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Research Article

Relative Buchweitz-Happel theorem respect to a self-orthogonal class

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Abstract

Let R be a ring, F a subbifunctor of the functor $\mathrm{Ext}^1_R(-,-)$, $\mathcal{W}_F$ a self-orthogonal class of left R-modules respect to F. We introduce $\mathcal{W}_F$-Gorenstein modules $\mathcal{G}(\mathcal{W}_F)$ as a generalization of $\mathcal{W}$-Gorenstein modules (Geng and Ding, 2011, [14]), F-Gorenstein projective and F-Gorenstein injective modules (Tang, 2014 [27]). We introduce the notion of relative singularity category $D_{\mathcal{W}_F}(R)$ with respect to $\mathcal{W}_F$. Moreover, we give a necessary and sufficient condition such that the stable category $\underline{\mathcal{G}(\mathcal{W}_F)}$ and the relative singularity category $D_{\mathcal{W}_F}(R)$ are triangle-equivalence.

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1. Introduction

Let R be a left-noetherian ring with a unit. Denote by R-mod the category of finitely-generated left R-modules and R-proj the full subcategory of finitely-generated projective modules. Denote by $K^b(R\text{-mod})$ and $D^b(R\text{-mod})$ the bounded homotopy category and the bounded derived category of R, respectively. The singularity category of the ring R is the quotient category $D_{sg}(R) := D^b(R\text{-mod})/K^b(\text{proj-}R)$, where $K^b(\text{proj})$ is the bounded homotopy category of finitely generated projective modules. The Buchweitz-Happel Theorem [5,8,15] says that there is a triangle-embedding $\Phi : \underline{\mathcal{GP}}(R) \to D_{sg}(R)$, and Φ is an equivalence if R is Gorenstein. Recently, similar quotient triangulated categories were also studied, such as $D_{Sg}(R) := D^b(R\text{-Mod})/K^b(\text{Proj})$ [4], $D^b(R\text{-Mod})/K^b(\text{Inj})$ [4], $D^b(R\text{-mod})/K^b(\mathcal{I})$ [15], where R-Mod (respectively, Proj, Inj, $\mathcal{I}$) denotes the category of all (respectively, projective, injective, finitely generated injective) R-modules over a ring R. In [10], a more general notion of relative singularity categories $D_\omega(\mathcal{A}) := D^b(\mathcal{A})/K^b(\omega)$ was introduced, where $\mathcal{A}$ is an arbitrary abelian category and $\omega \subseteq \mathcal{A}$ is a self-orthogonal additive subcategory. This notion unifies previous various quotient triangulated categories. The author also introduced the Frobenius category of ω-Cohen-Macaulay objects, and under certain conditions, he showed that the stable category of ω-Cohen-Macaulay objects

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is triangle-equivalent to $D_\omega(\mathcal{A})$. If $\mathcal{A}$ is a module category, then the notion of ω-Cohen-Macaulay objects coincides with the ω-Gorenstein modules, which was defined and studied by Geng and Ding [14]. In this case, ω is a self-orthogonal class of modules.

In their series of papers [1–3], Auslander and Solberg applied relative homological algebra to the study of representation theory of artin algebras. These three papers provide for us certain foundational aspects of the theory of relative homological algebra in terms of subbifunctors of the functor $\mathrm{Ext}^1_\Lambda(-,-)$. Recently, Tang [27] defined F-Gorenstein projective and F-Gorenstein injective modules, and characterized F-Gorenstein algebra via some relative Gorenstein dimensions. Buan has generalized in [7] the construction of the Verdier quotient category to get a relative derived category $D^b_F(R)$, where he localized with respect to F-exact complexes.

Let F be a subbifunctor of the functor $\mathrm{Ext}^1_R(-,-)$, $\mathcal{W}_F$ a self-orthogonal class of modules respect to F. The main purpose of this paper is to obtain the relative version of the Buchweitz-Happel Theorem respect to the self-orthogonal class $\mathcal{W}_F$. Inspired by the above work, we introduce the notion of $\mathcal{W}_F$-Gorenstein modules, which is a generalization of $\mathcal{W}$-Gorenstein modules, F-Gorenstein projective and F-Gorenstein injective modules. Then we define the relative singularity category respect to $\mathcal{W}_F$ to be the following Verdier quotient triangulated category $D_{\mathcal{W}_F}(R) := D^b_F(R)/K^b(\mathcal{W}_F)$. Let θ be the composite of natural functors: $\mathcal{G}(\mathcal{W}_F) \hookrightarrow R\text{-Mod} \longrightarrow D^b_F(R) \longrightarrow D_{\mathcal{W}_F}(R)$. It will induce a functor $\underline{\theta} : \underline{\mathcal{G}(\mathcal{W}_F)} \longrightarrow D_{\mathcal{W}_F}(R)$. We show that $\underline{\theta}$ is a triangle-equivalence if and only if the global $\mathcal{G}(\mathcal{W}_F)$-$F$ dimension of R is finite (see Theorem 4.4).

The highlight of the present paper is that we prove the sufficiency of our main theorem in a completely different way than in [5]. Let R be an Artin ring. If every finitely generated injective left R-module has finite projective dimension and every finitely generated projective right R-module has finite injective dimension, then R is a Gorenstein ring. Using this fact, Bergh, Oppermann and Jorgensen proved in [5] that R is Gorenstein if the functor $\Phi : \underline{\mathcal{GP}}(R) \to D_{sg}(R)$ is dense. In this paper, we introduce the $\mathcal{G}(\mathcal{W}_F)$ dimension for complexes. We prove that if the functor θ is dense, then $\mathcal{G}(\mathcal{W}_F)\text{-dim}M^\bullet < \infty$ for any $M^\bullet \in D^b_F(R)$. Finally, by the fact that R is Gorenstein if and only if every finitely generated left R-module has finite Gorenstein dimension (see [18, Theorem]), we recover the known results.

The paper is organized as follows. In Section 2, we collect preliminary notions that will be used throughout the paper. In Section 3, we give the definition and study some elementary properties of $\mathcal{W}_F$-Gorenstein modules $\mathcal{G}(\mathcal{W}_F)$. In Section 4, we prove our main result. Finally, we give some applications of our main result for Gorenstein homological modules with respect to a semidualizing bimodule.

2. Preliminaries

In this section, we recall some known notions and facts needed in the sequel. Throughout this paper, R is a ring with identity and all modules considered will be unital modules. For any ring R, we denote the category of left R-modules by R-Mod.

2.1. Complexes

In this paper, the category of chain complexes of R-modules will be denoted by $C(R)$. A complex $X^\bullet$ will be denoted by

$$X^\bullet = \cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots .$$

Denote by $f : X^\bullet \to Y^\bullet$ the cochain map in $C(R)$. Given a complex $X^\bullet$ and an integer m, $X^\bullet[m]$ denotes the complex such that $X^\bullet[m]^n = X^{m+n}$ with boundary operators $(-1)^m d^{m+n}$. The n-th *cycle module* of $X^\bullet$ is defined as $Z^n X^\bullet := \mathrm{Ker}d^n$. The n-th *boundary module* of $X^\bullet$ is defined as $B^n X^\bullet := \mathrm{Im}d^{n-1}$. Recall that $X^\bullet$ is called *acyclic* (or *exact*) if $H^i(X^\bullet) = 0$ for any $i \in \mathbb{Z}$, and f is called a *quasi-isomorphism* if $H^i(f)$ is an isomorphism for any $i \in \mathbb{Z}$. $K^*(R)$ is the homotopy category of R-Mod, where $* \in \{\text{blank}, -, b\}$. A bounded complex $X^\bullet = (X^n, d^n)_{n\in\mathbb{Z}}$ is said to be *negative* if $X^n = 0$ for all $n \geq 0$. Denote by $K^{<0}(R)$ the full subcategory of $K(R)$ whose objects are isomorphic to a negative complex in $C(R)$. Similarly, we have the subcategory $K^{>0}(R)$. We will use the formula $\mathrm{Hom}_{K(R)}(X^\bullet, Y^\bullet[n]) = H^n\mathrm{Hom}_R(X^\bullet, Y^\bullet)$ for any $X^\bullet, Y^\bullet \in C(R)$ and $n \in \mathbb{Z}$.

Given a complex $X^\bullet$ and an integer $i \in \mathbb{Z}$, we denote by

$$\sigma^{\geq i}X^\bullet : 0 \to X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \to \cdots$$

the hard left truncation of $X^\bullet$, and

$$\sigma^{\leq i}X^\bullet : \cdots \to X^{i-2} \xrightarrow{d^{i-2}} X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} 0$$

the hard right truncation of $X^\bullet$. Set $\sigma^{>i}X^\bullet := \sigma^{\geq i+1}X^\bullet$ and $\sigma^{<i}X^\bullet := \sigma^{\leq i-1}X^\bullet$. The soft left truncation, $\tau^{\geq i}X^\bullet$, of $X^\bullet$ at i and the soft right-truncation, $\tau^{\leq i}X^\bullet$, of $X^\bullet$ at i are given by:

$$\tau^{\geq i}X^\bullet := 0 \to \mathrm{Coker}d^{i-1} \xrightarrow{\overline{d^i}} X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \to \cdots$$

and

$$\tau^{\leq i}X^\bullet := \cdots \to X^{i-2} \xrightarrow{d^{i-2}} X^{i-1} \xrightarrow{d^{i-1}} \mathrm{Ker}d^i \to 0,$$

where the differential $\overline{d^i}$ is the induced map on residue classes.

2.2. F-exact sequences

Suppose F is an additive subbifunctor of the additive bifunctor $\mathrm{Ext}^1_R(-,-) : R\text{-Mod}^{op} \times R\text{-Mod} \to \mathrm{Ab}$, a short exact sequence $\eta : 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in R-Mod is said to be F-*exact* if η is in $\mathrm{F}(Z, X)$. Moreover, f is called an F-*monomorphism* and g is called an F-*epimorphism*. Each additive subbifunctor F corresponds a class of short exact sequences that is closed under the operations of pushout, pullback, Baer sums and direct sums. For more details, we refer the reader to [25].

In what follows, F always denotes an additive subbifunctor of the additive bifunctor $\mathrm{Ext}^1_R(-,-)$. An exact sequence $X^\bullet = \cdots \longrightarrow X^{n-1} \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1} \xrightarrow{f^{n+1}} \cdots$ in R-Mod is called an F-*exact sequence* provided that $0 \longrightarrow \mathrm{Im}f^i \longrightarrow X^{i+1} \longrightarrow \mathrm{Im}f^{i+1} \longrightarrow 0$ is F-exact for all $i \in \mathbb{Z}$. A left R-module P (respectively, I) is said to be F-*projective* (respectively, F-*injective*) if for each F-exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$, the sequence $0 \longrightarrow \mathrm{Hom}_R(P, X) \longrightarrow \mathrm{Hom}_R(P, Y) \longrightarrow \mathrm{Hom}_R(P, Z) \longrightarrow 0$ (respectively, $0 \longrightarrow \mathrm{Hom}_R(Z, I) \longrightarrow \mathrm{Hom}_R(Y, I) \longrightarrow \mathrm{Hom}_R(X, I) \longrightarrow 0$) is exact. The full subcategory of R-Mod consisting of all F-projective (respectively, F-injective) modules is denoted by $\mathcal{P}(F)$ (respectively, $\mathcal{I}(F)$). F is said to have *enough projectives* (respectively, *enough injectives*) if for any $D \in R$-Mod there is an F-exact sequence $0 \longrightarrow B \longrightarrow P \longrightarrow D \longrightarrow 0$ (respectively, $0 \longrightarrow D \longrightarrow I \longrightarrow C \longrightarrow 0$) with P in $\mathcal{P}(F)$ (respectively, I in $\mathcal{I}(F)$). For any $M \in \text{Mod-}R$, a *left* F-*projective resolution* of M is an F-exact sequence $\mathbf{P} = \cdots \to P_1 \to P_0 \to M \to 0$ with $P_i \in \mathcal{P}(F)$ for all i. Right F-injective resolutions are defined dually. If F has enough projectives and injectives, then for all left R-modules C and D, the right derived functors of $\mathrm{Hom}_R(C,-)$ and $\mathrm{Hom}_R(-,D)$ using right F-injective and left F-projective resolutions, respectively, coincide. We denote by $\mathrm{Ext}^i_F(C,-)$ (respectively, $\mathrm{Ext}^i_F(-,D)$) the right derived functors of $\mathrm{Hom}_R(C,-)$ (respectively, $\mathrm{Hom}_R(-,D)$).

Recall from [9,21] that an *exact category* is a pair $(\mathcal{E}, \mathcal{S})$ where $\mathcal{E}$ is an additive category and $\mathcal{S}$ is a class of "short exact sequences": That is, triples of objects connected by arrows $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ such that i is the kernel of p and p is the cokernel of i. The class $\mathcal{S}$ of short exact sequences must satisfy the axioms in [9, Definition 2.1], which are inspired by the properties of short exact sequences in any abelian category.

Recall from [7] that an additive subbifunctor F is said to be *closed* if the following equivalent statements hold.

(1) The composition of F-epimorphisms is an F-epimorphism.
(2) The composition of F-monomorphisms is an F-monomorphism.
(3) For each object X the functor $F(X,-)$ is half exact on F-exact sequences.
(4) For each object X the functor $F(-,X)$ is half exact on F-exact sequences.
(5) The category R-Mod with respect to the F-exact sequences is an exact category.

A subcategory $\mathcal{X}$ of R-Mod is said to be closed under F-extensions, if $0 \to A \to B \to C \to 0$ is F-exact and A and C are in $\mathcal{X}$, then B is in $\mathcal{X}$. If F is closed, then the class of F-exact complexes is a null-system (see [20, Definition 1.6.6] for definition) in $K(R)$. A map h in $K(R)$ is called an F*-quasi-isomorphism* if the mapping cone is an F-exact sequence. Since the class of F-quasi-isomorphisms is a multiplicative system, it follows that we localize with respect to this system. Set

$$\mathcal{N} = \{X^\bullet \in K(R) \mid X^\bullet \text{ is an } F\text{-exact sequence}\}.$$

Then it is easy to see $\mathcal{N}$ is a thick subcategory of $K(R)$. Let $S(\mathcal{N})$ be the following set of morphisms

$$S(\mathcal{N}) = \{X^\bullet \to Y^\bullet \mid \ X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1] \ \text{ is a distinguished triangle in } K(R) \text{ with } Z^\bullet \in \mathcal{N}\}.$$

By [25], the relative derived category of R related to the subbifunctor F is defined to be the Verdier quotient, that is,

$$D_F(R) := K(R)/\mathcal{N} = S(\mathcal{N})^{-1}K(R).$$

2.3. F-(co)resolution and dimension

Let $\mathcal{C}$, $\mathcal{D}$ be classes of R-modules and M a left R-module.

(1) The left $\mathcal{C}$-F *resolution* of M is an F-exact sequence $C^\bullet = \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow M \longrightarrow 0$ with $C_i \in \mathcal{C}$ for all i.

(2) The $\mathcal{D}$-proper left $\mathcal{C}$-F *resolution* is a left $\mathcal{C}$-F resolution $C^\bullet$ of M such that $\mathrm{Hom}_R(D, C^\bullet)$ is exact for all $D \in \mathcal{D}$.

Dually, we can define the ($\mathcal{D}$-coproper) right $\mathcal{C}$-F *coresolution* of M.

(3) The left $\mathcal{C}$-F dimension of M, written $\mathcal{C}$-$\mathrm{pd}_F(M)$, is defined as the minimal nonnegative integer n such that M has a left $\mathcal{C}$-F resolution of length n. If no such an integer exists, then set $\mathcal{C}$-$\mathrm{pd}_F(M) = \infty$.

(4) The $\mathcal{D}$-proper left $\mathcal{C}$-F dimension of M, written $\mathcal{D}$-$\mathcal{C}$-$\mathrm{pd}_F(M)$, is defined as the minimal nonnegative integer n such that M has a $\mathcal{D}$-proper left $\mathcal{C}$-F resolution of length n. If no such an integer exists, then set $\mathcal{D}$-$\mathcal{C}$-$\mathrm{pd}_F(M) = \infty$.

Dually, we can define the ($\mathcal{D}$-coproper) right $\mathcal{C}$-F dimension of M, and denote it by $\mathcal{C}$-$\mathrm{id}_F(M)$ ($\mathcal{D}$-$\mathcal{C}$-$\mathrm{id}_F(M)$).

3. $\mathcal{W}_F$-Gorenstein modules

We begin this section with the following definitions. From now on, we always assume that F has enough projectives.

Definition 3.1. Let $\mathcal{W}$ be a class of modules in R-Mod and F an additive subbifunctor of the additive bifunctor $\mathrm{Ext}^1_R(-,-)$. Then $\mathcal{W}$ is called *self-orthogonal respect to* F if

$$\mathrm{Ext}^i_F(W, W') = 0$$

for all $W,\ W' \in \mathcal{W}$ and all $i \geq 1$.

In what follows, $\mathcal{W}_F$ always denotes a self-orthogonal class respect to F, which is closed under finite direct sums and direct summands.

Definition 3.2. A left R-module M is said to be $\mathcal{W}_F$-*Gorenstein* if there exists an F-exact sequence

$$W_\bullet = \cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots$$

of objects in $\mathcal{W}_F$ such that $M = \mathrm{Ker}(W^0 \to W^1)$ and $W_\bullet$ is $\mathrm{Hom}_R(\mathcal{W}_F, -)$ and $\mathrm{Hom}_R(-, \mathcal{W}_F)$ exact. In the following, we denote by $\mathcal{G}(\mathcal{W}_F)$ the class of $\mathcal{W}_F$-Gorenstein modules.

Remark 3.3. (1) It is clear that each module in $\mathcal{W}_F$ is $\mathcal{W}_F$-Gorenstein. If

$$W_\bullet = \cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots$$

is a $\mathrm{Hom}_R(\mathcal{W}_F, -)$ and $\mathrm{Hom}_R(-, \mathcal{W}_F)$ exact sequence of objects in $\mathcal{W}_F$, then by symmetry, all images, kernels and cokernels of $W_\bullet$ are $\mathcal{W}_F$-Gorenstein.

(2) If the subbifunctor F coincides with the bifunctor $\mathrm{Ext}^1_R(-,-)$, and $\mathcal{W}_F$ is the class of projective left R-modules (or the class of injective left R-modules), then $\mathcal{W}_F$-Gorenstein modules are exactly Gorenstein projective (injective) modules [13].

(3) A similar argument of [27, Lemma 2.4(2)] and [26, Proposition 4.4] yields that $\mathcal{G}(\mathcal{W}_F)$ is closed under F-extensions.

Let $\mathcal{W}_F$ be a self-orthogonal class of left R-modules respect to F. We define ${\mathcal{W}_F}^{\perp_F} := \bigcap {\mathcal{W}_F}^{\perp^i_F}$, where ${\mathcal{W}_F}^{\perp^i_F} = \{X \in A \mid \ \mathrm{Ext}^j_F(W, X) = 0 \text{ for all } W \in \mathcal{W}_F, j \geq i\}$, and

$$_{\mathcal{W}_F}\mathcal{X} := \{X \in A \mid \text{there exists an } F\text{-exact sequence} \cdots \to W^{-n} \overset{d^{-n}}{\to} W^{1-n} \to \cdots \to W^0 \overset{d^0}{\to} X \to 0, \text{ with each } W^i \in \mathcal{W}_F, \mathrm{Ker} d^{-i} \in {\mathcal{W}_F}^{\perp_F}\}.$$

Dually, we define ${}^{\perp_F}\mathcal{W}_F := \bigcap {}^{\perp^i_F}\mathcal{W}_F$, where ${}^{\perp^i_F}\mathcal{W}_F = \{X \in A \mid \mathrm{Ext}^j_F(X, W) = 0$ for all $W \in \mathcal{W}_F, j \geq i\}$, and

$$\mathcal{X}_{\mathcal{W}_F} := \{X \in A \mid \text{there exists an } F\text{-exact sequence } 0 \to X \overset{d^{-1}}{\to} W^0 \overset{d^0}{\to} W^1 \to \cdots \to W^{n-1} \overset{d^{n-1}}{\to} W^n \to \cdots, \text{ with each } W^i \in \mathcal{W}_F, \mathrm{Coker} d^i \in {}^{\perp_F}\mathcal{W}_F\}.$$

Since $\mathcal{W}_F$ is self-orthogonal respect to F, we have $\mathcal{W}_F \subseteq {}_{\mathcal{W}_F}\mathcal{X} \subseteq {\mathcal{W}_F}^{\perp_F}$ and $\mathcal{W}_F \subseteq \mathcal{X}_{\mathcal{W}_F} \subseteq {}^{\perp_F}\mathcal{W}_F$.

Lemma 3.4. [27, Lemma 2.4(2)] *Let $0 \to A \to B \to C \to 0$ be an F-exact sequence. Then for any $M \in$ Mod-R, there exists a long exact sequence* $0 \to \mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M) \to \mathrm{Ext}^1_F(C, M) \to \mathrm{Ext}^1_F(B, M) \to \mathrm{Ext}^1_F(A, M) \to \cdots$.

Using this lemma, we have the following result by definition.

Lemma 3.5. *A left R-module M is $\mathcal{W}_F$-Gorenstein if and only if $M \in {}_{\mathcal{W}_F}\mathcal{X} \cap \mathcal{X}_{\mathcal{W}_F}$, if and only if $M \in {}^{\perp_F}\mathcal{W}_F \cap \mathcal{W}_F^{\perp_F}$ and M has a $\mathcal{W}_F$-proper left $\mathcal{W}_F$-F resolution and a $\mathcal{W}_F$-coproper right $\mathcal{W}_F$-F coresolution.*

Lemma 3.6. *$\mathcal{G}(\mathcal{W}_F)$ is closed under direct summands.*

Proof. It is obtained by standard argument similar to the proof of [26, Proposition 4.11]. □

Proposition 3.7. *Let $0 \to M' \to M \to M'' \to 0$ be an F-exact sequence of left R-modules with $M \in \mathcal{G}(\mathcal{W}_F)$.*

(1) If $M' \in \mathcal{G}(\mathcal{W}_F)$ and $M'' \in {}^{\perp_F}\mathcal{W}_F$, then $M'' \in \mathcal{G}(\mathcal{W}_F)$.

(2) If $M'' \in \mathcal{G}(\mathcal{W}_F)$ and $M' \in \mathcal{W}_F^{\perp_F}$, then $M' \in \mathcal{G}(\mathcal{W}_F)$.

Proof. (1) Since $M' \in \mathcal{G}(\mathcal{W}_F)$, there is an F-exact sequence $0 \to M' \to W^0 \to L \to 0$ with $W^0 \in \mathcal{W}_F$ and $L \in \mathcal{G}(\mathcal{W}_F)$. Consider the following pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W^0 \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & M'' \arrow[r] & 0 \\
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

From the middle column, we get that D is $\mathcal{W}_F$-Gorenstein by Remark 3.3(3). Note that the middle row splits since $\mathrm{Ext}^1_F(M'', W^0) = 0$ by hypothesis. So M'' is $\mathcal{W}_F$-Gorenstein by Lemma 3.6.

(2) The proof is dual to that of (1). □

Recall from [7] that if F is closed, then the category R-Mod with respect to the F-exact sequences is an exact category. In the rest of the paper, we assume that F is a closed subbifunctor. Denote by ε the class of F-exact sequences of the form: $0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$ with $L,\ M,\ N \in \mathcal{G}(\mathcal{W}_F)$. By Remark 3.3(3), we directly obtain the following fact.

Proposition 3.8. *$(\mathcal{G}(\mathcal{W}_F), \varepsilon)$ is an exact category.*

By Proposition 3.8, we have the following result.

Corollary 3.9. *$(\mathcal{G}(\mathcal{W}_F), \varepsilon)$ is a Frobenius category, that is, $(\mathcal{G}(\mathcal{W}_F), \varepsilon)$ has enough projectives and enough injectives such that the projective objects coincide with the injective objects.*

Proof. Observe that objects in $\mathcal{W}_F$ are injective and projective in $\mathcal{G}(\mathcal{W}_F)$, the assertion follows from Proposition 3.8. □

Consider the stable category $\underline{\mathcal{G}(\mathcal{W}_F)}$ of $\mathcal{G}(\mathcal{W}_F)$ modulo $\mathcal{W}_F$. Then by [16, Theorem 2.6] the stable category $\underline{\mathcal{G}(\mathcal{W}_F)}$ is a triangulated category. We will make use of the following propositions.

Proposition 3.10. *Let M be a left R-module. Then M has a $\mathcal{W}_F$-proper left $\mathcal{W}_F$-F resolution if and only if M has a $\mathcal{W}_F$-proper left $\mathcal{G}(\mathcal{W}_F)$-$F$ resolution.*

Proof. It is enough to show the "if" part. Let $0 \to K_1 \to G_0 \to M \to 0$ be a $\mathrm{Hom}_R(\mathcal{W}_F, -)$-exact sequence with $G_0 \in \mathcal{G}(\mathcal{W}_F)$ and K_1 having a $\mathcal{W}_F$-proper left $\mathcal{G}(\mathcal{W}_F)$-$F$ resolution. Note that there exist an exact sequence $0 \to G' \to W \to G_0 \to 0$ such that

$G' \in \mathcal{G}(\mathcal{W}_F)$ and $W \in \mathcal{W}_F$. Then we have the following pullback diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & & \\
0 \arrow[r] & H \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

One can check that the first column is $\mathrm{Hom}_R(\mathcal{W}_F, -)$-exact. Since K_1 has a $\mathcal{W}_F$-proper left $\mathcal{G}(\mathcal{W}_F)$-$F$ resolution, we have an exact sequence $0 \to K_2 \to G_1 \to K_1 \to 0$, where K_2 has a left $\mathcal{G}(\mathcal{W}_F)$-$F$ resolution and $G_1 \in \mathcal{G}(\mathcal{W}_F)$. Consider the following pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K_2 \arrow[r, equal] \arrow[d] & K_2 \arrow[d] & \\
0 \arrow[r] & G' \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G' \arrow[r] & H \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0, &
\end{tikzcd}$$

It follows from Remark 3.3(3) that $L \in \mathcal{G}(\mathcal{W}_F)$. One can check that the F-exact sequence $0 \to K_2 \to L \to H \to 0$ is $\mathrm{Hom}_R(\mathcal{W}_F, -)$-exact. So H has a left $\mathcal{G}(\mathcal{W}_F)$-$F$ resolution which is $\mathrm{Hom}_R(\mathcal{W}_F, -)$-exact. By repeating the preceding process, we have that M has a $\mathcal{W}_F$-proper $\mathcal{W}_F$-F resolution. $\square$

Proposition 3.11. *Assume that each left R-module M admits a left $\mathcal{G}(\mathcal{W}_F)$-$F$ resolution. Then $\mathcal{W}_F$-$\mathcal{G}(\mathcal{W}_F)$-$\mathrm{pd}_F(M) \leqslant n$ if and only if in every F-exact sequence*

$$0 \longrightarrow K_n \longrightarrow W_{n-1} \longrightarrow \cdots \longrightarrow W_0 \longrightarrow M \longrightarrow 0$$

where all $W_i \in \mathcal{W}_F$, the left R-module K_n is $\mathcal{W}_F$-Gorenstein.

Proof. Note that every F-exact sequence

$$0 \longrightarrow K_n \longrightarrow W_{n-1} \longrightarrow \cdots \longrightarrow W_0 \longrightarrow M \longrightarrow 0$$

with all $W_i \in \mathcal{W}_F$ and $K_n \in \mathcal{G}(\mathcal{W}_F)$ is $\mathrm{Hom}_R(\mathcal{W}_F, -)$-exact. The proof follows from a similar argument to that in [14, Proposition 2.12]. $\square$

Note that for any left R-module M, $\mathcal{W}_F$-$\mathcal{G}(\mathcal{W}_F)$-$\mathrm{pd}_F(M) < \infty$ if and only if M has a finite $\mathcal{W}_F$-proper left $\mathcal{G}(\mathcal{W}_F)$-$F$ resolution. We define

$$gl\text{-}\mathcal{G}(\mathcal{W}_F)(R) := \sup\{\mathcal{W}_F\text{-}\mathcal{G}(\mathcal{W}_F)\text{-}\mathrm{pd}_F(M) \mid M \in R\text{-Mod}\}.$$

The value of gl-$\mathcal{G}(\mathcal{W}_F)(R)$ will be called the global $\mathcal{G}(\mathcal{W}_F)$-$F$ dimension of R. Dually, we can define

$$cogl\text{-}\mathcal{G}(\mathcal{W}_F)(R) := \sup\{\mathcal{W}_F\text{-}\mathcal{G}(\mathcal{W}_F)\text{-id}_F(M) \mid M \in R\text{-Mod}\}.$$

In the following, we will consider the $\mathcal{G}(\mathcal{W}_F)$ dimensions for complexes. Let

$$K^{-,Fb}(\mathcal{P}(F)) := \{X^\bullet \in K^-(\mathcal{P}(F)) \mid \exists\, n \in \mathbb{Z} \text{ s. t. } \mathrm{H}^i(\mathrm{Hom}_R(P, X^\bullet)) = 0 \text{ for } i \leq n, P \in \mathcal{P}(F)\}.$$

Since we always assume that F has enough projectives, By [25, Proposition 3.9], there is a triangle equivalence $D^b_F(R) \simeq K^{-,Fb}(\mathcal{P}(F))$. For any complex $X^\bullet \in D^b_F(R)$, we can without loss of generality suppose $X^\bullet$ is bounded. There is a complex $P^\bullet \in K^{-,Fb}(\mathcal{P}(F))$, such that $P^\bullet \simeq X^\bullet$ in $D^b_F(R)$. That is, there exists an F-quasi-isomorphism $P \overset{\pi}{\Longrightarrow} X$. Moreover, since F has enough projectives, π can be chosen as F-surjective and π is called an F-projective resolution of X.

Definition 3.12. For any $M^\bullet \in D^b_F(R)$, the Gorenstein $\mathcal{W}_F$ dimension is

$$\mathcal{G}(\mathcal{W}_F)\text{-dim}M^\bullet := \inf\left\{ n \in \mathbb{Z} \,\middle|\, \begin{array}{l} Z^{-n+1}(P^\bullet) \in \mathcal{G}(\mathcal{W}_F) \text{ for any } M^\bullet \simeq P^\bullet \\ \text{with } P^\bullet \in K^{-,Fb}(\mathcal{P}(F)) \end{array} \right\}.$$

By the definition, for a left R-module M, we show that the Gorenstein $\mathcal{W}_F$ dimension of M defined here is exactly the $\mathcal{W}_F$-proper left $\mathcal{G}(\mathcal{W}_F)$-$F$ dimension of M.

Lemma 3.13. *Let M be a left R-module. Then*

$$\mathcal{G}(\mathcal{W}_F)\text{-dim}M = \mathcal{W}_F\text{-}\mathcal{G}(\mathcal{W}_F)\text{-pd}_F M.$$

Proof. Suppose $\mathcal{G}(\mathcal{W}_F)$-dim$M = g$ and $\mathcal{W}_F$-$\mathcal{G}(\mathcal{W}_F)$-pd$_F M = g'$. Since $\mathcal{G}(\mathcal{W}_F)$-dim$M = g$, there is a complex $P^\bullet \in K^{-,Fb}(\mathcal{P}(F))$, such that $P^\bullet \simeq M$ in $D^b_F(R)$ and $Z^{-g+1}(P^\bullet) \in \mathcal{G}(\mathcal{W}_F)$. It follows that we have an F-exact sequence

$$0 \longrightarrow Z^{-g+1}(P^\bullet) \longrightarrow P^{-g+1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

with all $P_i \in \mathcal{P}(F)$ and $Z^{-g+1}(P^\bullet) \in \mathcal{G}(\mathcal{W}_F)$. Notice that this sequence is $\mathrm{Hom}_R(\mathcal{W}_F, -)$-exact. Hence $g' \leq g$.

Since $\mathcal{W}_F$-$\mathcal{G}(\mathcal{W}_F)$-pd$_F M = g'$, by Proposition 3.11, for F-exact sequence

$$0 \longrightarrow K^{-g'+1} \longrightarrow P^{-g'+1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

with all $P_i \in \mathcal{P}(F) \subseteq \mathcal{W}_F$, we have $K^{-g'+1} \in \mathcal{G}(\mathcal{W}_F)$. Therefore, we obtain an F-projective resolution $P^\bullet \longrightarrow M$ with complex $P^\bullet := 0 \to P^{-g'+1} \to \cdots \to P^0 \to 0$ and $Z^{-g'+1}(P^\bullet) = K^{-g'+1} \in \mathcal{G}(\mathcal{W}_F)$. Thus $g \leq g'$. Altogether, we have $g = g'$. □

Lemma 3.14. *Let $P^\bullet \in K^{-,Fb}(\mathcal{P}(F))$. Then for every cochain map $\alpha : P^\bullet \to N^\bullet$ and every F-quasi-isomorphism $\beta : M^\bullet \to N^\bullet$, there exists a cochain map $\gamma : P^\bullet \to M^\bullet$ such that $\alpha \sim \beta\gamma$ holds.*

Proof. Let $P^\bullet \in K^{-,Fb}(\mathcal{P}(F))$. Consider the triangle $M^\bullet \xrightarrow{\beta} N^\bullet \longrightarrow \mathrm{Cone}(\beta) \longrightarrow M^\bullet[1]$ in $K(R)$. Since $\mathrm{Hom}_{K(R)}(P^\bullet, -)$ is a cohomological functor, we get a long exact sequence

$$\cdots \to \mathrm{Hom}_{K(R)}(P^\bullet, M^\bullet) \to \mathrm{Hom}_{K(R)}(P^\bullet, N^\bullet) \to \mathrm{Hom}_{K(R)}(P^\bullet, \mathrm{Cone}(\beta)) \to \cdots.$$

Since β is an F-quasi-isomorphism, we obtain that the complex $\mathrm{Cone}(\beta)$ is an F-exact complex and $\mathrm{Hom}_R(P^v, \mathrm{Cone}(\beta))$ is exact for every $v \in \mathbb{Z}$. As $P^\bullet$ is bounded above, it follows from [11, Theorem 4.1.9] and [12, Proposition 2.4] that $\mathrm{ConeHom}_R(P^\bullet, \beta) \simeq \mathrm{Hom}_R(P^\bullet, \mathrm{Cone}(\beta))$ is acyclic. Hence $\mathrm{Hom}_{K(R)}(P, \mathrm{Cone}(\beta)) = \mathrm{H}^0\mathrm{Hom}_R(P, \mathrm{Cone}(\beta)) = 0$. This implies that

$$\mathrm{Hom}_{K(R)}(P^\bullet, \beta) : \mathrm{Hom}_{K(R)}(P^\bullet, M^\bullet) \to \mathrm{Hom}_{K(R)}(P^\bullet, N^\bullet)$$

is surjective. Thus, there exists a cochain map $\gamma : P^\bullet \longrightarrow M^\bullet$ such that $\alpha \sim \beta\gamma$. □

Lemma 3.15. *Consider an F-exact sequence*

$$\delta = 0 \longrightarrow Y^\bullet \xrightarrow{f} Z^\bullet \xrightarrow{g} X^\bullet \longrightarrow 0$$

of chain complexes over $C(R)$, that is,

$$0 \longrightarrow Y^i \longrightarrow Z^i \longrightarrow X^i \to 0$$

is an F-exact sequence for each i. If $Y^\bullet$ is F-exact, then δ gives rise to an F-exact sequence:

$$0 \longrightarrow Z^m(Y^\bullet) \longrightarrow Z^m(Z^\bullet) \longrightarrow Z^m(X^\bullet) \longrightarrow 0$$

in R-Mod.

Proof. Since δ is an F-exact sequence of complexes, for every $P \in \mathcal{P}(F)$, we have the short exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(P, Y^\bullet) \longrightarrow \mathrm{Hom}_R(P, Z^\bullet) \longrightarrow \mathrm{Hom}_R(P, X^\bullet) \longrightarrow 0.$$

By [1, Proposition 1.5], the complex $\mathrm{Hom}_R(P, Y^\bullet)$ is exact. Then the sequence

$$0 \longrightarrow Z^m(\mathrm{Hom}_R(P, Y^\bullet)) \longrightarrow Z^m(\mathrm{Hom}_R(P, Z^\bullet)) \longrightarrow Z^m(\mathrm{Hom}_R(P, X^\bullet)) \longrightarrow 0$$

is exact. Since $\mathrm{Hom}_R(P, -)$ preserves kernels, we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(P, Z^m(Y^\bullet)) \longrightarrow \mathrm{Hom}_R(P, Z^m(Z^\bullet)) \longrightarrow \mathrm{Hom}_R(P, Z^m(X^\bullet)) \longrightarrow 0.$$

By [1, Proposition 1.5] again, we complete the proof. □

Corollary 3.16. *Let $0 \to M_1^\bullet \to M_2^\bullet \to M_3^\bullet \to 0$ be an F-exact sequence of bounded complexes. If $M_1^\bullet$, $M_3^\bullet$ have finite $\mathcal{G}(\mathcal{W}_F)$ dimension, then so does $M_2^\bullet$.*

Proof. Since $0 \to M_1^\bullet \to M_2^\bullet \to M_3^\bullet \to 0$ is an F-exact sequence of bounded complexes in $C(R)$, by [25, Proposition 4.1], $M_1^\bullet \to M_2^\bullet \to M_3^\bullet \xrightarrow{\phi} M_1^\bullet[1]$ is a distinguished triangle in $D_F^b(R)$. Suppose $M_1^\bullet, M_3^\bullet$ have finite $\mathcal{G}(\mathcal{W}_F)$-dimension. By the definition, there exists an $n \in \mathbb{Z}$ and F-projective resolutions, $P_1^\bullet \xrightarrow{\pi_1} M_1^\bullet$ and $P_3^\bullet \xrightarrow{\pi_3} M_3^\bullet$ such that $Z_n(P_1^\bullet), Z_n(P_2^\bullet) \in \mathcal{G}(\mathcal{W}_F)$.

By Lemma 3.14, there exsits a cochain map α such that $\phi \circ \pi_3 = \pi_1[1] \circ \alpha$. This follows that, in the diagram:

$$\begin{tikzcd}
P_3^\bullet \arrow[r, "\alpha"] \arrow[d, "\pi_3"'] & P_1^\bullet[1] \arrow[r] \arrow[d, "\pi_1[1]"'] & \mathrm{Cone}(\alpha) \arrow[r] \arrow[d, "h"'] & P_3^\bullet[1] \arrow[d, "\pi_3[1]"'] \\
M_3 \arrow[r, "\phi"] & M_1[1] \arrow[r] & M_2[1] \arrow[r] & M_3[1]
\end{tikzcd}$$

we have a morphism $h : \mathrm{Cone}(\alpha) \longrightarrow M_2[1]$ such that $(\pi_3, \pi_1[1], h)$ is a morphism between triangles in $D_F^b(R)$. Since $P_1^\bullet[1]$, $P_3^\bullet \in K^{-,Fb}(\mathcal{P}(F))$ and $0 \longrightarrow P_1^\bullet[1] \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow P_3^\bullet[1] \longrightarrow 0$ is degreewise split, one can check that $\mathrm{Cone}(\alpha) \in K^{-,Fb}(\mathcal{P}(F))$. By the hypothesis, π_3 and $\pi_1[1]$ are isomorphisms in $D_F^b(R)$, so is h. Hence $\mathrm{Cone}(\alpha)$ is an F-projective resolution of $M_2[1]$. By Definition 3.12, choose n so that

$$H^i(\mathrm{Hom}_R(P, P_1^\bullet[1])) = H^i(\mathrm{Hom}_R(P, \mathrm{Cone}(\alpha))) = H^i(\mathrm{Hom}_R(P, P_3^\bullet)) = 0$$

for $i \geq n$, and all $P \in \mathcal{P}(F)$. By Lemma 3.15, for all $i \geq n$, the sequence of modules

$$0 \longrightarrow Z^i(P_1^\bullet) \longrightarrow Z^i(\mathrm{Cone}(\alpha)[-1]) \longrightarrow Z^i(P_3^\bullet) \longrightarrow 0$$

is F-exact. If $Z^i(P_1^\bullet)$ and $Z^i(P_3^\bullet)$ are $\mathcal{W}_F$-Gorenstein, then Remark 3.3(3) shows that $Z^i(\mathrm{Cone}(\alpha)[-1])$ is $\mathcal{W}_F$-Gorenstein, this implies that $M_2^\bullet$ has finite $\mathcal{G}(\mathcal{W}_F)$-dimension. □

4. Triangle-equivalences respect to $\mathcal{W}_F$

Suppose $\varphi : K^b(\mathcal{W}_F) \longrightarrow D^b_F(R)$ is the composition of the embedding functor from $K^b(\mathcal{W}_F)$ into $K^b(R)$, and the localization functor $Q^b : K^b(R) \longrightarrow D^b_F(R)$. We show that this composite functor is fully faithful. Recall that the cardinal of the set $\{i \mid X^i \neq 0,\ i \in \mathbb{Z}\}$ is called the width of X, and denoted by $w(X)$.

Lemma 4.1. *The functor φ is fully faithful.*

Proof. Let $M_1^\bullet, M_2^\bullet \in K^b(\mathcal{W}_F)$. We may assume that $M_2^i = 0$ for $i < 0$ and $M_2^0 = M_2 \neq 0$. We proceed by double induction on the widths of $M_1^\bullet$ and $M_2^\bullet$. If $w(M_1^\bullet) = w(M_2^\bullet) = 1$, then there exists $i \in \mathbb{Z}$ such that $M_1^\bullet = M_1[i]$ for some $M_1 \in \mathcal{W}_F$. If $i = 0$, then $\mathrm{Hom}_{K^b(\mathcal{W}_F)}(M_1^\bullet, M_2^\bullet) = \mathrm{Hom}_{D^b_F(R)}(M_1^\bullet, M_2^\bullet)$. Otherwise we have $\mathrm{Hom}_{C^b(R)}(M_1^\bullet, M_2^\bullet) = \mathrm{Hom}_{C^b(R)}(M_1[i], M_2) = 0$ for $i > 0$. It follows that $\mathrm{Hom}_{K^b(\mathcal{W}_F)}(M_1^\bullet, M_2^\bullet) = 0$, and $\mathrm{Hom}_{D^b_F(R)}(M_1^\bullet, M_2^\bullet) = 0$ for $i > 0$. For $i < 0$, we have $\mathrm{Hom}_{D^b_F(R)}(M_1^\bullet, M_2^\bullet) \simeq \mathrm{Ext}_F^{-i}(M_1, M_2)$ by [25, Lemma 3.8], hence the assertion follows by assumption. If $w(M_1^\bullet) = 1$ and $w(M_2^\bullet) = r$, then we consider the triangle $M_2^0[-1] \longrightarrow M_2' \to M_2^\bullet \to M_2^0$ where $M_2' = \sigma^{>0} M_2^\bullet$ is the hard truncated complex. We apply the cohomological functors $\mathrm{Hom}_{K^b(\mathcal{W}_F)}(M_1^\bullet, -)$, and $\mathrm{Hom}_{D^b_F(R)}(M_1^\bullet, -)$ to this triangle, using induction and the Five Lemma we infer that $\mathrm{Hom}_{K^b(\mathcal{W}_F)}(M_1^\bullet, M_2^\bullet) \cong \mathrm{Hom}_{D^b_F(R)}(M_1^\bullet, M_2^\bullet)$ under φ. The proof of the case $w(M_1^\bullet) = r'$ and $w(M_2^\bullet) = r$ for any positive integers r and r' is dual. $\square$

In this case, we can view $K^b(\mathcal{W}_F)$ as a triangle subcategory of $D^b_F(R)$, and note that $K^b(\mathcal{W}_F)$ is closed under direct summands, and hence is thick. It is of interest to consider the quotient triangulated category $D^b_F(R)/K^b(\mathcal{W}_F)$.

Definition 4.2. We define the *$\mathcal{W}_F$-singularity category* of R respect to $\mathcal{W}_F$ to be the following Verdier quotient triangle category $D_{\mathcal{W}_F}(R) := D^b_F(R)/K^b(\mathcal{W}_F)$.

Example 4.3. If $\mathcal{W}_F = \mathrm{Proj}(R)$ and if the subbifunctor F coincides with the bifunctor $\mathrm{Ext}^1_R(-,-)$, then $D^b_F(R)$ is the usual bounded derived category $D^b(R)$ and the $\mathcal{W}_F$-singularity category $D_{\mathcal{W}_F}(R)$ is the big singularity category $D_{Sg}(R)$ (see [4]).

Consider the following composite of natural functors.

$$\theta : \mathcal{G}(\mathcal{W}_F) \hookrightarrow R\text{-Mod} \xrightarrow{i_R} D^b_F(R) \longrightarrow D_{\mathcal{W}_F}(R)$$

where the first functor is the inclusion, the second is the functor which sends left R-modules to the corresponding stalk complexes concentrated in degree zero, and the last is the quotient functor $Q_{\mathcal{W}_F} : D^b_F(R) \longrightarrow D^b_F(R)/K^b(\mathcal{W}_F)$. By [25, Lemma 3.10], the functor $i_R : R\text{-Mod} \to D^b_F(R)$ is fully faithful. Note that $\theta(\mathcal{W}_F) \simeq 0$, and thus θ induces a unique functor $\underline{\theta} : \underline{\mathcal{G}(\mathcal{W}_F)} \to D_{\mathcal{W}_F}(R)$.

The following theorem is our main result.

Theorem 4.4. *Let R be a ring, $\mathcal{W}_F$ a self-orthogonal class of R-modules respect to F. Then the functor*

$$\underline{\theta} : \underline{\mathcal{G}(\mathcal{W}_F)} \longrightarrow D_{\mathcal{W}_F}(R)$$

is a triangle-equivalence if and only if $gl\text{-}\mathcal{G}(\mathcal{W}_F)(R) < \infty$.

In order to show that the functor $\underline{\theta}$ is an equivalence, we start with the following useful lemmas.

Lemma 4.5. *(1) For $M \in {}^{\perp_F}\mathcal{W}_F$ and $X^\bullet \in K^{<0}(\mathcal{W}_F)$, we have* $\mathrm{Hom}_{D_F(R)}(M, X^\bullet) = 0$.

(2) For $N \in \mathcal{W}_F^{\perp_F}$ and $Y^\bullet \in K^{>0}(\mathcal{W}_F)$, we have $\mathrm{Hom}_{D_F(R)}(Y, N) = 0$.

Proof. We only show (1). Consider $\mathcal{L} := \{Z^\bullet \in D_F^b(R) \mid \mathrm{Hom}_{D_F^b(R)}(M, Z^\bullet) = 0\}$. Since $M \in {}^{\perp_F}\mathcal{W}_F$ and $\mathrm{Ext}_F^i(M, W) \cong \mathrm{Hom}_{D_F(R)}(M, W[i])$ for any $W \in \mathcal{W}_F$, we have that $\mathcal{W}_F[i] \subseteq \mathcal{L}$ for all $i > 0$. Observe that the subcategory $\mathcal{L}$ is closed under extensions, and complexes in $K^{<0}(\mathcal{W}_F)$ are obtained by iterated extensions from objects in $\bigcup_{i>0} \mathcal{W}_F[i]$, thus we infer that $K^{<0}(\mathcal{W}_F) \subseteq \mathcal{L}$. □

In what follows, morphisms in $D_F^b(R)$ will be denoted by arrows, and those whose cones lie in $K^b(\mathcal{W}_F)$ will be denoted by doubled arrows. Morphisms in $D_{\mathcal{W}_F}(R)$ from $X^\bullet$ to $Y^\bullet$ will be denoted by right fractions g/f of the form $X^\bullet \overset{f}{\Longleftarrow} Z^\bullet \xrightarrow{g} Y^\bullet$, where $Z^\bullet \in D_F^b(R)$, $f : Z^\bullet \Leftarrow X^\bullet$ is an F-quasi-isomorphism, and $g : Z^\bullet \to Y^\bullet$ is a morphism in $D_F^b(R)$ (for the definition, see [28]). Let $M,\ N \in R$-Mod, we consider the natural map

$$\theta_{M,N} : \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_{D_{\mathcal{W}_F(R)}}(M, N),\ f \longmapsto f/Id_M.$$

Set $\mathcal{W}_F(M, N) = \{f \in \mathrm{Hom}_R(M, N) \mid f \text{ factors through an object in } \mathcal{W}_F\}$. It is easy to see $\theta_{M,N}$ vanished on $\mathcal{W}_F(M, N)$.

Lemma 4.6. *For any $M \in \mathcal{G}(\mathcal{W}_F)$ and $N \in \mathcal{W}_F^{\perp_F}$, we have an isomorphism*

$$\mathrm{Hom}_R(M, N)/\mathcal{W}_F(M, N) \simeq \mathrm{Hom}_{D_{\mathcal{W}_F}(R)}(M, N).$$

Proof. The proof is similar to the one in [10, Lemma 2.3], for completeness we give the detailed proof.

First, we show that the map $\theta_{M,N}$ is surjective. For this, consider a morphism a/s : $M \overset{s}{\Longleftarrow} Z^\bullet \xrightarrow{a} N$ in $D_{\mathcal{W}_F}(R)$, where $Z^\bullet$ is a bounded complex, both a and s are morphisms in $D_F^b(R)$, and the cone of s, $C^\bullet = \mathrm{Cone}(s)$, lies in $K^b(\mathcal{W}_F)$. Hence we have a distinguished triangle in $D_F^b(R)$

$$Z^\bullet \overset{s}{\Longrightarrow} M \longrightarrow C^\bullet \longrightarrow Z^\bullet[1]. \tag{4.1}$$

Since $M \in \mathcal{G}(\mathcal{W}_F)$, we have a long F-exact sequence

$$0 \longrightarrow M \xrightarrow{\varepsilon} T^0 \xrightarrow{d^0} T^1 \longrightarrow \cdots \longrightarrow T^n \xrightarrow{d^n} T^{n+1} \longrightarrow \cdots$$

where each $T^i \in \mathcal{W}_F$ and $\mathrm{Ker}d^i \in {}^{\perp_F}\mathcal{W}_F$. Hence in $D_F^b(R)$, M is isomorphic to the following complex

$$T^\bullet := 0 \longrightarrow T^0 \xrightarrow{d^0} T^1 \longrightarrow \cdots \longrightarrow T^n \xrightarrow{d^n} T^{n+1} \longrightarrow \cdots$$

and furthermore, M is isomorphic to the complex $\tau^{\leq l}T^\bullet := 0 \to T^0 \to \cdots \to T^{l-1} \to \mathrm{Ker}d^l \to 0$ for each $l \geqslant 1$ in $D_F^b(R)$. Consider the following natural triangle in $K^b(R)$

$$(\sigma^{<l}T^\bullet)[-1] \longrightarrow \mathrm{Ker}d^l[-l] \overset{s''}{\Longrightarrow} \tau^{\leq l}T^\bullet \longrightarrow \sigma^{<l}T^\bullet. \tag{4.2}$$

Take s' to be the following composite in $D_F^b(R)$

$$\mathrm{Ker}d^l[-l] \overset{s''}{\Longrightarrow} \tau^{\leq l}T^\bullet \longrightarrow T^\bullet \overset{\varepsilon}{\Longleftarrow} M$$

where $\tau^{\leq l}T^\bullet \longrightarrow T^\bullet$ is the natural chain map. Note that the composite $\tau^{\leq l}T^\bullet \longrightarrow T^\bullet \overset{\varepsilon}{\Longleftarrow} M$ is an isomorphism in $D_F^b(R)$. Thus from the triangle (4.2), we get a triangle in $D_F^b(R)$

$$(\sigma^{<l}T^\bullet)[-1] \longrightarrow \mathrm{Ker}d^l[-l] \overset{s'}{\Longrightarrow} M \xrightarrow{\varepsilon} \sigma^{<l}T^\bullet, \tag{4.3}$$

Since $C^\bullet \in K^b(\mathcal{W}_F)$, we may assume that

$$C^\bullet = \cdots \longrightarrow 0 \longrightarrow W^{-t'} \longrightarrow \cdots \longrightarrow W^{t-1} \longrightarrow W^t \longrightarrow 0 \longrightarrow \cdots$$

where $W^i \in \mathcal{W}_F, t, t' \geq 0$. Set $l_0 = t+1, E = \mathrm{Ker}(T^{l_0} \to T^{l_0+1})$. We can easily get that $E \in {}^{\perp_F}\mathcal{W}_F$ and $C^\bullet[l_0] \in K^{<0}(\mathcal{W}_F)$, by Lemma 4.5(1), we get

$$\mathrm{Hom}_{D_F^b(R)}(E[-l_0], C^\bullet) = \mathrm{Hom}_{D_F^b(R)}(E, C^\bullet[l_0]) = 0.$$

Hence, the morphism $E[-l_0] \overset{s'}{\Longrightarrow} M \longrightarrow C^\bullet$ is zero. By the triangle (4.1), we infer that there exists $h : E[-l_0] \longrightarrow Z^\bullet$ such that $s' = s \circ h$, and thus $a/s = (a \circ h)/s'$.

Note that $N \in \mathcal{W}_F^{\perp_F}$ and $(\sigma^{<l_0}T^\bullet)[-1] \in K^{>0}(\mathcal{W}_F)$, by Lemma 4.5(2), we have

$$\mathrm{Hom}_{D_F^b(R)}((\sigma^{<l_0}T^\bullet)[-1], N) = 0.$$

Applying the cohomological functor $\mathrm{Hom}_{D_F^b(R)}(-, N)$ to the triangle (4.3), we obtain the following exact sequence (here, take $l = l_0$)

$$\mathrm{Hom}_{D_F^b(R)}(M, N) \xrightarrow{\mathrm{Hom}_{D_F^b(R)}(s',N)} \mathrm{Hom}_{D_F^b(R)}(E[-l_0], N) \longrightarrow \mathrm{Hom}_{D_F^b(R)}((\sigma^{<l_0}T^\bullet)[-1], N)\ .$$

Thus there exists $f : M \longrightarrow N$ such that $f \circ s' = a \circ h$ in $D_F(R)$. Hence, we have

$$a/s = (a \circ h)/s' = (f \circ s')/s' = \theta_{M,N}(f),$$

proving that $\theta_{M,N}$ is surjective.

Next we will show that $\mathrm{Ker}\theta_{M,N} = \mathcal{W}_F(M, N)$, then we are done. It is already known that $\mathcal{W}_F(M, N) \subseteq \mathrm{Ker}\theta_{M,N}$. Conversely, consider $f : M \longrightarrow N$ such that $\theta_{M,N}(f) = 0$. Hence there exists $s : Z^\bullet \Longrightarrow M$ such that $f \circ s = 0$, where s is a morphism in $D_F^b(R)$ whose cone $C^\bullet = \mathrm{Cone}(s) \in K^b(\mathcal{W}_F)$. Using the notation above, we obtain that $s' = s \circ h$. Thus $f \circ s' = 0$. By the triangle (4.3), we infer that there exists $f' : \sigma^{<l_0}T^\bullet \longrightarrow N$ such that $f' \circ \varepsilon = f$.

Consider the following natural triangle in $K^b(R)$

$$T^0[-1] \longrightarrow \sigma^{>0}(\sigma^{<l_0}T^\bullet) \Longrightarrow \sigma^{<l_0}T^\bullet \xrightarrow{\pi} T^0 \tag{4.4}$$

Since $N \in \mathcal{W}_F^{\perp_F}$ and $\sigma^{>0}(\sigma^{<l_0}T^\bullet) \in K^{>0}(\mathcal{W})$, by Lemma 4.5(2), we have

$$\mathrm{Hom}_{D_F^b(R)}(\sigma^{>0}(\sigma^{<l_0}T^\bullet), N) = 0.$$

Thus the composite morphism $\sigma^{>0}(\sigma^{<l_0}T^\bullet) \Longrightarrow \sigma^{<l_0}T^\bullet \xrightarrow{f'} N$ is zero, and furthermore, by the triangle (4.4), we infer that there exists $g : T^0 \longrightarrow N$ such that $g \circ \pi = f'$. So we get $f = g \circ (\pi \circ \varepsilon)$, which proves that f factors through $\mathcal{W}_F$ inside $D_F^b(R)$. Note that $i_R : R\text{-Mod} \longrightarrow D_F^b(R)$ is fully faithful, and we infer that f factors through $\mathcal{W}_F$ inside R-Mod, i.e., $f \in \mathcal{W}_F(M, N)$. This finishes the proof. □

Let (α, η) be an exact category and let $\mathcal{C}$ be a triangulated category. Recall from [22, Section 1] that an additive functor $G : \alpha \longrightarrow \mathcal{C}$ is said to be a *∂-functor*, if for each short exact sequence $0 \to X \overset{i}{\to} Y \overset{d}{\to} Z \to 0 \in \eta$, there exists a morphism $\omega_{(i,d)} : G(Z) \longrightarrow G(X)[1]$ such that the following triangle in C is distinguished

$$G(X) \overset{G(i)}{\longrightarrow} G(Y) \overset{G(d)}{\longrightarrow} G(Z) \overset{\omega_{(i,d)}}{\longrightarrow} G(X)[1],$$

moreover, the morphisms ω are "functorial" in the sense that given any morphism between two short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "d"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & X' \arrow[r, "i'"] & Y' \arrow[r, "d'"] & Z' \arrow[r] & 0, \end{tikzcd}$$

the following is a morphism of triangles

$$\begin{array}{ccccccc} G(X) & \xrightarrow{G(i)} & G(Y) & \xrightarrow{G(d)} & G(Z) & \xrightarrow{\omega_{(i,d)}} & G(X)[1] \\ {\scriptstyle G(f)}\downarrow & & {\scriptstyle G(g)}\downarrow & & {\scriptstyle G(h)}\downarrow & & {\scriptstyle G(f)[1]}\downarrow \\ G(X') & \xrightarrow{G(i')} & G(Y') & \xrightarrow{G(d')} & G(Z') & \xrightarrow{\omega_{(i',d')}} & G(X')[1]. \end{array}$$

The next proposition shows that the functor $\underline{\theta}$ is a fully faithful triangle functor. Compare [19, Proposition 1.21] and [10, Lemma 2.3].

Proposition 4.7. *θ induces a fully faithful triangle functor $\underline{\theta}:\underline{\mathcal{G}(\mathcal{W}_F)}\longrightarrow D_{\mathcal{W}_F}(R)$.*

Proof. By Lemma 4.6, the functor $\underline{\theta}:\underline{\mathcal{G}(\mathcal{W}_F)}\longrightarrow D_{\mathcal{W}_F}(R)$ is fully faithful. It remains to show $\underline{\theta}$ is a triangle functor. It is easy to see that $i_R: R\text{-Mod}\longrightarrow D^b_F(R)$ is a ∂-functor (by [25, Proposition 4.1]). In fact, let $0\longrightarrow L\xrightarrow{f}M\xrightarrow{g}N\longrightarrow 0$ be an F-exact sequence with all terms in $\mathcal{G}(\mathcal{W}_F)$. Then it induces a distinguished triangle in $D_{\mathcal{W}_F}(R)$, saying $\theta(L)\xrightarrow{\theta(f)}\theta(M)\xrightarrow{\theta(g)}\theta(N)\xrightarrow{\omega(f,\theta)}\theta(L)[1]$. This shows that θ is a ∂-functor. Note that every object in $\mathcal{W}_F$ is zero in $D_{\mathcal{W}_F}(R)$, so θ vanishes on the projective-injective objects in $\mathcal{G}(\mathcal{W}_F)$. It follows from [10, Lemma 2.5] that the induced functor $\underline{\theta}$ is a triangle functor. □

It is of interest to make sense when θ is dense. We have the following result. The proof is similar to the one in [23, Theorem 4.11], for completeness we give the detailed proof.

Proposition 4.8. *If gl-$\mathcal{G}(\mathcal{W}_F)(R)<\infty$, then the natural functor $\theta:\mathcal{G}(\mathcal{W}_F)\longrightarrow D_{\mathcal{W}_F}(R)$ is dense.*

Proof. Let $X^\bullet\in D^b_F(R)$. By [25, Proposition 3.9], $D^b_F(R)\cong K^{-,Fb}(\mathcal{P}(F))$. Then there exists a complex $C^\bullet_0=(C^i_0,d^i_{C_0})\in K^{-,Fb}(\mathcal{P}(F))$ such that $X^\bullet\cong C^\bullet_0$ in $D^b_F(R)$. Using the fact that $C^\bullet_0$ is F-exact if and only if $\mathrm{Hom}_R(P,C^\bullet_0)$ is exact for any $P\in\mathcal{P}(F)$, we get that there exists $n_0\in\mathbb{Z}$, such that $\mathrm{H}^i(\mathrm{Hom}_R(P,C^\bullet_0))=0$ for all $i\le n_0, P\in\mathcal{P}(F)$.

Let $K^i=\mathrm{Ker}d^i_{C^\bullet_0}$. Then $C^\bullet_0$ is isomorphic to the complex:

$$0\longrightarrow K^i\longrightarrow C^i_0\longrightarrow C^{i+1}_0\longrightarrow C^{i+2}_0\longrightarrow\cdots$$

in $D^b_F(R)$ for any $i\le n_0$. It induces a distinguished triangle in $D^b_F(R)$, hence a distinguished triangle in $D_{\mathcal{W}_F}(R)$ of the following form

$$K^i[-i]\longrightarrow\sigma^{\ge i}C^\bullet_0\longrightarrow C^\bullet_0\longrightarrow K^i[-i+1].$$

Since $\sigma^{\ge i}C^\bullet_0\in K^b(\mathcal{P}(F))\subseteq K^b(\mathcal{W}_F), C^\bullet_0\cong K^i[-i+1]$ in $D_{\mathcal{W}_F}(R)$. Take $l_0=i$, and $Y=K^i$. Then $C^\bullet_0\cong Y[-l_0+1]$ in $D_{\mathcal{W}_F}(R)$. By assumption, we may assume that gl-$\mathcal{G}(\mathcal{W}_F)(R)=m_0<\infty$. By Proposition 3.10, M has a $\mathcal{W}_F$-proper $\mathcal{W}_F$-F resolution. Let $C^\bullet_1\longrightarrow Y$ be the left $\mathcal{W}_F$-resolution of Y. We claim that for any $n\le -m_0+1, \mathrm{Ker}d^n_{C_1}\in\mathcal{G}(\mathcal{W}_F)$, where $d^n_{C^\bullet_1}$ is the n-th differential of $C^\bullet_1$.

Since gl-$\mathcal{G}(\mathcal{W}_F)(R)<\infty$, we have a $\mathcal{W}_F$-proper left $\mathcal{G}(\mathcal{W}_F)$-$F$ resolution

$$0\longrightarrow G^{-m_0}\longrightarrow G^{-m_0+1}\longrightarrow\cdots\longrightarrow G^{-1}\longrightarrow G^0\longrightarrow Y\longrightarrow 0$$

with $G^i\in\mathcal{G}(\mathcal{W}_F)$ for any $-m_0<j\le 0$. Let $G^\bullet$ be the complex

$$0\longrightarrow G^{-m_0}\longrightarrow G^{-m_0+1}\longrightarrow\cdots\longrightarrow G^{-1}\longrightarrow G^0\longrightarrow 0$$

Then there exists an F-quasi-isomorphism $C^\bullet_1\longrightarrow G^\bullet$ lying over id_Y,

$$\begin{array}{ccccccccccccccccc} 0 & \to & \mathrm{Ker}d^n_{C_1} & \to & C^n_1 & \to & \cdots & \to & C^{-m_0}_1 & \to & \cdots & \to & C^{-1}_1 & \to & C^0_1 & \to & Y & \to & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & 0 & \to & 0 & \to & \cdots & \to & G^{-m_0} & \to & \cdots & \to & G^{-1} & \to & G^0 & \to & Y & \to & 0 \end{array}$$

and hence its mapping cone is F-exact. So for any $n \leq -m_0 + 1$, we get the following F-exact complex:

$$0 \to \mathrm{Ker}d^n_{C_1} \to C^n_1 \to \cdots \to C^{-m_0}_1 \to C^{-m_0+1}_1 \oplus G^{-m_0} \to \cdots \to C^0_1 \oplus G^{-1} \to G^0 \to 0.$$

Note that this complex is acyclic, because F-exact is acyclic. Put $K = \mathrm{Ker}(C^0_1 \oplus G^{-1} \to G^0)$, we get an F-exact sequence $0 \to K \to C^0_1 \oplus G^{-1} \to G^0 \to 0$. By [27, Lemma 2.4(2)], we have an exact sequence:

$$\mathrm{Ext}^1_F(G^0, W) \to \mathrm{Ext}^1_R(C^0_1 \oplus G^{-1}, W) \to \mathrm{Ext}^1_R(K, W) \to \mathrm{Ext}^2_F(G^0, W)$$

for any $W \in \mathcal{W}_F$. Since $G^0 \in \mathcal{G}(\mathcal{W}_F)$, $\mathrm{Ext}^i_F(G^0, \mathcal{W}) = 0$ for $i \geq 1$. Because both $C^0_1 \oplus G^{-1}$ and G^0 are in $\mathcal{G}(\mathcal{W}_F)$, $K \in \mathcal{G}(\mathcal{W}_F)$ by Lemma 3.5 and Proposition 3.7. Iterating this process, we get that $\mathrm{Ker}d^n_{C_1} \in \mathcal{G}(\mathcal{W}_F)$ for any $n \leq -m_0 + 1$. Choose a left $\mathcal{W}_F$-proper resolution $C^\bullet_1$ of Y and put $X = \mathrm{Ker}d^{-m_0+1}_{C^\bullet_1}$. By the above claim we have an F-exact sequence:

$$0 \to X \to C^{-m_0+1}_1 \to C^{-m_0+2}_1 \to \cdots \to C^0_1 \to Y \to 0$$

with $X \in \mathcal{G}(\mathcal{W}_F)$. Then $Y \simeq X[m_0]$ in $D_{\mathcal{W}_F}(R)$, and $X \simeq C_0 \simeq Y[-l_0+1] \simeq X[m_0-l_0+1]$ in $D_{\mathcal{W}_F}(R)$. We may assume that $X \cong C^\bullet_0 \cong X[r_0]$, for $r_0 > 0$. Because $X \in \mathcal{G}(\mathcal{W}_F)$, we get a $\mathrm{Hom}(\mathcal{W}_F, -)$ and $\mathrm{Hom}(-, \mathcal{W}_F)$ exact F-exact sequence $0 \longrightarrow X \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots \longrightarrow W^{r_0-1} \longrightarrow X' \longrightarrow 0$ with $X' \in \mathcal{G}(\mathcal{W}_F)$ and $\mathcal{W}^i \in \mathcal{W}_F$ for any $0 \leq i \leq r_0$. It follows that $X \simeq X'[-r_0]$ and $X^\bullet \simeq C^\bullet_0 \simeq X[r_0] \simeq X'$ in $D_{\mathcal{W}_F}(R)$. This completes the proof. $\square$

In this following we show that the converse of Proposition 4.8 is true. Put

$$D^b_F(R)_{fg(\mathcal{W}_F)} := \{M^\bullet \in D^b_F(R) \mid \mathcal{G}(\mathcal{W}_F)\text{-dim}M^\bullet < \infty\}.$$

Lemma 4.9. (1) *$D^b_F(R)_{fg(\mathcal{W}_F)}$ is a triangulated subcategory of $D^b_F(R)$.*

(2) *$K^b(\mathcal{W}_F)$ is a triangulated subcategory of $D^b_F(R)_{fg(\mathcal{W}_F)}$.*

Proof. (1) Clearly $D^b_F(R)_{fg(\mathcal{W}_F)}$ is closed under shift functors [1] and [-1]. Let $M^\bullet$ and $N^\bullet$ be two complexes with $M^\bullet \cong N^\bullet$ in $D^b_F(R)$. Then we have $\mathcal{G}(\mathcal{W}_F)$-dim$M^\bullet < \infty$ if and only if $\mathcal{G}(\mathcal{W}_F)$-dim$N^\bullet < \infty$. Hence $D^b_F(R)_{fg(\mathcal{W}_F)}$ is closed under isomorphisms. Assume that $M^\bullet \longrightarrow N^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet[1]$ is a triangle in $D^b_F(R)$ such that $M^\bullet$ and $N^\bullet$ are in $D^b_F(R)_{fg(\mathcal{W}_F)}$. Then there exists some triangle $X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow X^\bullet[1]$ in $K^b(R)$ such that $M^\bullet \longrightarrow N^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet[1]$ is its image under the canonical functor. Thus we have an isomorphism of triangles

$$\begin{tikzcd}
M^\bullet \arrow[r] \arrow[d] & N^\bullet \arrow[r] \arrow[d] & L^\bullet \arrow[r] \arrow[d] & M^\bullet[1] \arrow[d] \\
X^\bullet \arrow[r] & Y^\bullet \arrow[r] & \mathrm{Cone}(f) \arrow[r] & X^\bullet[1]
\end{tikzcd}$$

in $D^b(R)$, and then $L^\bullet \simeq \mathrm{Cone}(f)$. Since $D^b_F(R)_{fg(\mathcal{W}_F)}$ is closed under isomorphisms and $M^\bullet$, $N^\bullet \in D^b_F(R)_{fg(\mathcal{W}_F)}$, we have $X^\bullet, Y^\bullet \in D^b_F(R)_{fg(\mathcal{W}_F)}$. By the F-exactness of the sequence of complexes $0 \longrightarrow Y^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow X^\bullet[1] \to 0$, we have that $\mathrm{Cone}(f) \in D^b_F(R)_{fg(\mathcal{W}_F)}$ by Corollary 3.16. Hence $L^\bullet \in D^b_F(R)_{fg(\mathcal{W}_F)}$.

(2) Let $X^\bullet$ be a complex in $K^b(\mathcal{W}_F)$. We proceed by induction on the cardinal of the finite set $w(X^\bullet)$. If $w(X^\bullet) = 1$, then the assertion follows from the fact that $\mathcal{W}_F \subseteq \mathcal{G}(\mathcal{W}_F)$. Now suppose that $w(X^\bullet) \geq 2$ with $X^j \neq 0$, $j < 0$, and $X^i = 0$ for any $i > j$. Then we have a distinguished triangle in $D^b_F(R)$:

$$X^\bullet_2[-1] \xrightarrow{u} X^\bullet_1 \longrightarrow X^\bullet \longrightarrow X^\bullet_2$$

in $K^b(\mathcal{W}_F)$, where $X_1^\bullet = X^j[j]$ and $X_2^\bullet = \sigma^{<j}X^\bullet$. By the induction hypothesis and the fact that $D_F^b(R)_{fg(\mathcal{W}_F)}$ is a triangulated subcategory of $D_F^b(R)$, we get that $X^\bullet \in D_F^b(R)_{fg(\mathcal{W}_F)}$. $\square$

Proposition 4.10. *If $\underline{\theta} : \underline{\mathcal{G}(\mathcal{W}_F)} \longrightarrow D_{\mathcal{W}_F}(R)$ is dense, then $\mathcal{G}(\mathcal{W}_F)$-dim$M^\bullet < \infty$ for any $M^\bullet \in D_F^b(R)$. Moreover, in this case, gl-$\mathcal{G}(\mathcal{W}_F)(R) < \infty$.*

Proof. Assume that $\underline{\theta} : \underline{\mathcal{G}(\mathcal{W}_F)} \longrightarrow D_{\mathcal{W}_F}(R)$ is dense and any $M^\bullet \in D_F^b(R)$. It follows that $M^\bullet \cong F(G)$ in $D_{\mathcal{W}_F}(R)$ for some $G \in \mathcal{G}(\mathcal{W}_F)$. Let $s \setminus f : M^\bullet \xrightarrow{f} Z^\bullet \xLeftarrow{s} G$ be an isomorphism in $D_{\mathcal{W}_F}(R)$ with Cone$(s) \in K^b(\mathcal{W}_F)$, then Cone$(f) \in K^b(\mathcal{W}_F)$. Consider the triangle $G \xRightarrow{f} Z^\bullet \longrightarrow \text{Cone}(s) \longrightarrow G[1]$ in $D_F^b(R)$. By Lemma 4.9(2), both G and Cone(s) lie in $D_F^b(R)_{fg(\mathcal{W}_F)}$, so $Z^\bullet \in D_F^b(R)_{fg(\mathcal{W}_F)}$ by Lemma 4.9(1). It follows that $M^\bullet \in D_F^b(R)_{fg(\mathcal{W}_F)}$. Therefore $\mathcal{G}(\mathcal{W}_F)$-dim$M^\bullet < \infty$ for any $M^\bullet \in D_F^b(R)$. By Lemma 3.13, we have $\mathcal{G}(\mathcal{W}_F)$-dim$M < \infty$ for any R-module M. Hence, gl-$\mathcal{G}(\mathcal{W}_F)(R) < \infty$ follows by a argument similar to that of [24, Corollary 2.10]. $\square$

Therefore, Theorem 4.4 follows directly from Propositions 4.8 and 4.10.

Recall that a Gorenstein ring is a left and right Noetherian ring of finite left and right self-injective dimensions. We denote by $\mathcal{G}(\mathcal{P})$ the class of Gorenstein-projective R-modules. Set $F = \text{Ext}_R^1(-,-)$ and $\mathcal{W}_F = R$-Proj, then we have $\mathcal{G}(\mathcal{W}_F) = \mathcal{G}(\mathcal{P})$. In this case, the relative singularity category is the big singularity category of R : $D_{Sg}(R) = D^b(R\text{-Mod})/K^b(R\text{-Proj})$. We then apply the obtained results and we have the following theorem by Beligiannis ([4, Theorem 6.9]), which says that for a Gorenstein ring, the big singularity category is triangle-equivalent to the stable category of Gorenstein-projective modules.

Corollary 4.11. *Let R be a left and right noetherian ring. Then the natural functor*

$$\underline{\theta} : \underline{\mathcal{G}(\mathcal{P})} \to D_{Sg}(R)$$

is triangle-equivalence if and only if R is Gorenstein.

Proof. Set $F = \text{Ext}_R^1(-,-)$ and $\mathcal{W}_F = R$-Proj. From [6, Corollary 3.13] we know that R is Gorenstein if and only if gl-$\mathcal{G}(R\text{-Proj})(R) < \infty$. The assertion follows by Theorem 4.4. $\square$

Remark 4.12. Let R be an Artin algebra. In fact, if all involved modules are finitely generated,Theorem 4.4 still holds by similar arguments. Therefore, we can use this “finitely generated” version of Theorem 4.4 freely.

Using the “finitely generated” version of Theorem 4.4 and [18, Theorem], we have the following result. Compare [5, Theorem 3.6].

Corollary 4.13. *Let R be an Artin algebra. Then the natural functor*

$$\underline{\theta} : \underline{\mathcal{G}(R\text{-proj})} \to D_{sg}(R)$$

is dense if and only if R is Gorenstein.

Let R and S be rings. Following [17], an (S,R)-bimodule $C = {}_SC_R$ is semidualizing if:

(1) ${}_SC$ admits a degreewise finite S-projective resolution.
(2) C_R admits a degreewise finite R-projective resolution.
(3) The homothety map ${}_SS_S \to \text{Hom}_R(C,C)$ is an isomorphism.
(4) The homothety map ${}_RR_R \to \text{Hom}_S(C,C)$ is an isomorphism.
(5) $\text{Ext}_S^{\geqslant 1}(C,C) = 0$.
(6) $\text{Ext}_R^{\geqslant 1}(C,C) = 0$.

Let R and S be rings, and let ${}_SC_R$ be a semidualizing bimodule. Let

$$\mathcal{W}_P = \{C \otimes_R P \mid P \text{ is a projective left } R\text{-module}\},$$
$$\mathcal{W}_I = \{\mathrm{Hom}_S(C, I) \mid I \text{ is an injective left } S\text{-module}\}.$$

The $\mathcal{W}_P$-Gorenstein and $\mathcal{W}_I$-Gorenstein modules which will be called C-Gorenstein projective and C-Gorenstein injective modules respectively. By [14, Corollary 3.2], $\mathcal{W}_P$ and $\mathcal{W}_I$ are self-orthogonal. If we set $F = \mathrm{Ext}^1_S(-,-)$ and $\mathcal{W}_F = \mathcal{W}_P$, then denoted by *gl*-$\mathcal{G}(\mathcal{W}_P)(S)$ the C-Gorenstein projective global dimension of S. Similarly, if we set $F = \mathrm{Ext}^1_R(-,-)$ and $\mathcal{W}_F = \mathcal{W}_I$, then denoted by *cogl*-$\mathcal{G}(\mathcal{W}_I)(R)$ the C-Gorenstein injective global dimension of S. By Theorem 4.4 and its dual, we have the following result.

Corollary 4.14. *Let R and S be rings and ${}_SC_R$ a semidualizing bimodule. Then*

(1) *the functor*
$$\underline{\theta} : \underline{\mathcal{G}(\mathcal{W}_P)} \longrightarrow D_{\mathcal{W}_P}(S)$$
is a triangle-equivalence if and only if gl-$\mathcal{G}(\mathcal{W}_P)(S) < \infty$.

(2) *the functor*
$$\underline{\theta} : \underline{\mathcal{G}(\mathcal{W}_I)} \longrightarrow D_{\mathcal{W}_S}(R)$$
is a triangle-equivalence if and only if cogl-$\mathcal{G}(\mathcal{W}_I)(R) < \infty$.

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