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Generalized Shehu Transform to Ψ-Hilfer-Prabhakar Fractional Derivative and its Regularized Version

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Abstract

In this manuscript, authors interested on the generalized Shehu transform of Ψ-Riemann-Liouville, Ψ-Caputo, Ψ-Hilfer fractional derivatives. Moreover, Ψ-Prabhakar, Ψ-Hilfer-Prabhakar fractional derivatives and its regularized version presented in terms of the Ψ-Mittag-Leffler type function. They are also utilised to solve several Cauchy type problems involving Ψ-Hilfer-Prabhakar fractional derivatives and their regularised form, such as the space-time fractional advection-dispersion equation and the generalized fractional free-electron laser (FEL) equation.

Keywords: Ψ-Prabhakar integral Ψ-Hilfer-Prabhakar derivative Ψ-Mittag-Leffler function Ψ-Shehu transform .
2010 MSC: 44A35, 26A33, 33E12.

1. Introduction

Several authors have recently concentrated on fractional calculus and the generalization of integral transformations in the context of Ψ-fractional operators [15, 20, 27]. Due to its unique properties, the Hilfer-Prabhakar fractional derivative operator is used by numerous academics to model physical phenomena, especially when combined with several integral transforms presented in the literature of fractional differentiations and integrations such as Fourier, Elzaki, Laplace and others. Integral transforms techniques are important because they give an efficient approach for solving a variety of mathematical models, initial value

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problems, and boundary value problems that exist in differential equations [6]. Ghadle et al. [8] proposed a novel Sumudu-type integral transform, applied it to conformable derivative, and solved some applications involving conformable derivative. Sausa and Oliviera introduced the Ψ-Hilfer fractional derivative as a novel fractional derivative in the situation of the Ψ-fractional operator in [27]. Magar et al. recently published a work in [20] in which they gave numerous innovative concepts of fractional derivatives in the context of the Ψ-fractional operator, like a Ψ-Prabhakar integral, Ψ-Prabhakar derivative, Ψ-Hilfer-Prabhakar fractional derivatives and its regularized version in terms of Ψ-Mittag-Leffler function and applied generalised integral transforms such as Ψ-Laplace and Ψ-Sumudu to it.

The Shehu transform introduced by Maitama and Zhao [21] is an extension of of the Laplace and Sumudu integral transforms with some useful properties. Recently, the authors of [2] and [3] applied the Shehu transform to obtain solutions of differential equations involving Caputo and Atangana-Baleanu fractional derivative.

The major goal of this study is to introduce the generalized Shehu transform called the Ψ-Shehu transform and provide its useful properties based on Ψ-function such as Ψ-Riemann-Liouville, Ψ-Caputo, Ψ-Hilfer, Ψ-Prabhakar integral, Ψ-Prabhakar derivative, Ψ-Hilfer-Prabhakar fractional derivatives and its regularized version in terms of Ψ-Mittag-Leffler function. The Ψ-Shehu transform applied to solved some Cauchy type problems, such as the space-time fractional advection-dispersion equation and the generalised fractional free electron laser (FEL) equation, where the Ψ-Hilfer-Prabhakar fractional derivative and its regularised version are involved.

2. Preliminaries

Definition 2.1. *[1, 11, 12, 13, 14, 17, 18, 25] Let $\mu \in \mathbb{R}^+$ such that $-\infty \leq a < b \leq \infty$, $m = \mu + 1$, f be an integrable function on $[a, b]$ and $\Psi \in C^1([a, b])$ be increasing function such that $\Psi'(t) \neq 0$ for all $t \in [a, b]$. Then, the Ψ-Riemann-Liouville fractional integral is defined as*

$$I_0^{\mu,\Psi} f(t) = \frac{1}{\Gamma(\mu)} \int_0^{\infty} (\Psi(t) - \Psi(r))^{\mu-1} \Psi'(r) f(r) dr \tag{1}$$

The Ψ-Riemann-Liouville fractional derivative is defined as

$$D_0^{\mu,\Psi} = \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^m I_0^{m-\mu,\Psi} f(t) \tag{2}$$

The Ψ-Caputo fractional derivative [27] is defined as

$$^C D_0^{\mu,\Psi} = I_0^{m-\mu,\Psi} f(t) \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^m \tag{3}$$

The Ψ-Hilfer fractional derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by

$$D_0^{\mu,\nu,\Psi} = I_0^{\nu(m-\mu,\Psi)} \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^m I_0^{(1-\nu)(m-\mu),\Psi} f(t). \tag{4}$$

The Ψ-Prabhakar fractional integral and derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by

$$\begin{aligned} \left(P_{\alpha,\mu,\omega}^{\gamma,\Psi} f \right)(x) &= \int_0^x (\Psi(x) - \Psi(t))^{\mu-1} E_{\alpha,\mu}^{\gamma} [\omega(\Psi(x) - \Psi(t))^{\alpha}] \Psi'(t) f(t) dt, \\ &= (\varepsilon_{\alpha,\mu,\omega}^{\gamma} *_{\Psi} f)(x). \end{aligned} \tag{5}$$

where $*_{\Psi}$ denotes the convolution operation; $\alpha, \mu, \omega, \gamma \in \mathcal{C}$; $Re(\alpha) > 0$, $Re(\mu) > 0$ and

$$\varepsilon_{\alpha,\mu,\omega}^{\gamma} \Psi(t) = \begin{cases} (\Psi(t))^{\mu-1} E_{\alpha,\mu}^{\gamma} (\omega(\Psi(t))^{\alpha}) & t > 0, \\ 0 & t \leq 0. \end{cases} \tag{6}$$

For $\gamma = 0$, $\left(P^{0}_{\alpha,\mu,\omega} f\right)(x) = \left(I^{\mu,\Psi} f\right)(x)$ *and for* $\gamma = \mu = 0$, $\left(P^{0}_{\alpha,0,\omega} f\right)(x) = f(x)$.

$$D^{\gamma,\Psi}_{\rho,\mu,\omega} f(t) = \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{m} P^{-\gamma,\Psi}_{\rho,m-\mu,\omega} f(t). \tag{7}$$

The regularized version of Ψ*-Prabhakar fractional derivative of a function* f *of order* μ *is defined as*

$$^{C}D^{\gamma,\Psi}_{\rho,\mu,\omega} f(t) = P^{-\gamma,\Psi}_{\rho,m-\mu,\omega}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{m} f(t). \tag{8}$$

The Ψ*-Hilfer-Prabhakar fractional derivative of a function* f *of order* μ *defined as*

$$\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = \left(P^{-\gamma\nu,\Psi}_{\alpha,\nu(m-\mu),\omega,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{m}\left(P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(m-\mu,\omega,0^+} f\right)\right)(t). \tag{9}$$

The regularized version of Ψ*-Hilfer-Prabhakar fractional derivative of a function* f *of order* μ *is defined as,*

$$^{C}\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = \left(P^{-\gamma\nu,\Psi}_{\alpha,\nu(m-\mu),\omega,0^+} P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(m-\mu,\omega,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{m} f\right)(t). \tag{10}$$

Definition 2.2. *[7, 19, 22, 23, 26] The Prabhakar function is defined by*

$$E^{\gamma}_{\mu,\nu}(z) = \frac{1}{\Gamma(\gamma)}\sum_{i=0}^{\infty}\frac{\Gamma(\gamma+i)z^{i}}{i!\Gamma(\mu i+\nu)}, \qquad \mu,\nu,\gamma\in\mathbb{C}, \quad Re(\mu) > 0. \tag{11}$$

It is an entire function of order $\frac{1}{Re(\mu)}$, which is also known as three parameter Mittag-Leffler function.

Definition 2.3. *[21, 24, 28] The Shehu transform of the function* $v(t)$ *of exponential order is defined over the set of functions,*

$$A = \{v(t) : \exists Q, \tau_1, \tau_2 \geq 0, |v(t)| \leq Qe^{|t|/\tau_j},\ \textit{if}\ t \in (-1)^{j}\times[0,\infty)\},$$

by the following integral

$$\mathbb{S}[v(t)] = V(r,Z) = \int_0^{\infty} exp\left(-\frac{st}{Z}\right)v(t)dt; \quad r>0, Z>0.$$

Theorem 2.4. *[4] Let* f *and* g *in* A *be piecewise continuous functions of* Psi *order over* $[0,T]$*. The* Ψ*-convolution of* f *and* g *is therefore defined by*

$$(f *_{\Psi} g) = \int_0^{t=\Psi^{-1}(\Psi(t))} f(\Psi^{-1}(\Psi(t)-\Psi(\tau)))g(\tau)\Psi^{'}(\tau)d\tau. \tag{12}$$

Definition 2.5. *[16] A function* $f : [0,\infty) \to \mathbb{R}$ *is of* $\Psi-$*exponential order* $c > 0$ *if there exist positive constant* Q *such that for all* $t > T$

$$|f(t)| \leq Qe^{c\Psi(t)} \tag{13}$$

Symbolically, we write

$$f(t) = \vartheta(e^{c\Psi(t)}) \quad as \quad t\to\infty.$$

Lemma 2.6. *[2] In the complex plane* $\mathbb{C}$*, for any* $R_e(\alpha)$*,* $R_e(\beta) > 0$*,* $R_e(\gamma) > 0$ *and* $\omega \in \mathbb{C}$*, Shehu transform of* $E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha})$ *is given by*

$$\mathbb{S}\left(t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha})\right) = \left(\frac{Z}{r}\right)^{\beta}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}. \tag{14}$$

3. Main Results

Definition 3.1. *[4] Let $v(t) : [0,\infty) \to \mathbb{R}$ be a real valued function and Ψ be a non-negative increasing function such that $\Psi(0) = 0$. Then the Ψ-Shehu transform of the function $v(t)$ of exponential order is denoted by $\mathbb{S}_\Psi\{v(t)\}$ and is defined by*

$$\mathbb{S}_\Psi[v(t)] = V(r,Z) = \int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big)v(t)\Psi^{'}(t)dt; \qquad r>0, Z>0. \tag{15}$$

Theorem 3.2. *If $v(t)$ is a piecewise continuous in every finite interval $0 \le t \le \mu$ and of Ψ-exponential order, then the $\Psi-$Shehu transform of $v(t)$ exists for $t > c > 0$.*

Proof. For any positive number c, by definition of Ψ-Shehu transform and using equation (13) we have,

$$\begin{aligned}
|\mathbb{S}_\Psi f(t)| &= |\int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big)v(t)\Psi^{'}(t)dt| \\
&\le \int_0^\infty |exp\Big(-\frac{r\Psi(t)}{Z}\Big)v(t)\Psi^{'}(t)|dt \\
&= \int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big)|v(t)|\Psi^{'}(t)dt \\
&\le Q\int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big)\Psi^{'}(t)e^{c\Psi(t)}dt \\
&= \frac{ZQ}{r-cZ}.
\end{aligned}$$

The proof is complete. □

Property 1. Let the functions $\eta v(t)$ and $\xi w(t)$ be in set A, then $(\eta v(t) + \xi w(t)) \in A$, where η and ξ are nonzero arbitrary constants, and

$$\mathbb{S}_\Psi[(\eta v(t) + \xi w(t))] = \eta\mathbb{S}_\Psi[v(t)] + \xi\mathbb{S}_\Psi[w(t)].$$

Proof. By definition of Ψ-Shehu transform, we get

$$\begin{aligned}
\mathbb{S}_\Psi[(\eta v(t) + \xi w(t))] &= \int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big)(\eta v(t) + \xi w(t))\Psi^{'}(t)dt \\
&= \int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big)(\eta v(t))\Psi^{'}(t)dt \\
&+ \int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big)(\xi w(t))\Psi^{'}(t)dt \\
&= \eta\int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big)v(t)\Psi^{'}(t)dt \\
&+ \xi\int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big)w(t)\Psi^{'}(t)dt \\
&= \eta\mathbb{S}_\Psi[v(t)] + \xi\mathbb{S}_\Psi[w(t)]
\end{aligned}$$

□

Property 2. Let the function $v(t) = \frac{\Psi(t)^{\mu-1}}{\Gamma(\mu-1)}$. Then its Ψ-Shehu transform is given by

$$\mathbb{S}_\Psi\left[\frac{\Psi(t)^{\mu-1}}{\Gamma(\mu-1)!}\right] = \Big(\frac{Z}{r}\Big)^\mu\Gamma(\mu),\ \mu = 0,1,2,... \tag{16}$$

Proof. By definition of Ψ-Shehu transform and integration by parts, we get

$$\begin{aligned}\mathbb{S}_{\Psi}[\Psi(t)^{\mu-1}] &= \int_0^{\infty} \Psi(t)^{\mu-1} exp\bigg(\frac{-r\Psi(t)}{Z}\bigg)\Psi^{'}(t)dt \\ &= \frac{Z}{r}(\mu-1)\int_0^{\infty} \Psi(t)^{\mu-2} exp\bigg(\frac{-r\Psi(t)}{Z}\bigg)\Psi^{'}(t)dt \\ &= \bigg(\frac{Z}{r}\bigg)^2(\mu-1)(\mu-2)\int_0^{\infty} \Psi(t)^{\mu-3} exp\bigg(\frac{-r\Psi(t)}{Z}\bigg)\Psi^{'}(t)dt \\ &= \bigg(\frac{Z}{r}\bigg)^3(\mu-1)(\mu-2)(\mu-3)\int_0^{\infty} \Psi(t)^{\mu-4} exp\bigg(\frac{-r\Psi(t)}{Z}\bigg)\Psi^{'}(t)dt \\ &= ... = (\mu-1)!\bigg(\frac{Z}{r}\bigg)^{\mu} \\ &= \bigg(\frac{Z}{r}\bigg)^{\mu}\Gamma(\mu)\end{aligned}$$

□

Property 3. Let the function $v(t) = exp(\mu\Psi(t))$. Then its Ψ-Shehu transform is given by

$$\mathbb{S}_{\Psi}[exp(\mu\Psi(t))] = \frac{Z}{r-\mu Z}$$

Proof. By definition of Ψ-Shehu transform, we get

$$\begin{aligned}\mathbb{S}_{\Psi}[exp(\mu\Psi(t))] &= \int_0^{\infty} exp\bigg(-\frac{r\Psi(t)}{Z}\bigg)exp(\mu\Psi(t))\Psi^{'}(t)dt \\ &= \int_0^{\infty} exp\bigg(-\frac{\big(r-\mu Z\big)\Psi(t)}{Z}\bigg)\Psi^{'}(t)dt \\ &= \frac{Z}{r-\mu Z}\end{aligned}$$

□

Property 4. Let the function $v(t) = \Psi(t)exp(\mu\Psi(t))$. Then its Ψ-Shehu transform is given by

$$\mathbb{S}_{\Psi}[\Psi(t)exp(\mu\Psi(t))] = \frac{u^2}{(r-\mu Z)^2}$$

Proof. By definition of Ψ-Shehu transform and integration by parts, we get

$$\begin{aligned}\mathbb{S}_{\Psi}[\Psi(t)exp(\mu\Psi(t))] &= \int_0^{\infty} exp\bigg(-\frac{r\Psi(t)}{Z}\bigg)\Psi(t)exp(\mu\Psi(t))\Psi^{'}(t)dt \\ &= \int_0^{\infty} \Psi(t)exp\bigg(-\frac{(r-\mu Z)\Psi(t)}{Z}\bigg)\Psi^{'}(t)dt \\ &= -\frac{Z}{(r-\mu Z)}\lim_{\gamma\to\infty}\bigg[\Psi(t)exp\bigg(-\frac{(r-\mu Z)\Psi(t)}{Z}\bigg)\bigg]_0^{\gamma} \\ &\quad + \frac{Z}{(r-\mu Z)}\int_0^{\infty} exp\bigg(\frac{(r-\mu Z)\Psi(t)}{Z}\bigg)\Psi^{'}(t)dt \\ &= -\frac{Z^2}{(r-\mu Z)^2}\lim_{\gamma\to\infty}\bigg[exp\bigg(-\frac{(r-\mu Z)\Psi(t)}{Z}\bigg)\bigg]_0^{\gamma} \\ &= \frac{Z^2}{(r-\mu Z)^2}\end{aligned}$$

The proof is complete. □

Property 5. Let the function $v(t) = sin(\mu\Psi(t))$. Then its Ψ-Shehu transform is given by

$$\mathbb{S}_\Psi[sin(\mu\Psi(t))] = \frac{\mu Z^2}{r^2 + \mu^2 Z^2}.$$

Proof. By definition of Ψ-Shehu transform, we get

$$\begin{aligned}\mathbb{S}_\Psi[sin(\mu\Psi(t))] &= \int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big) sin(\mu\Psi(t))\Psi^{'}(t)dt \\ &= \lim_{\gamma\to\infty}\left[\frac{e^{-\frac{-r\Psi(t)}{Z}}}{\frac{r^2}{Z^2}+\mu^2}\Big\{-\frac{r}{Z}sin\mu\Psi(t) - \mu cos\mu\Psi(t)\Big\}\right]_0^\gamma \\ &= \frac{\mu Z^2}{r^2+\mu^2 Z^2}\end{aligned}$$

□

Property 6. Let the function $v(t) = cos(\mu\Psi(t))$. Then its Ψ-Shehu transform is given by

$$\mathbb{S}_\Psi[cos(\mu\Psi(t))] = \frac{Zr}{r^2 + \mu^2 Z^2}$$

Proof. By definition of Ψ-Shehu transform, we get

$$\begin{aligned}\mathbb{S}_\Psi[cos(\mu\Psi(t))] &= \int_0^\infty exp\Big(-\frac{r\Psi(t)}{Z}\Big) cos(\mu\Psi(t))\Psi^{'}(t)dt \\ &= \lim_{\gamma\to\infty}\left[\frac{e^{-\frac{-r\Psi(t)}{Z}}}{\frac{r^2}{Z^2}+\mu^2}\Big\{-\frac{r}{Z}cos\mu\Psi(t) + \mu sin\mu\Psi(t)\Big\}\right]_0^\gamma \\ &= \frac{rZ}{r^2+\mu^2 Z^2}\end{aligned}$$

□

Lemma 3.3. *Let $f(t)$ and $g(t)$ be a functions in A, and $\mathbb{S}_\Psi[f(t)](Z) = V(r, Z)$, $\mathbb{S}_\Psi[g(t)](Z) = W(r, Z)$ where, Ψ-Shehu transforms $V(r, Z)$ and $W(r, Z)$ respectively. The Ψ-Shehu transform of convolution $(f *_\Psi g)$ is given by*

$$\mathbb{S}_\Psi[(f *_\Psi g)(t)](Z) = V(r, Z)W(r, Z). \tag{17}$$

Proof. By definition of Ψ-Shehu transform and Ψ-convolution equation (12), we get,

$$\begin{aligned}&\mathbb{S}_\Psi[(f *_\Psi g)](t) \\ &= \int_0^\infty e^{-\frac{r\Psi(t)}{Z}}\Big\{\int_0^t f(\Psi^{-1}(\Psi(t)-\Psi(\tau)))g(\tau)\Psi^{'}(\tau)d\tau\Big\}\Psi^{'}(t)dt \\ &= \int_0^\infty e^{-\left[\frac{r(\Psi(t)-\Psi(\tau)+\Psi(\tau))}{Z}\right]}\Psi^{'}(t)\int_0^\infty f(\Psi^{-1}(\Psi(t)-\Psi(\tau)))g(\tau)\Psi^{'}(\tau)d\tau dt \\ &= \int_0^\infty e^{-\frac{r\Psi(t)-\Psi(\tau)}{Z}}e^{-\frac{r\Psi(\tau)}{Z}}\Psi^{'}(t)dt\int_0^\infty f(\Psi^{-1}(\Psi(t)-\Psi(\tau)))g(\tau)\Psi^{'}(\tau)d\tau\end{aligned}$$

by changing order of integration and substituting above we get,

$$\begin{aligned}&= \int_0^\infty e^{-\frac{r\Psi(t)}{Z}}g(\tau)\Psi'(\tau)d\tau\int_0^\infty e^{-\frac{r\Psi(\nu)}{Z}}\Psi^{'}(\nu)f(\nu)d\nu \\ &= \mathbb{S}_\Psi[(f)]\mathbb{S}_\Psi[(g)] \\ &= V(r, Z)W(r, Z)\end{aligned}$$

□

4. Ψ-Shehu transform of generalized fractional derivatives and its regularized versions

In this section, we obtain the generalized Shehu transform for Ψ−Riemann-Liouville, Ψ−Caputo Ψ-Hilfer, Ψ-Prabhakar, Ψ-Hilfer-Prabhakar fractional derivatives and its regularized versions.

Lemma 4.1. *The Ψ-Shehu transform of Ψ-Riemann-Liouville fractional integral equation* (1) *is*

$$\mathbb{S}_\Psi\left\{(I_0^{\mu,\Psi}f)(v(t))\right\} = \left(\frac{Z}{r}\right)^{\mu}\mathbb{S}_\Psi\{f(v(t))\}.$$

Proof. By definition of Ψ-Shehu transform of Ψ-Riemann-Liouville fractional integral (1) and using (16) and (17), we get

$$\begin{aligned}\mathbb{S}_\Psi\{I^{\mu,\Psi}(t)\}(Z) &= \frac{1}{\Gamma(\mu)}\mathbb{S}_\Psi\{\Psi(t)^{\mu-1} *_\Psi f(v(t))\}(Z,r)\\ &= \frac{1}{\Gamma(\mu)}\left(\frac{Z}{r}\right)^{\mu}\Gamma(\mu)\mathbb{S}_\Psi\{f(v(t))\}\\ &= \left(\frac{Z}{r}\right)^{\mu}\mathbb{S}_\Psi f(v(t))\end{aligned}$$

□

Lemma 4.2. *The Ψ-Shehu transform of Ψ-Riemann-Liouville fractional derivative* (2) *is*

$$\mathbb{S}_\Psi\left\{D_0^{\mu,\Psi}f(V(t))\right\} = \left(\frac{Z}{r}\right)^{-\mu}\mathbb{S}_\Psi\{f(v(t))\} - \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-m+k+1}\left(I^{m-k-\mu,\Psi}f\right)(0).$$

Proof. By definition of Ψ-Shehu transform of Ψ-Riemann-Liouville fractional derivative (2), we get

$$\begin{aligned}\mathbb{S}_\Psi\{D^{\mu,\Psi}f(v(t))\}(r,Z) &= \mathbb{S}_\Psi\left\{\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{m}\left(I^{m-\mu,\Psi}f\right)(v(t))\right\}\\ &= \left(\frac{Z}{r}\right)^{-m}\mathbb{S}_\Psi[I^{m-\mu,\Psi}f(v(t))]\\ &- \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-m+k+1}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{k}\left(I^{m-\mu,\Psi}f\right)(0)\\ &= \left(\frac{Z}{r}\right)^{-\mu}\mathbb{S}_\Psi\{f(v(t))\} - \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-m+k+1}\left(I^{m-k-\mu,\Psi}f\right)(0).\end{aligned}$$

□

Lemma 4.3. *The Ψ-Shehu transform of Ψ-Caputo fractional derivative* (3) *is*

$$\mathbb{S}_\Psi\left\{{}^{C}D_0^{\mu,\Psi}f(v(t))\right\} = \left(\frac{Z}{r}\right)^{-\mu}\mathbb{S}_\Psi\{f(v(t))\} - \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-\mu+k+1}(D^{k,\Psi})f(v)^{(k)}(0).$$

Proof. By definition of Ψ-Shehu transform of Ψ-Caputo fractional derivative (3), and using (16), we get

$$\begin{aligned}\mathbb{S}_\Psi\left\{{}^C D_0^{\mu,\Psi} f(v(t))\right\} &= \mathbb{S}_\Psi\left\{I^{m-\mu,\Psi}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m f(v(t))\right\}\\ &= \left(\frac{Z}{r}\right)^{m-\mu}\left\{\mathbb{S}_\Psi\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m f(t)\right\}\\ &= \left(\frac{Z}{r}\right)^{m-\mu}\left\{\left(\frac{Z}{r}\right)^{-m} V(r,Z) - \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-m+k+1} f_\Psi^{(k)}(0)\right\}\end{aligned}$$

where,

$$\begin{aligned}f(v(t))_\Psi^{(k)} &= \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k f(v(t))\\ &= \left(\frac{Z}{r}\right)^{-\mu} V(r,Z) - \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-\mu+k+1}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k f_\Psi^k(0)\\ &= \left(\frac{Z}{r}\right)^{-\mu} S_\Psi\{f(t)\} - \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-\mu+k+1}(D^{k,\Psi}) f_\Psi^k(0).\end{aligned}$$

□

Lemma 4.4. *The Ψ-Shehu transform of Ψ-Hilfer fractional derivative equation* (4)*.*

$$\begin{aligned}\mathbb{S}_\Psi\left\{D_0^{\mu,\nu,\Psi} f(v(t))\right\} = &\left(\frac{Z}{r}\right)^{-\mu} S_\Psi\{f(v(t))\}\\ &- \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{m(\nu-1)-\nu\mu-\mu+k+1}(I_0^{(1-\nu)(m-\mu)-k,\Psi} f(v))(0).\end{aligned}$$

Proof. By definition of Ψ-Shehu transform of Ψ-Hilfer fractional derivative (4), and using (16) and (17), we get

$$\begin{aligned}\mathbb{S}_\Psi\left\{D_0^{\mu,\nu,\Psi} f(v(t))\right\} =& \mathbb{S}_\Psi\left\{I_0^{\nu(m-\mu,\Psi)}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m I_0^{(1-\nu)(m-\mu),\Psi} f(v(t))\right\}.\\ \mathbb{S}_\Psi\left\{D_0^{\mu,\nu,\Psi} f(t)\right\} =& \left(\frac{Z}{r}\right)^{\nu(m-\mu)}\mathbb{S}_\Psi\left\{\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m I_0^{(1-\nu)(m-\mu),\Psi} f(v(t))\right\}\\ =& \left(\frac{Z}{r}\right)^{\nu(m-\mu)}\left[\left(\frac{Z}{r}\right)^{-m}\mathbb{S}_\Psi\left\{I_0^{(1-\nu)(m-\mu),\Psi} f(t)\right\}\right.\\ &\left.- \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-m+k+1}(D^{k,\Psi} I_0^{(1-\nu)(m-\mu),\Psi} f)(0)\right]\\ =& \left(\frac{Z}{r}\right)^{\nu(m-\mu)}\left[\left(\frac{Z}{r}\right)^{-m}\left(\frac{Z}{r}\right)^{(1-\nu)(m-\mu)}\mathbb{S}_\Psi\left\{I_0^{(1-\nu)(m-\mu),\Psi} f(v(t))\right\}\right.\\ &\left.- \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-m+k+1}(D^{k,\Psi} I_0^{(1-\nu)(m-\mu),\Psi} f)(0)\right]\\ =& \left(\frac{Z}{r}\right)^{-\mu}\mathbb{S}_\Psi\left\{f(v(t))\right\}\\ &- \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{m(\nu-1)-\nu\mu+k+1}(I_0^{(1-\nu)(m-\mu)-k,\Psi} f)(0)\end{aligned}$$

□

Lemma 4.5. *The Ψ-Shehu transform of Ψ-Prabhakar fractional integral* (5)

$$\mathbb{S}_\Psi\left\{\left(P^{\gamma,\Psi}_{\alpha,\mu,\omega} *_\Psi f\right)v(t)\right\} = \left(\frac{Z}{r}\right)^{\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{-\gamma} V(r,Z). \tag{18}$$

Proof. By definition of Ψ-Shehu transform of Ψ-Prabhakar fractional integral (5), and using (6), (16) and (17), we get

$$\begin{aligned}
\mathbb{S}_\Psi\left\{\left(P^{\gamma,\Psi}_{\alpha,\mu,\omega} *_\Psi f\right)v(t)\right\} &= \mathbb{S}_\Psi\left\{\left(\epsilon^{\gamma,\Psi}_{\alpha,\mu,\omega} *_\Psi f\right)v(t)\right\}\\
\mathbb{S}_\Psi\left\{\epsilon^{\gamma,\Psi}_{\alpha,\mu,\omega}\Psi(t) *_\Psi f\right\} &= \mathbb{S}_\Psi\left\{\Psi(t)^{\mu-1}E^{\gamma,\Psi}_{\alpha,\mu,}\omega(\Psi(t)^{\alpha})\right\}\\
&= \sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\mu)}\frac{\omega^k}{n!}\mathbb{S}_\Psi\left\{\Psi(t)^{\alpha k+\mu-1}\right\}\\
&= \sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\mu)}\frac{\omega^k}{n!}\Gamma(\alpha k+\mu)\left(\frac{Z}{r}\right)^{\alpha k+\mu}\\
&= \left(\frac{Z}{r}\right)^{\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{-\gamma} V(r,Z).
\end{aligned}$$

□

Lemma 4.6. *The Ψ-Shehu transform of Ψ-Prabhakar fractional derivative* (7) *is given by*

$$\begin{aligned}
\mathbb{S}_\Psi\left\{D^{\gamma,\Psi}_{\rho,\mu,\omega}f(t)\right\}(r,Z) &= \left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma} V(r,Z)\\
&\quad -\sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-m+k+1}\left(P^{\gamma,\Psi}_{\alpha,(m-\mu)-k,\omega}f\right)(0).
\end{aligned} \tag{19}$$

For the case $[\mu]+1=m=1$,

$$\mathbb{S}_\Psi\left(D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\right)(r) = \left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma} V(r,Z) - \left[P^{-\gamma,\Psi}_{\alpha,(1-\mu),\omega}f(t)\right]_{t=0^+}. \tag{20}$$

with $|\omega(r)^{-\alpha}|<1$.

Proof. By definition of Ψ-Shehu transforms of Ψ-Prabhakar fractional derivative in (7) and using (17) and (18), we get,

$$\begin{aligned}
\mathbb{S}_\Psi\left(D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\right)(r,Z) &= \mathbb{S}_\Psi\left(\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega}f(t)\right)(r,Z),\\
&= \left(\frac{r}{Z}\right)^m \mathbb{S}_\Psi\left(\left(\varepsilon^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega}*f\right)(t)\right)(r,Z)\\
&- \sum_{k=0}^{m-1}\left(\frac{r}{Z}\right)^{m-k-1}\left[\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega}f(t)\right]_{t=o+},\\
&= \left(\frac{r}{Z}\right)^m \mathbb{S}_\Psi\left((\Psi(t))^{(m-\mu)-1}E^{-\gamma,\Psi}_{\alpha,(m-\mu)}\left(\omega(\Psi(t))^\alpha\right)V(r,Z)\right)\\
&- \sum_{k=0}^{m-1}\left(\frac{r}{Z}\right)^{m-k-1}\left[\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega}f(0^+)\right],\\
&= \left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma V(r,Z)\\
&- \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-m+k+1}\left(P^{-\gamma,\Psi}_{\alpha,(m-\mu)-k,\omega}f(0^+)\right).
\end{aligned}$$

For the case $[\mu]+1=m=1$, we have,

$$\mathbb{S}_\Psi\left(D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\right)(r,Z) = \left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma V(r,Z) - \left[P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega}f(t)\right]_{t=o+}.$$

□

Lemma 4.7. *The Ψ-Shehu transform of regularized version of Ψ-Prabhakar fractional derivative equation* (8) *is,*

$$\begin{aligned}
\mathbb{S}_\Psi\left({}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\right)(r,Z) &= \left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma V(r,Z)\\
&- \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-\mu+k+1}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma f^{(k)}(0^+),
\end{aligned} \tag{21}$$

with $|\omega\left(\frac{Z}{r}\right)^\alpha| < 1$.

Proof. Taking Ψ-Shehu transform of regularized version of Ψ-Prabhakar fractional derivative in (8) and using (17) and (18), we get

$$\begin{aligned}
\mathbb{S}_\Psi\left({}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\right)(r,Z), &= \mathbb{S}_\Psi\left(\varepsilon^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega}\left(\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m f(t)\right)\right)(r,Z)\\
&= \mathbb{S}_\Psi\left(\varepsilon^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} *_\Psi \left(\frac{1}{\Psi'(t)}\frac{d}{dt}f\right)^m(t)\right)(r,Z),\\
&= \left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma V(r,Z)\\
&- \sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-\mu+k+1}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma f^k(0^+).
\end{aligned}$$

□

Lemma 4.8. *The Ψ-Shehu transform of Ψ-Hilfer-Prabhakar fractional derivative equation* (9) *is,*

$$\mathbb{S}_\Psi\left(\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\right)(r,Z)=\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}V(r,Z)$$
$$-\sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{m(\nu-1)-\nu\mu+k+1}[1-\omega\left(\frac{Z}{r}\right)^{\alpha}]^{\gamma\nu}$$
$$\times\left(P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(m-\mu)-k,\omega}f(0^+)\right). \tag{22}$$

Proof. By definition of Ψ-Shehu transform of Ψ-Hilfer-Prabhakar fractional derivative in (9) and using (17), (18) we have,

$$\mathbb{S}_\Psi\left(\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\right)(r,Z)$$
$$=\mathbb{S}_\Psi\left[\left(\varepsilon^{-\gamma\nu}_{\alpha,\nu(m-\mu),\omega}*_\Psi\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m\left(P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(m-\mu),\omega,0^+}f\right)\right)(t)\right](r,Z),$$
$$=\left(\frac{r}{Z}\right)^m\left(\frac{r}{Z}\right)^{-\nu(m-\mu)}\left(1-\omega\left(\frac{r}{Z}\right)^{-\alpha}\right)^{\gamma\nu}\mathbb{S}_\Psi\left(\left(\varepsilon^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(m-\mu),\omega}*f\right)(t)\right)(r,Z)$$
$$-\sum_{k=0}^{m-1}\left(\frac{r}{Z}\right)^m\left(\frac{r}{Z}\right)^{-\nu(m-\mu)}\left(1-\omega\left(\frac{r}{Z}\right)^{-\alpha}\right)^{\gamma\nu}\left[P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu),(m-\mu)-k,\omega}f(0^+)\right],$$
$$=\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}V(r,Z)$$
$$-\sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{m(\nu-1)-\nu\mu+k+1}\left[1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right]^{\gamma\nu}\left(P^{-\gamma(1-\nu)(m-\mu),\psi}_{\alpha,(1-\nu)(m-\mu)-k,\omega}f(0^+)\right).$$

□

Lemma 4.9. *The Ψ-Shehu transforms of the regularized version of Ψ-Hilfer-Prabhakar fractional derivative equation* (10) *of order* μ *is,*

$$\mathbb{S}_\Psi\left({}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\right)(r,Z)=\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}V(r,Z)$$
$$-\sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-\mu+k+1}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}f^k(0^+).$$

Proof. By definition of Ψ-Shehu transforms of regularized version of Ψ-Hilfer-Prabhakar fractional derivative in (10) and using (17), (18) we have,

$$\mathbb{S}_\Psi\left({}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\right)(r)$$
$$=\mathbb{S}_\Psi\left(\left(\varepsilon^{-\gamma\nu}_{\alpha,\nu(m-\mu),\omega}*_\Psi\left(P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(k-\mu),\omega,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^m f\right)\right)(t)\right)(r,Z),$$
$$=\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}V(r,Z)$$
$$-\sum_{k=0}^{m-1}\left(\frac{Z}{r}\right)^{-\mu+k+1}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}f^k(0^+).$$

□

5. Applications

In this section, we using Ψ-Shehu transform to find the solutions of Cauchy problems such as space-time fractional convection-dispersion equation and the generalized fractional free electron laser (FEL) equation involving Ψ-Hilfer-Prabhakar fractional derivative with order $\mu \in (0,1)$ [9, 10].

Theorem 5.1. *The solution of Cauchy problem*

$$\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}(Z(r,t)) = -\beta D_x Z(r,t) + \rho\Delta^{\frac{\lambda}{2}}(Z(r,t)), \tag{23}$$

Subject to constraints

$$\left(P^{-\gamma(1-\nu)(m-\mu),\psi}_{\alpha,(1-\nu)(m-\mu)-k,\omega} Z(r,0^+)\right) = g(x), \quad \omega,\gamma,x \in \mathcal{R}, \quad \alpha \geq 0 \tag{24}$$

$$\lim_{x\to\infty} Z(r,t) = 0, \quad t \geq 0 \tag{25}$$

is

$$Z(r,t) = \sum_{m=0}^{\infty} \frac{t^{\nu(1-\mu)+m\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{ikx} g(k)(i\beta k - \rho|k|^{\lambda})^m P^{\gamma,(\beta-\nu)}_{\alpha,\nu(1-\mu)+m\mu}(\omega t^{\alpha})dk \tag{26}$$

Proof. Applying Fourier transform of (23) with respect to space variable x, we get

$$\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}(Z^*(k,t)) = \beta ik Z^*(k,t) - \rho|k|^{\lambda} Z^*(k,t), \tag{27}$$

Fourier transform of $\Delta^{\frac{\lambda}{2}}$ is given in [5], as

$$F\{\Delta^{\frac{\lambda}{2}} Z(r,t); k\} = -|k|^{\lambda} F\{Z(r,t)\}, \qquad \lambda \in (0,2),$$

Taking Ψ-Shehu transform on left hand side of (27) with respect to space variable t and using (22), we get

$$\begin{aligned}\mathbb{S}_{\Psi}\left\{\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} Z^*(k,t)\right\} &= \left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma} V^*(k,s,Z) \\ &\quad - \left(\frac{Z}{r}\right)^{\nu(1-\mu)}\left[1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right]^{\gamma\nu}\left[P^{-\gamma(1-\nu),\psi}_{\alpha,(1-\nu)(1-\mu),\omega} Z^*(k,t)\right]_{t=0}.\end{aligned}$$

Where, $V^*(k,s,Z)$ represent Ψ-Shehu transform of $Z^*(k.t)$.

Again, apply Ψ-Shehu transform on right hand side of (27) and using initial condition (25), we get

$$\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}V^{*}(k,s,Z)$$
$$-\left(\frac{Z}{r}\right)^{\nu(1-\mu)}\left[1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right]^{\gamma\nu}g(k)=\beta ik-\rho|k|^{\lambda}V^{*}(k,s,Z)$$
$$\left(\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}-\beta ik+\rho|k|^{\lambda}\right)V^{*}(k,s,Z)$$
$$=\left(\frac{Z}{r}\right)^{-\nu(1-\mu)}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma\nu}g(k)$$
$$V^{*}(k,s,Z)=\left(\frac{Z}{r}\right)^{-\nu(1-\mu)}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma\nu}g(k)$$
$$\times\frac{1}{\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}-\beta ik+\rho|k|^{\lambda}}$$
$$=\left(\frac{Z}{r}\right)^{-\nu(1-\mu)}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma\nu}g(k)$$
$$\times\frac{1}{1-\frac{\rho|k^{\lambda}|-i\beta k}{\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{\gamma}}}$$

so it gives

$$V^{*}(k,s,Z)=\sum_{m=0}^{\infty}\left(\frac{Z}{r}\right)^{\nu(1-\mu)+\mu m}\left[1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right]^{-\gamma,(m-\nu)}(\rho|k|^{\lambda}-\beta ik)^{m}g(k). \tag{28}$$

now taking inverse Ψ-Shehu transform and Fourier transform of equation (28) using (14), we have

$$z(k,t)=\sum_{m=0}^{\infty}\frac{t^{\nu(1-\mu)+m\mu-1}}{2\pi}\int_{-\infty}^{\infty}exp^{-ikx}g(k)(\beta ik-\rho|k|^{\lambda})^{m}E_{\alpha,\nu(1-\mu)+m\mu}^{\gamma(m-\nu)}(\omega\Psi(t)^{\alpha})dk. \tag{29}$$

□

Example 5.1. If $\beta=0$, $\rho=\frac{ih}{2\pi}$ in above theorem 5.1, the solution of the resulting equation called one dimensional space-time Schrödinger equation of fractional order, for a free nature particle of mass m with h Planck constant, is

$$Z(k,t)=\sum_{m=0}^{\infty}\frac{t^{\nu(1-\mu)+m\mu-1}}{2\pi}\int_{-\infty}^{\infty}exp^{-ikx}g(k)\big(-\frac{ih}{2m}|k|^{\lambda}\big)^{m}E_{\alpha,\nu(1-\mu)+m\mu}^{\gamma(m-\nu)}(\omega\Psi(t)^{\alpha})dk. \tag{30}$$

Where λ,x,t,μ,ν and $\Delta^{\frac{\lambda}{2}}$ are the same as use identified previously.

Here we study the following generalization of the FEL, equation, involving Ψ-Hilfer-Prabhakar fractional derivative

Theorem 5.2. *The solution of Cauchy problem*

$$\mathcal{D}_{\alpha,\omega,0^{+}}^{\gamma,\mu,\nu,\Psi}y(t)=\lambda P_{\alpha,\mu,\omega,0^{+}}^{\delta}y(t)+f(t), \tag{31}$$

$$\big(P_{\alpha,(1-\nu)(m-\mu)-k,\omega}^{-\gamma(1-\nu)(m-\mu),\psi}Z(r,0^{+})f\big)_{t=0_{+}}=K, \tag{32}$$

where $f(x) \in L_1[0,\infty); \mu \in (0,1), \nu \in [0,1]; \omega, \lambda \in \mathbb{C}; t, \alpha > 0, K, \gamma, \delta \geq 0,$ *is given by*

$$y(t) = K \sum_{m=0}^{\infty} \lambda^m \Psi(t)^{\nu(1-\mu)+\mu(2m+1)-1} E^{\gamma-\gamma\nu+m(\delta+\gamma)}_{\alpha,(1-\nu)(1-\mu)+\mu(2m+1)}(\omega\Psi(t)^\alpha)$$
$$+ \sum_{m=0}^{\infty} E^{\gamma+m(\delta+\gamma)}_{\alpha,\mu(2m+1),\omega,0^+} f(t). \tag{33}$$

Proof. We denote by $Y(s,Z)$ and $V(s,Z)$ the Ψ-Shehu transform of $y(t)$ and $f(t)$, respectively. Applying Ψ-Shehu transform of (31) and using (5), (12), (6) and (14), (32), we have

$$\begin{aligned}\mathbb{S}_\Psi\big(D^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} y(t)\big)(r,Z) &= \mathbb{S}_\Psi\big(\lambda P^\delta_{\alpha,\mu,\omega,0^+} y(t) f(t)\big)\\ &= \lambda \mathbb{S}_\Psi\big(P^\delta_{\alpha,\mu,\omega,0^+} y(t)\big)(r,Z) + V(r,Z),\\ &= \lambda \mathbb{S}_\Psi\big((\varepsilon^\delta_{\alpha,\mu,\omega,0^+} *_\Psi y)(t)\big)(r,Z) + V(r,Z),\\ &= \lambda \mathbb{S}_\Psi\big(\Psi(t)^{\mu-1} E^\delta_{\alpha,\mu}(\omega\Psi(t))\big)(r,Z) Y(r,Z) + V(r,Z),\\ &= \lambda\left(\frac{Z}{r}\right)^\mu \left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^{-\delta} Y(r,Z) + V(r,Z),\end{aligned}$$

and from (22), we get

$$\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma Y(r,Z) - K\left(\frac{Z}{r}\right)^{\nu(1-\mu)}\left[1-\omega\left(\frac{Z}{r}\right)^\alpha\right]^{\gamma\nu}$$
$$= \lambda\left(\frac{Z}{r}\right)^\mu\left(1-\omega(\frac{Z}{r})^\alpha\right)^{-\delta} Y(r,Z) + V(r,Z),$$

so that

$$Y(r,Z) = \frac{V(r,Z) + K\left(\frac{Z}{r}\right)^{\nu(1-\mu)}\left[1-\omega\left(\frac{Z}{r}\right)^\alpha\right]^{\gamma\nu}}{\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma} \frac{1}{1-\frac{\lambda\left(\frac{Z}{r}\right)^\mu\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^{-\delta}}{\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma}},$$

$$\begin{aligned}&= \left[\frac{V(r,Z) + K\left(\frac{Z}{r}\right)^{\nu(1-\mu)}\left[1-\omega\left(\frac{Z}{r}\right)^\alpha\right]^{\gamma\nu}}{\left(\frac{Z}{r}\right)^{-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma}\right]\\ &\times \sum_{m=0}^{\infty} \lambda^m \left(\frac{Z}{r}\right)^{(2\mu)m}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^{-(\delta+\gamma)m},\\ &= V(r,Z)\sum_{m=0}^{\infty} \lambda^m\left(\frac{Z}{r}\right)^{\mu(2m+1)}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^{-(\delta+\gamma)m-\gamma}\\ &+ K\sum_{m=0}^{\infty}\lambda^m\left(\frac{Z}{r}\right)^{\mu(2m+1)+\nu(1-\mu)}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^{-(\delta+\gamma)m+\gamma\nu-\gamma}\end{aligned}$$

Last step is valid for

$$\left|\frac{\lambda\left(\frac{Z}{r}\right)^\mu\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^{-\delta}}{\left(\frac{Z}{r}\right)^{1-\mu}\left(1-\omega\left(\frac{Z}{r}\right)^\alpha\right)^\gamma}\right| < 1$$

The required solution (33) is obtained by applying the inverse of Ψ-Shehu transform on both side of last equation,

$$\begin{aligned}
y(t) &= \mathbb{S}_{\Psi}^{-1}\left[V(r,Z)\sum_{m=0}^{\infty}\lambda^m\left(\frac{Z}{r}\right)^{\mu(2m+1)}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{-(\delta+\gamma)m-\gamma}\right] \\
&+ \mathbb{S}_{\Psi}^{-1}\left[K\sum_{m=0}^{\infty}\lambda^m\left(\frac{Z}{r}\right)^{\mu(2m+1)+\nu(1-\mu)}\left(1-\omega\left(\frac{Z}{r}\right)^{\alpha}\right)^{-(\delta+\gamma)m+\gamma\nu-\gamma}\right], \\
&= K\sum_{m=0}^{\infty}\lambda^m\Psi(t)^{\nu(1-\mu)+\mu(2m+1)-1}E_{\alpha,(1-\nu)(1-\mu)+\mu(2m+1)}^{\gamma-\gamma\nu+m(\delta+\gamma)}(\omega\Psi(t)^{\alpha}) \\
&+ \sum_{m=0}^{\infty}E_{\alpha,\mu(2m+1),\omega,0^+}^{\gamma+m(\delta+\gamma)}f(t).
\end{aligned}$$

□

6. Conclusion

Finding a new integral transform for solving ordinary and partial fractional differential equations is always beneficial in the subject of fractional calculus. In this manuscript, we introduced new integral transform called Ψ-Shehu transform, applied it on some basic properties and elementary functions. Moreover, we obtained Ψ-Shehu transform of generalized fractional derivatives in sense of Ψ function such as Ψ−Riemann-Liouville, Ψ-Caputo Ψ- Hilfer fractional derivative also, Ψ-Prabhakar integral, Ψ-Prabhakar fractional derivative, Ψ-Hilfer-Prabhakar fractional derivative and its regularised version in terms of Mittag-Leffler function. The space-time fractional convection-dispersion equation and the generalized fractional free electron laser (FEL) equation including the Ψ-Hilfer-Prabhakar fractional derivative and its regularized form were also solved.

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