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# The Period and Rank of the Complex-type Padovan-$p$ Numbers Modulo $m$

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**Abstract.** In this paper, we study the complex-type Padovan-$p$ sequence modulo $m$ and then we give some results concerning the periods and ranks of this sequence for any $p$ and $m$. Furthermore, we produce the cyclic groups using the multiplicative orders of the generating matrix of the complex-type Padovan-$p$ sequence when read modulo $m$. Finally, we give the relationships between the periods of the complex-type Padovan-$p$ sequence modulo $m$ and the orders of the cyclic groups produced.

## 1. Introduction

It is well-known that the Padovan sequence $\{P(n)\}$ is defined recursively by the equation:

$$P(n) = P(n-2) + P(n-3)$$

for $n \geq 3$, where $P(0) = P(1) = P(2) = 1$.

The Padovan $p$-sequence $\{Pap(n)\}$ is defined [6] by initial values $Pap(1) = Pap(2) = \cdots = Pap(p) = 0$, $Pap(p+1) = 1$, $Pap(p+2) = 0$ and the following homogeneous linear recurrence relation

$$Pap(n+p+2) = Pap(n+p) + Pap(n)$$

for any given $p\,(p = 2, 3, 4, \ldots)$ and $n \geq 1$. Note that the $(2n+1)\,th$ term of the Padovan 2-sequence $\{Pa2(n)\}$, is equal to $nth$ Fibonacci number.

The complex-type Padovan $p$-sequence $\left\{Pa_p^{(i)}(n)\right\}$ is defined [11] as follows:

$$Pa_p^{(i)}(n+p+2) = i^2 \cdot Pa_p^{(i)}(n+p) + i^{p+2} \cdot Pa_p^{(i)}(n) \tag{1}$$

for any given $p\,(p = 3, 5, 7, \ldots)$ and $n \geq 1$, where $Pa_p^{(i)}(1) = \cdots = Pa_p^{(i)}(p) = 0$, $Pa_p^{(i)}(p+1) = 1$, $Pa_p^{(i)}(p+2) = 0$ and $\sqrt{-1} = i$.

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. For example,

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the sequence $a, b, c, d, b, c, d, b, c, d, \ldots$ is periodic after the initial element $a$ and has period 3. A sequence is simply periodic with period $k$ if the first $k$ elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, \ldots$ is simply periodic with period 4.

The study of the behavior of the linear recurrence sequences under a modulus began with the earlier work of Wall [17] where the periods of the ordinary Fibonacci sequences modulo $m$ were investigated. Recently, the theory extended to some special linear recurrence sequences by several authors; see, for example, [3, 4, 12, 15, 16]. In the first part of this paper, we consider the complex-type Padovan-$p$ sequence modulo $m$ and then we derive some interesting results concerning the periods and ranks of the complex-type Padovan-$p$ sequence for any $p$ and $m$.

The relationships between the periods of the linear recurrence sequences modulo $m$ and the cyclic groups which are produced using the multiplicative orders of the generating matrices of these sequences when read modulo $m$ have been studied recently by many authors; see, for example, [1, 2, 5, 7–10, 13, 14, 18]. In the second part, we derive the cyclic groups using the multiplicative orders of the generating matrix of the complex-type Padovan-$p$ numbers when read modulo $m$. Then, we give the relationships between the periods of the complex-type Padovan-$p$ sequence modulo $m$ and the orders of the cyclic groups produced.

## 2. The Main Results

If we reduce the complex-type Padovan-$p$ sequence $\left\{Pa_p^{(i)}(n)\right\}$ by a modulus $m$, taking least nonnegative residues, then we get the following recurrence sequence:

$$\left\{Pa_p^{(i,m)}(n)\right\} = \left\{Pa_p^{(i,m)}(0),\ Pa_p^{(i,m)}(1),\ \ldots, Pa_p^{(i,m)}(j),\ \ldots\right\}$$

where $Pa_p^{(i,m)}(j)$ is used to mean the $j$th element of the complex-type Padovan-$p$ sequence when read modulo $m$. We note here that the recurrence relations in the sequences $\left\{Pa_p^{(i,m)}(n)\right\}$ and $\left\{Pa_p^{(i)}(n)\right\}$ are the same.

**Theorem 2.1.** *For any given $p$ $(p = 3, 5, 7, \ldots)$, the sequence $\left\{Pa_p^{(i,m)}(n)\right\}$ is simply periodic.*

*Proof.* Consider the set

$$\begin{aligned} C \quad = \quad & \left\{\left(c_1, c_2, \ldots, c_{p+2}\right) \mid c_n\text{'s are complex numbers } a_n + ib_n \text{ where} \right. && (2) \\ & \left. a_n \text{ and } b_n \text{ are integers such that } 0 \le a_n, b_n \le m-1 \text{ and } 1 \le n \le p+2\right\}. && (3) \end{aligned}$$

Let the notation $|C|$ indicate the cardinality of the set $C$. Since the set $C$ is finite, there are $|C|$ distinct $(p+2)$-tuples of the complex-type Padovan-$p$ numbers modulo $m$. Thus, it is clear that at least one of these $(p+2)$-tuples appears twice in the sequence $\left\{Pa_p^{(i,m)}(n)\right\}$. Therefore, the subsequence following this $(p+2)$-tuple repeats; that is, $\left\{Pa_p^{(i,m)}(n)\right\}$ is a periodic sequence. Let us consider $Pa_p^{(i,m)}(u) \equiv Pa_p^{(i,m)}(v)$, $Pa_p^{(i,m)}(u+1) \equiv Pa_p^{(i,m)}(v+1), \ldots, Pa_p^{(i,m)}(u+p+2) \equiv Pa_p^{(i,m)}(v+p+2)$ and $v \ge u$. Then we have $v \equiv u\,(mod(p+2))$. From the recurrence relation in (1), we can write the following recursive equations:

$$Pa_p^{(i)}(u) = i^{2-p} \cdot Pa_p^{(i)}(u+p+2) + i^{3-p} \cdot Pa_p^{(i)}(u+p)$$

and

$$Pa_p^{(i)}(v) = i^{2-p} \cdot Pa_p^{(i)}(v+p+2) + i^{3-p} \cdot Pa_p^{(i)}(v+p).$$

So we get $Pa_p^{(i,m)}(u-1) \equiv Pa_p^{(i,m)}(v-1)$, $Pa_p^{(i,m)}(u-2) \equiv Pa_p^{(i,m)}(v-2)$, $\ldots$, $Pa_p^{(i,m)}(2) \equiv Pa_p^{(i,m)}(v-u+2)$, $Pa_p^{(i,m)}(1) \equiv Pa_p^{(i,m)}(v-u+1)$, which implies that the complex-type Padovan-$p$ sequence modulo $m$ is simply periodic. □

Let the notation $lP_p^i(m)$ denote the smallest period of the sequence $\left\{Pa_p^{(i,m)}(n)\right\}$.

Given an integer matrix $A=\left[a_{ij}\right]$, $A\,(mod m)$ means that all entries of $A$ are modulo $m$, that is, $A\,(mod m)=\left(a_{ij}\,(mod m)\right)$. Let us consider the set $\langle A\rangle_m=\{(A)^n\,(mod m)\mid n\geq 0\}$. If $(\det A, m)=1$, then the set $\langle A\rangle_m$ is a cyclic group; if $(\det A, m)\neq 1$, then the set $\langle A\rangle_m$ is a semigroup.

In [11], the generating matrix of the complex-type Padovan-$p$ sequence had been given as:

$$D_p=\left[d_{jk}^{(p)}\right]_{(p+2)\times(p+2)}=\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & 0 & i^{p+2} \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

The matrix $D_p$ is said to be the complex-type Padovan-$p$ matrix. Then they had been written the following matrix relation:

$$\begin{bmatrix} Pa_p^{(i)}(n+p+2) \\ Pa_p^{(i)}(n+p+1) \\ \vdots \\ Pa_p^{(i)}(n+2) \\ Pa_p^{(i)}(n+1) \end{bmatrix}=D_p\cdot\begin{bmatrix} Pa_p^{(i)}(n+p+1) \\ Pa_p^{(i)}(n+p) \\ \vdots \\ Pa_p^{(i)}(n+1) \\ Pa_p^{(i)}(n) \end{bmatrix}.$$

It can be readily established by mathematical induction that for $n\geq p+1$,

$$\left(D_p\right)^n=\begin{bmatrix} Pa_p^{(i)}(n+p+1) & Pa_p^{(i)}(n+p+2) & i^{p+2}\cdot Pa_p^{(i)}(n+1) & i^{p+2}\cdot Pa_p^{(i)}(n+2) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n+p) \\ Pa_p^{(i)}(n+p) & Pa_p^{(i)}(n+p+1) & i^{p+2}\cdot Pa_p^{(i)}(n) & i^{p+2}\cdot Pa_p^{(i)}(n+1) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n+p-1) \\ Pa_p^{(i)}(n+p-1) & Pa_p^{(i)}(n+p) & i^{p+2}\cdot Pa_p^{(i)}(n-1) & i^{p+2}\cdot Pa_p^{(i)}(n) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n+p-2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Pa_p^{(i)}(n+1) & Pa_p^{(i)}(n+2) & i^{p+2}\cdot Pa_p^{(i)}(n-p+1) & i^{p+2}\cdot Pa_p^{(i)}(n-p+2) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n) \\ Pa_p^{(i)}(n) & Pa_p^{(i)}(n+1) & i^{p+2}\cdot Pa_p^{(i)}(n-p) & i^{p+2}\cdot Pa_p^{(i)}(n-p+1) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n-1) \end{bmatrix}. \quad (4)$$

Since $\det D_p=i^{p+2}$, the set $\left\langle D_p\right\rangle_m$ is a cyclic group for every positive integer $m\geq 2$. From Theorem 2.1 and the equation (**??**), it is easy to see that $lP_p^i(m)=\left|\left\langle D_p\right\rangle_m\right|$ for any given $p\,(p=3,5,7,\ldots)$.

Clearly,

$$i^{p+2}=\begin{cases} i, & p\equiv -1\,(mod 4), \\ -i, & p\equiv 1\,(mod 4). \end{cases}$$

Since also $\det D_p=i^{p+2}$ and $lP_p^i(m)=\left|\left\langle D_p\right\rangle_m\right|$,

$$\left(i^{p+2}\right)^{lP_p^i(m)}=\left(\det D_p\right)^{lP_p^i(m)}=\det D_p^{lP_p^i(m)}\equiv 1\,(mod m).$$

From this we see that $4\left|lP_p^i(m)\right.$.

The rank of the sequence $\left\{Pa_p^{(i,m)}(n)\right\}$ is the least positive integer $r$ such that $Pa_p^{(i,m)}(r+1)\equiv Pa_p^{(i,m)}(r+2)\equiv Pa_p^{(i,m)}(r+p)\equiv 0(mod m)$, $Pa_p^{(i,m)}(r+p+1)\equiv u(mod m)$ $(u\in\mathbb{C})$, $Pa_p^{(i,m)}(r+p+2)\equiv 0(mod m)$, and we denote the rank of $\left\{Pa_p^{(i,m)}(n)\right\}$ by $RP_p^i(m)$. If $Pa_p^{(i,m)}(r+p+1)\equiv u(mod m)$ $(u\in\mathbb{C})$, then the terms of the sequence

$\left\{Pa_p^{(i,m)}(n)\right\}$ starting with index $RP_p^i(m)$, namely $\underbrace{0,0,\ldots,0}_{p},u,0,-u,0,u,\ldots,$ are exactly the initial terms of $\left\{Pa_p^{(i,m)}(n)\right\}$ multiplied by a factor $u$.

Let the notation $I$ denote the identity matrix of size $(p+2)$. The exponents $n$ for which $\left(D_p\right)^n \equiv I(modm)$ form a simple aritmetic progression. Then we have

$$\left(D_p\right)^n \equiv I(modm) \Longleftrightarrow lP_p^i(m) \mid n.$$

Similarly, the exponents $n$ for which $\left(D_p\right)^n \equiv cI(modm)$ for some $c \in \mathbb{C}$ form a simple aritmetic progression, and hence

$$\left(D_p\right)^n \equiv cI(modm) \Longleftrightarrow RP_p^i(m) \mid n.$$

Consequently, we can see that $RP_p^i(m)$ divides $lP_p^i(m)$ for any given $p\,(p=3,5,7,\ldots)$ and $m \geq 3$.

The order of the sequence $\left\{Pa_p^{(i,m)}(n)\right\}$, $(m \geq 3)$ is defined by $\frac{lP_p^i(m)}{RP_p^i(m)}$ and we denote it by $OP_p^i(m)$. Let $\left(D_p\right)^{RP_p^i(m)} \equiv cI(modm)$ $(c \in \mathbb{C})$, then $ord_m(c)$ is the least positive value of $\lambda$ such that $\left(D_p\right)^{\lambda RP_p^i(m)} \equiv I(modm)$. So it is confirm that $ord_m(c)$ is the least positive integer $\lambda$ with $lP_p^i(m) \mid \lambda RP_p^i(m)$ for $m \geq 3$. As a direct consequence of this we see that the smallest such $\lambda$ is $OP_p^i(m)$ for $m \geq 3$. Therefore, we obtain $OP_p^i(m) = ord_m(c)$, $(m \geq 3)$ when $\left(D_p\right)^{RP_p^i(m)} \equiv cI(modm)$. As a result, we may easily deduce that $OP_p^i(m)$ is always a positive integer, and that $OP_p^i(m) = ord_m\left(Pa_p^{(i)}\left(RP_p^i(m)+p+1\right)\right)$ for $m \geq 3$, the multiplicative order of $Pa_p^{(i,m)}\left(RP_p^i(m)+p+1\right)$.

**Example 2.2.** *The sequence* $\left\{Pa_3^{(i,2)}(n)\right\}$ *is as follows:*

$$\left\{\begin{array}{c} 0,0,0,1,0,1,0,1,i,1,0,1,i,,0,0,0,i,1,i,1,i, \\ 0,0,1,i,0,i,0,0,1,0,0,0,0,i,0,i,0,i,1,i,0, \\ i,1,0,0,0,1,i,1,i,1,0,0,i,1,0,1,0,0,i,0, \\ 0,0,0,1,0,1,0,1,i,\ldots. \end{array}\right\}$$

*Thus it is verified that* $lP_3^i(2) = 62$, $RP_3^i(2) = 31$ *and* $OP_3^i(2) = 2$.

**Example 2.3.** *The sequence* $\left\{Pa_3^{(i,4)}(n)\right\}$ *is as follows:*

$$\left\{\begin{array}{c} 0,0,0,1,0,3,0,1,i,3,2i,1,3i,2,0,0,i,1,i,3,3i,0,2i,3,i,2,3i,0,0,3,2i, \\ 2,2i,2,i,0,i,2,i,1,3i,2,3i,1,2i,0,0,1,i,1,3i,3,2i,0,3i,1,0,1,0,0,i,0, \\ 0,0,0,3,0,1,2i,3,3i,1,2i,3,i,2,0,0,3i,3,3i,1,i,0,2i,1,3i,2,i,0,0,1,2i, \\ 2,2i,2,3i,0,3i,2,3i,3,i,2,i,3,2i,0,0,3,3i,3,i,1,2i,0,i,3,0,3,0,0,3i,0, \\ 0,0,0,1,0,3,0,1,i,\ldots. \end{array}\right\}$$

*Thus it is verified that* $lP_3^i(4) = 124$, $RP_3^i(4) = 62$ *and* $OP_3^i(4) = 2$.

**Theorem 2.4.** *Let $\rho$ be a prime. Then we have the following results for any given $p\,(p=3,5,7,\ldots)$:*

*i. If $t$ is the smallest positive integer such that $lP_p^i\left(\rho^{t+1}\right) \neq lP_p^i\left(\rho^t\right)$, then $lP_p^i\left(\rho^{t+1}\right) = \rho lP_p^i\left(\rho^t\right)$.*

*ii. If $t$ is the smallest positive integer such that $RP_p^i\left(\rho^{t+1}\right) \neq RP_p^i\left(\rho^t\right)$, then $RP_p^i\left(\rho^{t+1}\right) = \rho RP_p^i\left(\rho^t\right)$.*

*Proof.* i. Let $n$ be a positive integer such that $\left(D_p\right)^{lP_p^i\left(\rho^{n+1}\right)} \equiv I(mod\rho^{n+1})$. Then we can easily derive $\left(D_p\right)^{lP_p^i\left(\rho^{n+1}\right)} \equiv I(mod\rho^n)$, which implies that $lP_p^i\left(\rho^{n+1}\right)$ is divided by $lP_p^i\left(\rho^n\right)$. On the other hand, we may write $(D_p)^{lP_p^i(\rho^n)} = I + \left(\left(d_{jk}^{(p)}\right)^n \cdot \rho^n\right)$. Thus, we get the following matrix equation by using binomial expansion

$$(D_p)^{\rho \cdot lP_p^i(\rho^n)} = \left(I + \left(\left(d_{jk}^{(p)}\right)^n \cdot \rho^n\right)\right)^\rho = \sum_{k=0}^{\rho} \binom{\rho}{k} \left(\left(d_{jk}^{(p)}\right)^n \cdot \rho^n\right)^k \equiv I\left(\mathrm{mod}\rho^{n+1}\right),$$

which yields that $\rho \cdot lP_p^i(\rho^n)$ is divided by $lP_p^i\left(\rho^{n+1}\right)$. Hence, $lP_p^i\left(\rho^{n+1}\right) = lP_p^i(\rho^n)$ or $lP_p^i\left(\rho^{n+1}\right) = \rho \cdot lP_p^i(\rho^n)$, and the latter holds if and only if there is a $\left(d_{jk}^{(p)}\right)^n$ which is not divisible by $\rho$. Due to fact that we assume $t$ is the smallest positive integer such that $lP_p^i\left(\rho^{t+1}\right) \neq lP_p^i(\rho^t)$, there is an $\left(d_{jk}^{(p)}\right)^n$ which is not divisible by $\rho$. This shows that $lP_p^i\left(\rho^{t+1}\right) = \rho lP_p^i(\rho^t)$.

ii. The proof is similar to the above and is omitted. □

**Theorem 2.5.** *Let $m_1$ and $m_2$ be positive integers with $m_1, m_2 \geq 2$, then $RP_p^i(lcm[m_1, m_2]) = lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right]$ and $lP_p^i(lcm[m_1, m_2]) = lcm\left[lP_p^i(m_1), lP_p^i(m_2)\right]$ for any given $p$ $(p = 3, 5, 7, \ldots)$.*

*Proof.* Let us consider the ranks $RP_p^i(m_1)$ and $RP_p^i(m_2)$. Suppose that $lcm[m_1, m_2] = m$. Then we may write

$$Pa_p^{(i)}\left(RP_p^i(m_1) + 1\right) \equiv Pa_p^{(i)}\left(RP_p^i(m_1) + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(RP_p^i(m_1) + p\right) \equiv 0(modm),$$

$$Pa_p^{(i)}\left(RP_p^i(m_1) + p + 1\right) \equiv u_1(modm), Pa_p^{(i)}\left(RP_p^i(m_1) + p + 2\right) \equiv 0(modm),$$

$$Pa_p^{(i)}\left(RP_p^i(m_2) + 1\right) \equiv Pa_p^{(i)}\left(RP_p^i(m_2) + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(RP_p^i(m_2) + p\right) \equiv 0(modm),$$

$$Pa_p^{(i)}\left(RP_p^i(m_2) + p + 1\right) \equiv u_2(modm), Pa_p^{(i)}\left(RP_p^i(m_2) + p + 2\right) \equiv 0(modm)$$

and

$$Pa_p^{(i)}\left(RP_p^i(m) + 1\right) \equiv Pa_p^{(i)}\left(RP_p^i(m) + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(RP_p^i(m) + p\right) \equiv 0(modm),$$

$$Pa_p^{(i)}\left(RP_p^i(m) + p + 1\right) \equiv u(modm), Pa_p^{(i)}\left(RP_p^i(m) + p + 2\right) \equiv 0(modm)$$

where $u_1$, $u_2$ and $u$ are complex numbers. Using the least common multiple operation this implies that

$$Pa_p^{(i)}\left(RP_p^i(m) + 1\right) \equiv Pa_p^{(i)}\left(RP_p^i(m) + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(RP_p^i(m) + p\right) \equiv 0(modm_j),$$

$$Pa_p^{(i)}\left(RP_p^i(m) + p + 1\right) \equiv u(modm_j), Pa_p^{(i)}\left(RP_p^i(m) + p + 2\right) \equiv 0(modm_j)$$

for $j = 1, 2$. So we get $RP_p^i(m_1) \mid RP_p^i(m)$ and $RP_p^i(m_2) \mid RP_p^i(m)$, which means that $RP_p^i(lcm[m_1, m_2])$ is divided by $lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right]$. We also know that

$$Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + 1\right) \equiv Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p\right) \equiv 0(modm_j),$$

$$Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p + 1\right) \equiv u_j(modm_j), Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p + 2\right) \equiv 0(modm_j)$$

for $j = 1, 2$. Then we can write

$$Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + 1\right) \equiv Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p\right) \equiv 0(modm),$$

$$Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p + 1\right) \equiv u(modm), Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p + 2\right) \equiv 0(modm),$$

which yields that $lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right]$ is divided by $RP_p^i(lcm[m_1, m_2])$. So we have the conclusion.

There is a similar proof for the periods $lP_p^i(m_1)$ and $lP_p^i(m_2)$. □

## 3. Conclusion

We have examined the complex-type Padovan-$p$ sequence modulo $m$ and then we give some results concerning the periods and ranks of this sequence for any $p$ and $m$. In addition, we have considered the complex-type Padovan-$p$ matrix and we obtained cyclic groups by taking the multiplicative order of this matrix according to $m$. Finally, we have reached that the periods of the complex-type Padovan-$p$ sequence according to modulo $m$ are equal to the order the cyclic groups obtained.

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