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MINIMUM TSALLIS PORTFOLIO

MİNİMUM TSALLİS PORTFÖYÜ

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Abstract

Mean-variance portfolio optimization model has been shown to have serious drawbacks. The model assumes that assets returns are normally distributed that is not valid for most of the markets and portfolios. It also relies on assets' covariance matrices for the calculation of portfolio's risk that is open to estimation errors. Moreover, these optimization errors are maximized by the method that result in poor out-of-sample performances. In order to address these issues, we propose a new portfolio optimization method based on minimization of Tsallis entropy, which is valid for any underlying distribution. First, we show that the Tsallis entropy can be employed as a risk measure for portfolio analysis. Then we demonstrate the validity of the model by comparing its performance with those of mean-variance and minimum-variance portfolios using BIST 30 data. The results show that Minimum Tsallis portfolio achieve similar Sharpe Ratios to mean-variance and minimum-variance portfolios but is more diversified that indicates a better out-of-sample performance.

Keywords: Portfolio optimization, entropy, minimum Tsallis portfolio
JEL Classification: G11

Öz

Ortalama-varyans portföy optimizasyon modelinin ciddi dezavantajları olduğu gösterilmiştir. Model, çoğu piyasa ve portföy için geçerli olmayan varlık getirilerinin normal dağıldığını varsaymaktadır. Ayrıca model portföy riskinin hesaplanmasında tahmin hatalarına açık olan varlık kovaryans matrislerini kullanmaktadır. Üstelik, bu optimizasyon hataları, model tarafından maksimize edilerek zayıf örneklem dışı performanslara neden olmaktadır. Bu sorunları aşmak için, bu çalışmada, herhangi bir dağılım için geçerli olan Tsallis entropisinin minimizasyonuna dayalı yeni bir portföy optimizasyonu modeli önerilmektedir. İlk olarak, Tsallis entropisinin portföy analizi için bir risk ölçüsü olarak kullanılabileceği gösterilmektedir. Ardından, modelin geçerliliği BIST 30 verileri kullanılarak ortalama-varyans ve minimum-varyans portföyleri ile

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karşılaştırmalı olarak gösterilmektedir. Sonuçlar Minimum Tsallis portföyün ortalama-varyans ve minimum varyans portföylerine benzer Sharpe Ratio değerlerine ulaştığını fakat daha fazla çeşitlendirilmiş olduğunu görtermiştir. Bu Minimum Tsallis portföyün örneklem dışı veri ile daha iyi performans gösterebileceğine işaret etmektedir.
Keywords: Portföy optimizasyonu, entropi, minimum Tsallis portföyü
JEL Sınıflandırması: G11

1. Introduction

Mean-variance portfolio optimization model assumes a normal distribution of assets returns resulting in estimation errors as the real distribution deviates from normality. Moreover, the model uses historical data directly to estimate the mean and the variances of the returns as if they were true parameters. But statistical theory states that when the number of parameters is greater than two, sample mean is not an admissible estimator for the true parameters (Stein, 1956). There has been a great deal of research focusing on this issue most of which based on shrinkage estimators. For instance, Jorion (1968) used Bayes-Stein shrinkage method for estimating the portfolio means and obtained better portfolio performances compared to using historical data directly. Michaud (1989) although mentioned the benefits of mean-variance optimization he pointed out that mean-variance optimization maximizes estimation errors which results in overweighting (underweighting) assets with high (low) estimated returns and low (high) variances. Best & Grauer (1991) showed that small changes in the means of individual assets could result in extreme changes in the weights, the means and the variances of the portfolios. They also found out that a small increase in one the asset's mean could push half of the assets out of the portfolio, which indicated the importance of the estimation errors. Black & Litterman (1992) introduced a model in which investors can include their subjective views for estimating excess returns of the assets. Instead of using historical returns investors could adjust the weights of the assets referencing to a neutral CAPM (Capital Asset Pricing Model) equilibrium point as to reflect their views. Black-Litterman model received a great deal of interest since it was published in 1992 both from portfolio investors and researchers. Many researchers have proposed different versions of the model so far. One may refer to Walters (2014) for a thorough discussion.

Because of the difficulty of estimating expected returns some researchers proposed minimum-variance instead of mean-variance portfolios (Green & Hollifield, 1992; Chan, Karceski & Lakonishok, 1999; Jagannathan & Ma, 2003). This model, which is based on only the risk minimization of portfolios, was shown to have better out-of-sample results. Since the estimation of covariance matrix is still based on historical return data, some researchers started questioning the minimum-variance model too. Ledoit & Wolf (2004) proposed a shrinkage estimator for the covariance matrix, which was based on a trade-off between the sample covariance matrix and a highly structured estimator by defining a shrinkage constant. They showed that their shrinkage estimator portfolios performed better than sample covariance matrix portfolios for most of the cases they studied. Covariance matrix shrinkage estimator of Ledoit-Wolf has drawn substantial attention since then and has been employed in portfolio analysis and other application areas. Although shrinkage estimators for covariance matrix have provided better out-of-sample results they still don't account for the non-normality of asset returns.

As the variance does not account for all the moments of non-normal distributions, entropy emerged as a measure in portfolio analyses. Philippatos & Wilson (1972) found that their entropy-based model was consistent with the classical mean-variance portfolio. But they were also convinced that a new risk measure, which is free from the distribution of asset returns, was necessary, namely entropy. Many researchers have deployed entropy in portfolio analysis thus far, in which different entropy measures were used. For example, Smimou, Bector & Jacoby (2007) used maximum entropy for the minimum risk portfolio. Bera & Park (2008) deployed cross entropy measure in portfolio objective function and used it as a shrinkage estimator to a predefined target portfolio. Usta & Kantar (2011) added the entropy measure to the mean-variance-skewness model (MVSM) to generate a well-diversified portfolio. Zhang, Liu & Xu (2012) presented possibilistic entropy concept for multi-period portfolio selection. Xu, Wu, Long & Song (2014) introduced continuous maximum entropy for portfolio optimization with transaction costs and dividends. Aksarayli & Pala (2018) used entropy together with higher moments for a polynomial goal portfolio optimization. Rotela Junior et al. (2017) introduced entropic data envelopment analysis for portfolio optimization. Zhou et al. (2019) and Zhang & Li (2019) used entropy and semi-entropy based on credibility measure for multi-period portfolio selection. All of the proposed methods mentioned use Shannon entropy. On the other hand, other forms of entropy for portfolio optimization are rather rare. For instance, Lassance & Vrins (2019) employed Rényi entropy in the portfolio objective function and used the parameter α of the Rényi entropy for tuning contributions from the central and tail parts of the asset return distributions. Batra & Taneja (2020) maximized Rényi and Tsallis entropies in the objective functions for a given return value and compared the results with those of Mean-Variance model. Tsallis relative entropy was also proposed for portfolio optimization assuming q-Gaussian distribution of stock market data recently (Devi, 2019; Trindade et al., 2020).

In this study, we investigate Tsallis entropy as a risk measure for portfolio analysis and propose a novel method for portfolio optimization by minimizing the Tsallis entropy without assuming any underlying market data distribution. The rest of the paper is organized as follows. Tsallis entropy and its relation to variance are presented in section 2. Mean-variance (MV) and minimum-variance (Minvar) optimization models, which are used for performance comparison, are briefly described in section 3. In section 4, minimum Tsallis portfolio optimization is described. Comparative portfolio analysis and results are included and discussed in section 5. Finally, section 6 concludes the study.

2. Tsallis Entropy

Tsallis entropy (Tsallis, 1988) is a generalized form of Shannon entropy, which is defined as

$$H_q = \frac{1-\sum_{i=1}^{K} p_i^q}{q-1}, \qquad for \;\; q>0 \;\; and \;\; q \neq 1 \qquad (1)$$

where,

p_i = Probability of discrete events,

$q \in \mathbb{R}$ = Generalization constant.

Tsallis entropy can be shown to diverge to Shannon entropy as $q \to 1$ by using L'Hospital's rule;

$$\lim_{q\to 1}\frac{1-\sum_{i=1}^{K}p_i^q}{q-1}=\lim_{q\to 1}\frac{\frac{d}{dq}\left(1-\sum_{i=1}^{K}p_i^q\right)}{\frac{d}{dq}(q-1)}=\lim_{q\to 1}\frac{-\sum_{i=1}^{K}p_i^q ln\, p_i}{1}=-\sum_{i=1}^{K}p_i ln\, p_i \tag{2)}$$

The maximum value of Tsallis entropy is obtained when all the event probabilities are equal, $p_i = \frac{1}{K}$, through the equation (1) as

$$\frac{1-K^{1-q}}{q-1} \tag{3}$$

2.1. Tsallis entropy as a portfolio risk measure

Although Tsallis entropy, like other forms of entropy, does not assume any underlying distribution for the portfolio optimization, Tsallis entropies and the corresponding variances are presented for normal and Binomial distribution in order to demonstrate the relation between them.

For normal distribution $\boldsymbol{\mathcal{N}}(\boldsymbol{\mu},\boldsymbol{\sigma^2})$, Tsallis entropy can be derived as (Appendix A)

$$\frac{\sqrt{q}-(\sigma\sqrt{2\pi})^{1-q}}{\sqrt{q}(q-1)} \tag{4}$$

As the equation (4) indicates variance and Tsallis entropy are positively correlated for normal distribution. Tsallis entropies for q=0.7 and q=1.5 are sketched in Figure 1. The figure demonstrates that Tsallis entropy with parameter can be employed as a risk measure in portfolio analysis.

As an example of discrete case, Tsallis entropy for binomial distribution $\binom{n}{x}p^x(1-p)^{n-x}$ with variance $n\,p\,(1-p)$ can be deduced, assuming large values of n, as

$$\frac{\sqrt{q}-(\sqrt{2\pi np(1-p)})^{1-q}}{\sqrt{q}(q-1)} \tag{5}$$

which also demonstrates the relation between Tsallis entropy and the variance.

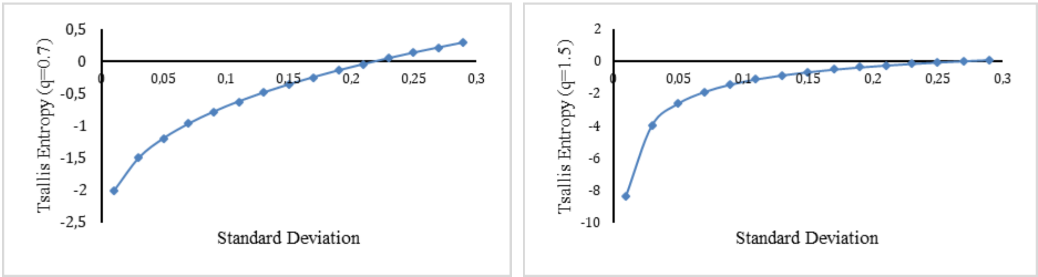

Figure 1: Tsallis Entropy vs Standard Deviation for Normal Distribution

3. Portfolio Optimization Methods for Comparison Analysis

Mean-Variance, minimum-variance portfolio optimization methods against which, the minimum Tsallis portfolio optimization is compared, are briefly described in this section.

3.1 Mean-Variance Portfolio Optimization

The Mean-Variance (MV) Portfolio optimization method, although criticized severely for not being optimal when the returns are not normally or elliptically distributed and very sensitive to estimation errors, still provides a good benchmark for portfolio optimization studies. MV optimization is a risk versus return optimization through the maximization of the following utility equation

$$\max_{\pi} \boldsymbol{\pi}' \boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{\pi}'\Sigma\boldsymbol{\pi} \tag{6}$$

subject to $\boldsymbol{\pi}'\boldsymbol{\mu} = \mu_0$

where,

$\boldsymbol{\pi}$ = Weights vector,

$\boldsymbol{\mu}$ = Expected return vector,

Σ = Covariance matrix,

γ = Investors' risk aversion

μ_0 = Predetermined value for expected return of the portfolio.

The maximization problem given in equation (6) is solved with the sample expected return $\widehat{\boldsymbol{\mu}}$ and the sample covariance matrix $\widehat{\Sigma}$, together with a short selling and a budget constraint of $\pi_i \geq 0, \sum_{i=1}^{N} \pi_i = 1$ respectively. The risk aversion factor is taken as 1.

3.2 Minimum-Variance Portfolio Optimization

The minimum-variance portfolio (MinVar) which, relies only on covariance matrix estimation and not on estimation of returns, is less sensitive to estimation errors and is shown to have better out-of-sample results compared to MV portfolios (Green & Hollifield, 1992; Chan, Karceski, & Lakonishok, 1999; Jagannathan & Ma, 2003). The MinVar optimization problem is given as

$$\min_{\boldsymbol{\pi}} \boldsymbol{\pi}'\Sigma\boldsymbol{\pi} \tag{7}$$

subject to $\boldsymbol{\pi}'\boldsymbol{\mu} = \mu_0$

where,

$\boldsymbol{\pi}$ = Weights vector,

Σ= Covariance matrix.

Short selling and the budget constraint $\pi_i \geq 0, \sum_{i=1}^{N} \pi_i = 1$ respectively are also imposed for MinVar optimization.

4. Minimum Tsallis Entropy Portfolio

As shown in section 2 Tsallis entropy and the variance are closely related. This enables the minimization of the Tsallis entropy in the objective function of the portfolio optimization problem. Hence the optimization problem can be written as follows;

$$\min_{\boldsymbol{\pi}} H_q = \frac{1 - \sum_{i=1}^{K} p_i^q}{q - 1} \tag{8}$$

subject to $\boldsymbol{\pi}'\boldsymbol{\mu} = \mu_0$

Calculation of the Tsallis entropy for the optimization problem in equation (8) is based upon combinatorial approach proposed by Mercurio, Wu and Xie (2020). If the return range of the portfolio is assumed to be divided into *K* number of discrete return levels then the probability p_i of any return level is given by

$$p_i = \frac{1}{T} \sum_{j=1}^{T} I\left(r_{k-1} < \boldsymbol{\pi}'\boldsymbol{r_j} < r_k\right), \ i = 1, \dots, K \tag{9}$$

where,

I= Indicator function,

T = Number of observations in time,

π = Weight vector,

r_j = Return vectors corresponding each observation in time.

Then the minimum Tsallis optimization equation which, will be used for the portfolio analysis, is obtained by substituting p_i from equation (9) into the equation (8). The short selling and the budget constraints are valid for this portfolio too.

5. Comparison Analysis and Results

The minimum Tsallis portfolio method is applied to BIST 30 stocks and the results are compared with those mean-variance, minimum-variance portfolios. BIST 30 stocks data, which is downloaded from Yahoo finance website (2021), is used for analysis. The data comprises 510 daily returns of 30 stocks between 01.01.2019 and 31.12.2020.

Since mean-variance and minimum-variance portfolio optimization methods assume the normality of asset returns, a Shapiro-Wilk normality test is used for checking the normality of the stock returns. Table1 lists the average returns, variances and Shapiro-Wilk scores and the corresponding p values of the BIST 30 stocks for the analysis period. All of the stocks have p values close to zero meaning returns are not normally distributed that contradicts the assumption of mean-variance portfolio model.

Table 1: Daily statistics of BIST 30 stocks between 01.01.2019 and 31.12.2020

	Return	Variance	Shapiro-Wilk	p
AKBNK	0.00031295	0.0005585	0.96466	9.913e-10
ARCLK	0.00140151	0.00052125	0.94293	4.276e-13
ASELS	0.00086152	0.00057858	0.94424	6.413e-13
BIMAS	0.00123693	0.00034048	0.95783	6.622e-11
EREGL	0.00240071	0.00052393	0.96123	2.453e-10
EKGYO	0.00079957	0.00060103	0.96827	4.745e-09
EREGL	0.00220841	0.00046709	0.94895	2.909e-12
FROTO	0.00201054	0.00060645	0.96601	1.761e-09
GARAN	0.00052107	0.0006421	0.95965	1.325e-10
GUBRF	0.00655461	0.00113117	0.95941	1.206e-10
HALKB	-0.000432	0.00051438	0.97087	1.565e-08
ISCTR	0.00085179	0.0005306	0.96973	9.194e-09
KCHOL	0.00084328	0.00041301	0.95694	4.762e-11
KOZAA	0.0015171	0.00080653	0.96484	1.068e-09
KOZAL	0.00128268	0.00052841	0.953	1.152e-11
KRDMD	0.00214944	0.00076295	0.9713	1.913e-08
PETKM	0.00084153	0.00055125	0.94493	7.968e-13
PGSUS	0.00229864	0.00134869	0.95323	1.25e-11
SAHOL	0.00098925	0.00045497	0.96771	3.697e-09

SASA	0.00245712	0.00095186	0.89453	2.2e-16
SISE	0.00058479	0.00056345	0.94913	3.089e-12
TAVHL	8.3603E-05	0.00061716	0.95845	8.365e-11
TCELL	0.00079637	0.00040775	0.95752	5.901e-11
THYAO	-0.0004315	0.00063712	0.94462	7.214e-13
TKFEN	-0.0001208	0.00054452	0.98954	0.001074
TTKOM	0.00168902	0.00066316	0.96818	4.561e-09
TUPRS	8.1811E-05	0.00046087	0.96367	6.583e-10
VAKBN	0.00039484	0.0006405	0.97013	1.106e-08
VESTL	0.00262026	0.00097148	0.92166	1.242e-15
YKBNK	0.00128464	0.00058623	0.96818	4.546e-09

5.1 Comparison Methodology and Performance Measures

Minimum Tsallis, Mean-variance, and Minvar portfolios are formed according to optimization problems given in equations (6), (7), and (8) for different μ_0 (daily average return of the portfolio) values using nonlinear programming solver function fmincon of MATLAB. The results are given Table 2. The table lists the portfolio's risk (σ_p , standard deviation of the portfolio assets), Sharpe ratios and diversity indexes of the portfolios. Sharpe ratio is one of the widely used portfolio measures and given by

$$\text{Sharpe Ratio} = \frac{\mu_p}{\sigma_p}$$

where,

μ_p = Portfolio's return,

σ_p = Portfolio's standard deviation.

Sharpe ratio is the measure of expected return of a portfolio per unit risk incurred. Another performance measure that is considered in this study is the diversity index that explains how well a portfolio is diversified. The diversity index, which is used, is the one defined by Woerheide and Persson (1993) and given as

$$\text{Diversity} = 1 - \sum_{i=1}^{n} \pi_i^2 \tag{11}$$

where,

π_i = Weights of the assets in the portfolio.

Diversity index values ranges between 0 and 1. Values close to zero mean that the portfolio weights are concentrated in a few assets and values close to 1 correspond to well diversified portfolios.

Table 2: Performance comparisons of Minimum Tsallis (q=0.7), Minvar and MV portfolios

	μ_0	σ_p	**Sharpe Ratio**	**Diversity**
Min. Tsallis (q=0.7)		0.0161187	0.1240795	0.95438142
Minvar	0.002	0.0129809	0.1540725	0.83828333
MV		0.01298285	0.1540494	0.83747097
Min. Tsallis (q=0.7)		0.01756294	0.1708142	0.90425764
Minvar	0.003	0.01503576	0.1995243	0.80475265
MV		0.01503592	0.1995223	0.80531343
Min. Tsallis (q=0.7)		0.02037463	0.1963226	0.78857881
Minvar	0.004	0.0189039	0.2115965	0.72974023
MV		0.01890499	0.2115844	0.73023654
Min. Tsallis (q=0.7)		0.02490222	0.2007853	0.57792719
Minvar	0.005	0.02398681	0.2084479	0.53370747
MV		0.02398681	0.2084479	0.53358763
Min. Tsallis (q=0.7)		0.03025009	0.1983465	0.24805598
Minvar	0.006	0.02991839	0.2005455	0.22547719
MV		0.03025009	0.1983465	0.22547724

As seen from Table 2, standard deviations, Sharpe ratios have very close values for three types of portfolios. Mean-Variance and Minvar portfolio values are inherently almost equal since they are two different of interpretations of the same optimization problem. However, diversity index values show that Minimum Tsallis portfolios are better diversified for all values of predefined portfolio expected return μ_0. This indicates that for similar values of expected return and risk, Minimum Tsallis portfolios are supposed to show better out-of-sample performances for they do not overweight (at least not as much as MV and MinVar portfolios do) the assets depending on the historical values.

Performance measure for Minimum Tsallis portfolio was calculated for q=0.7. Different values for q between 0 and 1, did not yield remarkable changes in portfolio composition. Hence optimizing the parameter q for different markets and data is not applicable. This may be considered as a limitation of the model.

6. Conclusion

Mean-variance and minimum variance portfolio optimization models have been criticized for the following drawbacks since they emerged. First, these models assume that asset returns, which constitute the portfolio, are normally distributed. Second, these models overweight some assets that results in very poor out-of-sample portfolio performances. Third, both models require the calculation of covariance matrices for the portfolio risk. This causes severe estimation errors especially when the number of assets in the portfolio is large. Many variations of these models and different approaches have been proposed to overcome these drawbacks so far. Minimum Tsallis portfolio, which we proposed in this study, also addressed these drawbacks. We first showed that Tsallis entropy can be used as risk measure for portfolio analysis. Since Tsallis entropy does not assume any underlying distribution for the asset returns, first of the drawbacks is intrinsically addressed. We showed that although the Sharpe ratios are comparable with those of mean-variance and minimum variance portfolios, minimum Tsallis portfolios are better diversified and hence more likely to have better out-of-sample performances. Minimum Tsallis entropy portfolio model does not require the calculation of covariance matrices for portfolio risk and is not prone to estimation errors unlike mean-variance and minimum-variance portfolio models. Consequently, minimum Tsallis portfolio proved itself to be a simple but a robust portfolio optimization model. Moreover, minimum Tsallis portfolio is easy to comprehend and calculate. Hence, it is a practical tool for ordinary investors as well as for finance professionals and academics. Although the out-of-sample performance of a minimum Tsallis portfolio is expected to be better compared to mean-variance and minimum variance portfolios since it is more diversified, further research should be conducted to prove it empirically. Another research subject would be a comparison analysis against other portfolio models, which are based on other entropy measures.

Author Contribution

CONTRUBUTION RATE	EXPLANATION	CONTRIBUTORS
Idea or Notion	Form the research idea or hypothesis	Erhan USTAOĞLU Atıf EVREN
Literature Review	Review the literature required for the study	Erhan USTAOĞLU
Research Design	Designing method, scale, and pattern for the study	Erhan USTAOĞLU Atıf EVREN
Data Collecting and Processing	Collecting, organizing, and reporting data	Atıf EVREN
Discussion and Interpretation	Taking responsibility in evaluating and finalizing the findings	Erhan USTAOĞLU Atıf EVREN

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Resume

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Appendix A

For the continuous case Tasllis entropy in equation (1) is written as

$$H_q = \frac{1 - \int_{-\infty}^{\infty} [\mathrm{F(x)}]^q}{q-1}$$

Let

$$\mathrm{I} = \int_{-\infty}^{\infty} [\mathrm{F(x)}]^q = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^q e^{-\frac{q}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\mathrm{I} = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{q-1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma/\sqrt{q}}\right)^2} dx$$

$$\mathrm{I} = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{q-1} \frac{1}{\sqrt{q}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma}{\sqrt{q}}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma/\sqrt{q}}\right)^2} dx$$

Let $\boldsymbol{\sigma}' = \frac{\boldsymbol{\sigma}}{\sqrt{\boldsymbol{q}}}$

$$\mathrm{I} = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{q-1} \frac{1}{\sqrt{q}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma'} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma'}\right)^2} dx$$

Integral part of the above equation sums up to 1 for a normal distribution with a mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}'$, then

$$H_q = \frac{1 - \frac{1}{\sqrt{q}}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{q-1}}{q-1} = \frac{\sqrt{q} - (\sigma\sqrt{2\pi})^{1-q}}{\sqrt{q}(q-1)}$$

Assuming large values of n with variance $n\,p\,(1-p)$ Tsallis entropy can be written for Binomial case as

$$\frac{\sqrt{q} - (\sqrt{2\pi np(1-p)})^{1-q}}{\sqrt{q}(q-1)}$$