \right)^{b''}
\]

Here we apply condition (3A). We have the following equalities.

\[
(b, a)^{(b'', a'')} + (b', a')^{(b'', a'')} = \left( b^{b''}, \left( a^{a''} \right)^{b''} + \left( b^{a''} \right)^{b''} \right) + \left( b^{b''}, \left( a^{a''} \right)^{b''} + \left( b^{a''} \right)^{b''} \right)
\]
\[
= \left( b^{b''} + b^{b'} a'' \right) + \left( b^{b''} \right)^{b''} + \left( b^{b'} \right)^{b''} + \left( b^{b'} a^{a''} \right)^{b''}
\]
\[
= \left( b^{b''} + b^{b'} a'' \right) + \left( b^{b''} \right)^{b''} + \left( b^{b'} \right)^{b''} + \left( b^{b'} a^{a''} \right)^{b''}
\]

Applying condition (4A) we obtain
\[
b \cdot \left( a^{a''} \right)^{b''} = b^{b''} \cdot \left( a^{a''} \right)^{b''}
\]
and
\[
b \cdot \left( b^{a''} \right)^{b''} = b^{b''} \cdot \left( b^{a''} \right)^{b''}
\]
which proves that we have condition (b).

Finally, we check condition (c).

\[
\text{(c) Here, if we apply the equalities}
\]
\[
\left( a^{0_A} \right)^{0_B} = a^{0_B} = a
\]
\[
\text{and}
\]
\[
\left( b^{0_A} \right)^{0_B} = 0_A^{0_B} = 0_A,
\]

then we get
\[
(b, a)^{0_B, 0_A} = \left( b^{0_B}, \left( a^{0_A} \right)^{0_B} + \left( b^{0_A} \right)^{0_B} \right) = (b, a).
\]

$(3) \Rightarrow (1)$: Suppose $B \ltimes A \in \text{Gr}^*$, then we have a split extension
\[
0 \longrightarrow A \overset{i}{\longrightarrow} B \ltimes A \overset{j}{\longrightarrow} B \longrightarrow 0 (3.7)
\]
where \( p(b, a) = b, i(a) = (0, a) \) and \( j(b) = (b, 0) \). Define derived actions from this extension in a usual way.

\[
\begin{align*}
    b \cdot a &= j(b) + a - j(b) \\
    &= (b, 0) + (0, a) - (b, 0) \\
    &= (b, b \cdot a),
\end{align*}
\]

therefore the derived action corresponding to the addition operation coincides with the given action.

Action corresponding to the action operation, denoted by \( \ast \), is defined by

\[
\begin{align*}
    a \ast b &= (0_B, a)(b, 0) \\
    &= (0_B b, (a^0_A) b + (0_B^0_A b)) \\
    &= (0, a b).
\end{align*}
\]

As we see this action also coincides with the given action.

For the dual to \( \ast \) operation, i.e. dual action we have

\[
\begin{align*}
    a \ast b &= (b, 0_A)(0_B, a) \\
    &= (b, b^a) - (b, 0_A) \\
    &= (b - b, b^a + b \cdot 0_A) \\
    &= (0_B, b^a).
\end{align*}
\]

Therefore this action also coincides with the given action of \( B \) on \( A \), which proves that the given action of \( B \) on \( A \) is a derived action, which concludes the proof of the theorem.

For the examples of derived actions in the category \( \text{Gr}^* \) see Section 4, Lemma 4.5 and Corollary 4.6.

4. The subcategory \( \text{rGr}^* \rightarrow \text{Gr}^* \)

Consider the objects \( A \in \text{Gr}^* \) which satisfy two conditions:

1. \( x^y + z = z + x^y, \ y \neq 0 \) and
2. \( x^{(y^r)} = x^y, \)

for any \( x, y, z \in A \). This kind of objects will be called reduced groups with action, and the corresponding full subcategory of \( \text{Gr}^* \) will be denoted by \( \text{rGr}^* \).

Derived actions are defined in \( \text{rGr}^* \) in analogous way as it is in \( \text{Gr}^* \).

**Example 4.1.** For any set \( X \) let \( F(X) \) be a free group with action with the basis \( X \) in \( \text{Gr}^* \) (see Example 2.2 in Section 2). Let \( R \) be a congruence relation on \( F(X) \) generated by the relations

\[
x^y + z \sim z + x^y
\]

for any \( y \neq 0 \) and

\[
x^{(y^r)} \sim x^y
\]

for any \( x, y, z \in F(X) \). Then the quotient object \( F(X)/R \) by the \( R \) obviously is an object of \( \text{rGr}^* \) and it is a free object in \( \text{rGr}^* \) with the basis \( X \).

**Example 4.2.** An easy checking shows that the object \( \mathbb{Z}^* \) in Example 2.3 in Section 2 is an object of \( \text{rGr}^* \).
Example 4.3. Any abelian group with trivial action on itself is an object of \( \text{rGr}^* \).

Theorem 4.4. Let \( A, B \in \text{rGr}^* \) and \( \beta = (\beta_+, \beta_\ast, \beta_\circ) : B \times A \to A \) be a triple of actions of \( B \) on \( A \) in \( \text{rGr}^* \). Then the following conditions are equivalent:

1. \( \beta \) is a triple of derived actions in \( \text{rGr}^* \).
2. \( \beta \) satisfies condition (2) of Theorem 3.2 and the following conditions

\[
\begin{align*}
    b \cdot a' &= a'' \quad \text{for} \quad a'' \neq 0 \\
    b \cdot a &= a'' \quad \text{for} \quad a'' \neq 0 \\
    b^b \cdot a &= a' \quad \text{for} \quad a' \neq 0 \\
    b^{(a')} &= b^a \\
    b^{(a'')} &= b^{a''}
\end{align*}
\]

(4.1)

for any \( a, a' \in A, b, b' \in B \). Note that under the conditions (4.1), (3_A) and (4_A) have simpler forms.

3. The semi-direct product \( B \ltimes A \) is an object in \( \text{rGr}^* \).

Proof. (1)\( \Rightarrow \) (2): We will check only the conditions \( a^{(b')} = a, b^{(a'')} = 0 \) and \( b^{(a')} = b^a \).

Other conditions are obvious.

(i) \( a^{(b')} = a \cdot j(b)^{a'} - j(b) = a \cdot j(b) - j(b) = a - a = 0 \);

(ii) \( b^{(a'')} = j(b) \cdot (a'') - j(b) = j(b) \cdot j(b') - j(b) = j(b) \cdot 0 - j(b) = 0 \);

(iii) \( b^{(a')} = j(b) \cdot (a') - j(b) = j(b)^a - j(b) = b^a \).

(2)\( \Rightarrow \) (3): By Theorem 3.2 we need to prove only that

\[
(b, a)^{(b', a')} + (b'', a'') = (b', a') + (b, a)^{(b', a')}
\]

and

\[
(b, a)^{(b', a')(a'', a'')} = (b, a)^{(b', a')}
\]

for any \( (b, a), (b', a'), (b'', a'') \in B \ltimes A \). We have

\[
(b, a)^{(b', a')} + (b'', a'') = \left( b^{b'}, \left( a^{a'} \right)^{b'} + \left( b^{a'} \right)^{b'} \right) + (b'', a'')
\]

\[
= \left( b^{b'} + b'', \left( a^{a'} \right)^{b'} + \left( b^{a'} \right)^{b'} + b'' \cdot a'' \right)
\]

\[
= \left( b^{b'} + b'', \left( a^{a'} \right)^{b'} + \left( b^{a'} \right)^{b'} + a'' \right)
\]

On the other hand

\[
(b'', a'') + (b, a)^{(b', a')} = \left( b'', a'' \right) + \left( b^{b'}, \left( a^{a'} \right)^{b'} + \left( b^{a'} \right)^{b'} \right)
\]

\[
= \left( b'' + b^{b'}, a'' + b'' \cdot \left( a^{a'} \right)^{b'} + b'' \cdot \left( b^{a'} \right)^{b'} \right)
\]

\[
= \left( b^{b'} + b'', a'' + \left( a^{a'} \right)^{b'} + \left( b^{a'} \right)^{b'} \right)
\]

\[
= \left( b^{b'} + b'', \left( a^{a'} \right)^{b'} + \left( b^{a'} \right)^{b'} + a'' \right)
\]
which proves the first identity. For the second identity we have

\[(b, a)\left((b', a')^{(b'' a'')}\right) = (b, a)\left(b^{b''} (a'' a')^{b''} + (b'' a')^{b''}\right)\]

\[= \left(b^{b''} \left( a\left((a'' a')^{b''}\right) + (b'' a')^{b''}\right) \right) + \left(b\left((a'' a')^{b''} + (b'' a')^{b''}\right)\right)\]

\[= \left(b^{b''}, (a^0 a')^{b''} + \left(b\left((a'' a')^{b''}\right)\right)^{b''}\right)\]

\[= \left(b^{b''} + (a^0 a')^{b''} + \left(b\left((a'' a')^{b''}\right)\right)^{b''}\right)\]

\[= \left(b^{b''} + (a^0 a')^{b''}\right)\]

\[= (b, a)^{(b', a')}\]

which proves the second identity. Here we applied that \(b\left((b'' a')^{b''}\right) = 0\), which follows from (4.1), where we have \(b^{b''} = 0\), for any \(a \in A\), in particular for \(a = a''\) in our case, and the fact that \(0^{b'} = 0\) (3.1 (b)).

(3)\(\Rightarrow\)(1): The proof is the same as of the one in Theorem 3.2 and therefore we omit.

\[\square\]

**Lemma 4.5.** Let \(A \in \text{Gr}^\bullet\) (resp. \(A \in \text{rGr}^\bullet\)). An action of \(A\) on itself defined by \(a \cdot a' = a + a' - a, a' \cdot a = a'' a = a'' a\) and \(a' \cdot a = a'' - a\), for \(a, a' \in A\), is a derived action in \(\text{Gr}^\bullet\) (resp. \(\text{rGr}^\bullet\)).

**Proof.** Easy but careful checking of the conditions given in Theorem 3.2 (resp. Theorem 4.4).

\[\square\]

Note, that an action of \(A\) on itself defined by \(a \cdot a' = a + a' - a, a'^0 a = a'^0 a\) and \(a'' a' = a'' a\), for \(a, a' \in A\), is not a derived action in \(\text{Gr}^\bullet\) and therefore in \(\text{rGr}^\bullet\). It is obvious that conditions (2\(_A\)) and (1\(_B\)) are not satisfied.

**Corollary 4.6.** Let \(A \in \text{Gr}^\bullet\) (resp. \(A \in \text{rGr}^\bullet\)) and let \(I \subset A\) be an ideal of \(A\). Then the action of \(A\) on \(I\) defined by \(a \cdot i = a + i - a, i^0 a = i^0 a\) and \(a'^i = a'^i - a, i \in I\), \(a \in A\) is a derived action in \(A \in \text{Gr}^\bullet\) (resp. in \(A \in \text{rGr}^\bullet\)).

Lemma 4.5 and Corollary 4.6 give examples of derived actions in the categories \(\text{Gr}^\bullet\) and \(\text{rGr}^\bullet\).
References