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Two-Dimensional Generalized Magneto-Thermo-Viscoelasticity Problem for a Spherical Cavity with One Relaxation Time Using Fractional Derivative

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Abstract:

The present paper is aimed to studying the two-dimensional generalised magneto-thermo-viscoelasticity problem for a spherical cavity with one relaxation time using fractional derivative. The formulation is applied to generalised thermoelasticity based on the theory of generalised thermoelastic diffusion with one relaxation time. The spherical cavity of the solid surface is assumed to be traction free and subjected to both heating and an external magnetic field. The Laplace transform technique is used to obtain the general solution. The inverse Laplace transform is carried out using a numerical inversion method based. The temperature, displacement, and stresses are obtained and represented graphically with the help of Mathcad software.

Keywords: *Fractional order; magneto-thermo-viscoelasticity; spherical cavity; electromagnetic field*.

1. Introduction

The classical theory of thermoelasticity has been generalised and modified into various thermoelastic models that run under the label of hyperbolic thermoelasticity. The notation hyperbolic reflects the fact that thermal waves are modelled, avoiding the physical paradox of the infinite propagation speed of the classical model. At present, there are several theories of hyperbolic thermoelasticity.

Biot [1] introduced the theory of coupled thermoelasticity, which predicts infinite speeds of wave propagation, which is physically unacceptable. Lord and Shulman [2] introduced the generalized dynamical theory of thermoelasticity with one relaxation time, for the isotropic body. Caputo [3] proposed viscoelastic energy dissipation mechanism based on a memory mechanism with two degrees of freedom for the problem. Ezzat [4] discussed the generalised magneto-thermoelastic waves by thermal shock in half-space. Ezzat [5] used the fractional order derivative to investigate magneto-thermoelasticity with thermoelectric properties. Roychoudhuri et al. [6, 7] investigated magneto-thermoelastic interactions in a viscoelastic cylinder of temperature rate dependent material subjected to periodic loading, as well as the effect of rotation and relaxation times in generalised thermoviscoelasticity. Sherief et al. [8] proposed the new theory of coupled thermoelasticity and generalised thermoelasticity with one relaxation time using the method of fractional calculus. Povstenko [9] solved some thermoelastic problems based on the heat conduction equation in one dimensional and two dimensional domains with a time fractional derivative and associted thermal stresses. Deswal and Kalkal [10] introduced the effects of viscosity and diffusion on thermoelastic interactions in thermally, isotropic and electrically conducting half-space solids whose surfaces are subjected to thermal and mechanical loads.

Zenkour et al. [11] studied the generalised thermodiffusion of an unbounded body for a spherical cavity subjected to periodic loading. Gaikwad et. al. [12] studied the quasi-static thermoelastic mathematical model for an infinitely long circular cylinder by using the integral transform technique. Gaikwad [13] analysed the thermoelastic deformation of a thin hollow circular disk due to a partially distributed heat supply. Gaikwad et. al. [14] studied the non-homogeneous heat conduction problem and its thermal deflection due to internal heat generation in a thin hollow circular disk. Gaikwad [15] analysed the thermoelastic deformation of a thin hollow circular disk due to partially distributed heat supply. H. Sherief and A. M. Abd El-Latief [16] discussed the application of fractional order theory of thermoelasticity problem for a half-space. Raslan [17] solved one dimensional problem of fractional order theory of thermoelasticity of an infinitely long cylindrical cavity using integral transform technique. Kalkal and Deswal [18] investigated the effects of fractional order parameter, viscosity, magnetic field, and diffusion on thermoelastic interaction in an infinite body with a mechanical load on its surface. Hussain [19] solved the fractional order thermoelastic problem for an infinitely long solid circular cylinder. Raslan [20] introduced the fractional-order theory of thermoelasticity to the two-dimensional problem of a thick plate whose lower and upper surfaces are traction-free and subjected to the given axi-symmetric temperature distribution. Gaikwad [21] proposed the two-dimensional study-state temperature distribution of a thin circular plate due to uniform internal energy generation.

Tripathi et al. [22] analyzed the fractional order thermoelastic problem for a thick circular plate with finite wave speeds. Gaikwad [23] discussed the axi-symmetric thermoelastic stress analysis of a thin circular plate due to heat generation. Gaikwad [24] studied the time-fractional

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heat conduction problem in a thin hollow circular disk and its thermal deflection. Khavale et al. [25] introduced the generalized theory of magneto-thermo-viscoelastic spherical cavity problem under fractional order derivative using the state space approach. Gaikwad et al.[26] analyzed the transient thermoelastic temperature distribution of a thin circular plate and its thermal deflection under uniform heat generation. Gaikwad et al.[27] proposed the fractional order thermoelastic problem for finite piezoelectric rod subjected to different types of thermal loading using direct approach. Gaikwad et al.[28] solved the fractional order transient thermoelastic problem using the integral transform technique and discussed stress analysis of a the thin circular sector disk.

In the present work, a new model of time-fractional derivative of order α has been considered in the context of a two-dimensional generalised magneto-thermoviscoelasticity problem for a spherical cavity with one relaxation time. The spherical cavity of the solid surface is assumed to be traction free and subjected to both heating and an external magnetic field. Laplace transform have been employed for the general solution of the problem. The results obtained theoretically have been computed numerically and are depicted graphically. It is believed that this particular problem has not been considered by anyone. This is a new and novel contribution to the field of thermoelasticity. Applications of this study are more useful in the fields of seismology, geomechanics, earthquakes engineering and soil dynamics etc,.

2. Basic Equations and Formulation

The constitutive equations and field equations for an isotropic, homogeneous elastic solid in the absence of body forces under the fractional order theory of generalized thermo-viscoelasticity with temperature-dependent modulus of elasticity can be written in the following form.

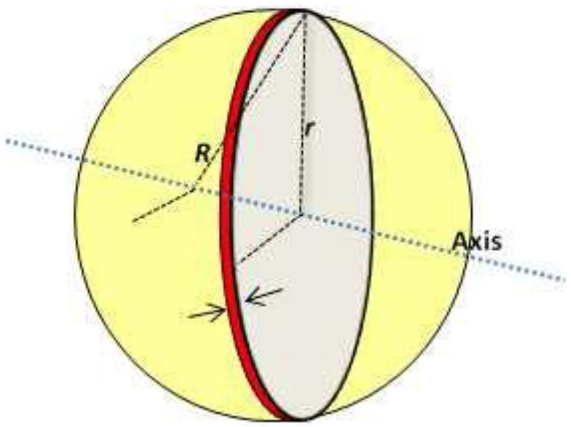

Figure 1. Geometrical representation of the problem.

(i) Maxwell governing equations:

$$curl\ \boldsymbol{h} = \boldsymbol{J} \tag{1}$$

$$curl\ \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{h}}{\partial t} \tag{2}$$

$$\boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}_0 \tag{3}$$

$$div\ \boldsymbol{h} = 0 \tag{4}$$

$$div\ \boldsymbol{E} = 0 \tag{5}$$

where F_i is the components of Lorentz force, whose expression is

$$F_i = \mu_0 (\boldsymbol{J} \times \boldsymbol{H})_i \tag{6}$$

(ii) Equation of motion:

$$\begin{aligned}\rho \frac{\partial^2 u_i}{\partial t^2} &= F_i + \mu\left(1 + \alpha_2 \frac{\partial}{\partial t}\right) u_{i,jj} \\ &+ \left(\lambda\left(1 + \alpha_1 \frac{\partial}{\partial t}\right) + \mu\left(1 + \alpha_2 \frac{\partial}{\partial t}\right)\right) u_{j,ij} \\ &-(3\lambda + 2\mu)\beta(\alpha_t \theta_{,i} - \alpha_c C_{,i})\end{aligned} \tag{7}$$

where α_1 and α_2 are the thermoviscoelastic relaxation times and $\beta = \left(1 + \frac{3\lambda\alpha_1 + 2\mu\alpha_2}{3\lambda + 2\mu}\frac{\partial}{\partial t}\right)$.

(iii) Heat conduction equation:

$$k'\theta_{,ii} = \left(\frac{\partial}{\partial t} + t_0 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)(\rho C_E \theta + (3\lambda + 2\mu)\alpha_t \beta T_0 e + a T_0 C) \tag{8}$$

The Caputo type fractional derivative given by [30]

$$D^\alpha f(t) = \begin{cases} \dfrac{1}{\overline{|n-\alpha}} \displaystyle\int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n; \\ \dfrac{df(t)}{dt}, & n = 1 \end{cases} \tag{9}$$

For finding the Laplace transform, the Caputo derivative requires information of the initial values of the function *f(t)* and its integer derivative of the order *k = 1, 2, ..., n – 1*

$$L\left\{D^\alpha f(t); s\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad n-1 < \alpha < n \tag{10}$$

(iv) Mass diffusion equation:

$$D(3\lambda + 2\mu)\alpha_c \beta e_{,ii} + Da\nabla^2 \theta_{,ii} + \left(\frac{\partial}{\partial t} + t_1 \frac{\partial^2}{\partial t^2}\right) C - Db\nabla^2 C_{,ii} = 0 \tag{11}$$

where D is diffusion coefficient, a is a coefficient describing the measure of thermoelastic diffusion effects and b is a coefficient describing the measure of diffusive effects.

(v) Constitutive equations:

$$\sigma_{ij} = 2\mu\left(1 + \alpha_2 \frac{\partial}{\partial t}\right) e_{ij} + \delta_{ij}\left(\lambda\left(1 + \alpha_1 \frac{\partial}{\partial t}\right) e_{kk} + (3\lambda + 2\mu)\beta(\alpha_t \theta - \alpha_c C)\right) \tag{12}$$

(vi) Chemical potential equation:

$$P = -(3\lambda + 2\mu)\alpha_c \beta e_{kk} + bC - a\theta \tag{13}$$

where P is the chemical potential per unit mass.

Consider the spherical polar coordinates (r, Θ, ϕ) are taken for any representative point of the body at at time t and the origin of the coordinate system is at the center of the spherical cavity of radius *R*. Considering radial variations of the medium, the only non-zero displacement component is $u = u(r, t)$, so that, the component of strain tensor are

$$e_{rr} = \frac{\partial u}{\partial r},\ e_{\theta\theta} = \frac{u}{r} = e_{\phi\phi},\ e_{r\phi} = e_{r\theta} = e_{\theta\phi} = 0 \tag{14}$$

$$e = div\, \mathrm{u} = e_{rr} + e_{\phi\phi} + e_{\theta\theta} = \frac{\partial u}{\partial r} + \frac{2u}{r} = \frac{1}{r^2}\frac{\partial (r^2 u)}{\partial r} \qquad (15)$$

From equation (12), we obtained the stress tensor components as

$$\sigma_{rr} = 2\mu\left(1 + \alpha_2 \frac{\partial}{\partial t}\right)\frac{\partial u}{\partial r} + \lambda\left(1 + \alpha_1 \frac{\partial}{\partial t}\right)e - (3\lambda + 2\mu)\beta(\alpha_t\theta - \alpha_c C) \qquad (16)$$

$$\sigma_{\theta\theta} = 2\mu\left(1 + \alpha_2 \frac{\partial}{\partial t}\right)\frac{u}{r} + \lambda\left(1 + \alpha_1 \frac{\partial}{\partial t}\right)e - (3\lambda + 2\mu)\beta(\alpha_t\theta - \alpha_c C) \qquad (17)$$

and from equation (11), the chemical potential is

$$P = -(3\lambda + 2\mu)\alpha_c\beta e + bC - a\theta \qquad (18)$$

Due to the application of the initial magnetic field $\mathbf{H_0}$, there results an induced magnetic field $\mathbf{h} = (0,0,\mathrm{h})$ which be small, so that, their products with $\mathrm{u_i}$ and their derivatives can be neglected for linearization and an induced electric field E. Applying an initial magnetic field vector $\mathbf{H_0} = (0,0,\mathrm{H_0})$ then equations (1),(2) and (3) yield.

$$J = \left(o, -\frac{\partial \boldsymbol{h}}{\partial u}, 0\right) \qquad (19)$$

$$\square = -\boldsymbol{H_0}\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) \qquad (20)$$

$$\boldsymbol{E} = \left(o, \mu_0 H_0 \frac{\partial u}{\partial t}, 0\right) \qquad (21)$$

The components of Lorentz force can be obtained from equation (19-21) in the form

$$F_r = \mu_0(\boldsymbol{J} \times \boldsymbol{H})_r = \mu_0 H_0^2 \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) \qquad (22)$$

The equation of motion, equation (7) can be written as:

$$\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = \rho\frac{\partial^2 u}{\partial t^2} \qquad (23)$$

Using equations (16),(17) and (23), we get

$$\rho\frac{\partial^2 u}{\partial t^2} = \left[\lambda\left(1 + \alpha_1\frac{\partial}{\partial t}\right) + 2\mu\left(1 + \alpha_2\frac{\partial}{\partial t}\right) + \mu_0 H_0\right]\frac{\partial e}{\partial r} - (3\lambda + 2\mu)\beta\left(\alpha_t\frac{\partial\theta}{\partial r} - \alpha_c\frac{\partial C}{\partial r}\right) \qquad (24)$$

Applying the operator $(\partial/\partial \mathrm{r} + 2/\mathrm{r})$ to both sides of equation (24), one obtains

$$\rho\frac{\partial^2 u}{\partial t^2} = \left[\lambda\left(1 + \alpha_1\frac{\partial}{\partial t}\right) + 2\mu\left(1 + \alpha_2\frac{\partial}{\partial t}\right) + \mu_0 H_0\right]\nabla^2 e - (3\lambda + 2\mu)\beta(\alpha_t\nabla^2\theta - \alpha_c\nabla^2) \qquad (25)$$

The heat conduction equation, equation (8) can be written as:

$$k\nabla^2\theta = \left(\frac{\partial}{\partial t} + t_0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)(\rho C_E\theta + (3\lambda + 2\mu)\alpha_t\beta T_0 e + aT_0 C) \qquad (26)$$

where ∇^2 is Laplaces operator in spherical coordinates which is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\Theta}\left(\sin\Theta\frac{\partial}{\partial\Theta}\right) + \frac{1}{r^2\sin^2\Theta}\frac{\partial^2}{\partial\phi^2} \qquad (27)$$

In case of dependence on only r, this reduce to

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \qquad (28)$$

Now, we will introduced the following non-dimensional variables:

$$r' = c\eta r, \quad u' = c\eta u, \quad P' = \frac{P}{(3\lambda + 2\mu)\alpha_c}, \quad t' = c^2\eta t,$$
$$\theta' = \frac{(3\lambda + 2\mu)\alpha_t}{\lambda + 2\mu}\theta, \quad t_0' = c^2\eta t_0, \quad C' = \frac{(3\lambda + 2\mu)\alpha_c}{\lambda + 2\mu}C, \qquad (29)$$
$$t_1' = c^2\eta t_1, \quad \sigma_{ij}' = \frac{1}{\lambda + 2\mu}\sigma_{ij}, \quad q_r' = \frac{3\lambda\alpha_1 + 2\mu\alpha_2}{k'(3\lambda + 2\mu)c(\lambda + 2\mu)}q_r.$$

where $\eta = \frac{\rho C_E}{k'}$, $c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ is the speed of propagation of isothermal elastic waves, q_r is the heat flux in the radial direction.

Using these non-dimensional variables, equations (16-18) and (25-27) takes the form (dropping the primes for convenience):

$$\frac{\partial^2 e}{\partial t^2} = \beta_1\nabla^2 e - \beta\nabla^2\theta - \beta\nabla^2 C \qquad (30)$$

$$\nabla^2\theta = \left(\frac{\partial}{\partial t} + t_0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)(\theta + \varepsilon\beta e + \varepsilon\beta_2 C) \qquad (31)$$

$$\beta_3\nabla^2 C = \beta\nabla^2 e + \beta_2\nabla^2\theta + \beta_4\left(\frac{\partial}{\partial t} + t_1\frac{\partial^2}{\partial t^2}\right)C \qquad (32)$$

$$\sigma_{rr} = \left(1 + \frac{(\lambda\alpha_1 + 2\mu\alpha_2)^2}{\rho(\lambda + 2\mu)}\frac{\partial}{\partial t}\right)e - \frac{4\mu}{\lambda + 2\mu}\left(1 + \alpha_2\frac{\partial}{\partial t}\right)\frac{u}{r} - \beta\theta - \beta C \qquad (33)$$

$$\sigma_{\theta\theta} = \left(1 - \frac{2\mu}{\lambda + 2\mu}\right)\left(1 + \alpha_1\frac{\partial}{\partial t}\right)e - \frac{2\mu}{\lambda + 2\mu}\left(1 + \alpha_2\frac{\partial}{\partial t}\right)\frac{u}{r} - \beta\theta - \beta C \qquad (34)$$

$$P = \beta_3 C - \beta e - \beta_2\theta \qquad (35)$$

here

$$\beta_1 = 1 + \frac{\mu_0^2 H_0^2}{\rho(\lambda + 2\mu)} + \frac{(\lambda\alpha_1 + 2\mu\alpha_2)^2}{\rho(\lambda + 2\mu)}\frac{\partial}{\partial t},$$

$$\beta_2 = \frac{a\rho c^2}{(3\lambda + 2\mu)^2\alpha_t\alpha_c}, \varepsilon = \frac{(3\lambda + 2\mu)^2\alpha_t^2 T_0}{\rho C_E(\lambda + 2\mu)},$$

$$\beta_3 = \frac{b\rho c^2}{(3\lambda + 2\mu)^2\alpha_c^2}, \quad \beta_4 = \frac{\rho c^2}{(3\lambda + 2\mu)^2\alpha_c^2\eta D}$$

(vii) The initial and regularity conditions:

$$u = 0 = \frac{\partial u}{\partial t}, \text{at } t = 0 \qquad (36)$$

$$\theta = 0 = \frac{\partial\theta}{\partial t}, \text{ at } t = 0 \qquad (37)$$

$$C = 0 = \frac{\partial C}{\partial t} \text{ at } t = 0 \qquad (38)$$

The homogeneous indictional conditions are supplemented by the following boundary conditions:

- The cavity surface is traction free:

$$\sigma_{rr} = 0 \quad \text{at } r = R \qquad (39)$$

- The cavity surface is subjected to a thermal shock:

$$\theta = \theta_0 H(t) \quad \text{at } r = R \tag{40}$$

• The chemical potential is also assumed to be a known function of time at the cavity surface

$$P = P_0 H(t) \quad \text{at } r = R \tag{41}$$

where θ_0 and P_0 are constants and $H(t)$ is heaviside unit step function.

3. Solution in the Laplace Transform Domain

Apply the Laplace transform defined by the relation.

$$\bar{f}(r,s) = L[f(r,t)] = \int_0^\infty e^{-st} f(r,t)dt \tag{42}$$

to equation (30) to (35) under the initial conditions given in equation (36) to (38) we obtain

$$(\nabla^2 - \beta_5 s^2)\bar{e} = \beta_6 s \nabla^2 \bar{\theta} + \beta_6 s \nabla^2 \bar{C} \tag{43}$$

$$\left(\nabla^2 - (s + t_0 s^{\alpha+1})\right)\bar{\theta} = \beta_7 \varepsilon s (s + t_0 s^{\alpha+1})\bar{e} + \beta_2 \varepsilon (s + t_0 s^{\alpha+1})\bar{C} \tag{44}$$

$$\beta_3 \nabla^2 \bar{C} = \beta_8 \nabla^2 \bar{e} + \beta_2 \nabla^2 \bar{\theta} + \beta_4 (s + t_0 s^2)\bar{C} \tag{45}$$

$$\bar{\sigma}_{rr} = \beta_9 \bar{e} - \frac{4\mu(1+\alpha_2 s)}{\lambda+2\mu}\frac{\bar{u}}{r} - \beta_8 \bar{\theta} - \beta_8 \bar{C} \tag{46}$$

$$\bar{\sigma}_{\theta\theta} = 1 + \frac{2\mu}{\lambda+2\mu}(1+\alpha_1 s)\bar{e} + \frac{2\mu(1+\alpha_2 s)}{\lambda+2\mu}\frac{\bar{u}}{r} - \beta_6 \bar{\theta} - \beta_6 \bar{C} \tag{47}$$

$$\bar{P} = \beta_3 \bar{C} - \beta_8 \bar{e} - \beta_2 \bar{\theta} \tag{48}$$

here,

$$\beta_5 = \frac{\rho(\lambda+2\mu)}{\rho(\lambda+2\mu)+\mu_0^2 H_0^2 + (\lambda\alpha_1+2\mu\alpha_2)^2},$$

$$\beta_6 = \frac{\rho(\lambda+2\mu)[(3\lambda+2\mu)+(3\lambda\alpha_1+2\mu\alpha_2)]}{(3\lambda+2\mu)[\rho(\lambda+2\mu)+\mu_0^2 H_0^2+(\lambda\alpha_1+2\mu\alpha_2)^2]}, \quad \beta_7 = \frac{(3\lambda\alpha_1+2\mu\alpha_2)}{3\lambda+2\mu},$$

$$\beta_8 = 1 + \frac{3\lambda\alpha_1+2\mu\alpha_2}{3\lambda+2\mu}s, \quad \beta_9 = 1 + \frac{\mu_0^2 H_0^2}{\rho(\lambda+2\mu)} + \frac{(\lambda\alpha_1+2\mu\alpha_2)^2 s}{\rho(\lambda+2\mu)}$$

Eliminating $\bar{e}$,$\bar{C}$ between equations (43)-(45), one obtained six-order partial differential equation satisfied by $\bar{\theta}$ in the form

$$(\nabla^6 - C_1 \nabla^4 + C_2 \nabla^2 - C_3)\bar{\theta} = 0 \tag{49}$$

here,

$$C_1 = \frac{b_2 a_2 + b_1 a_3 - \xi b_3 - a_1 b_4}{b_1 a_2 - a_1 b_3},$$

$$C_2 = \frac{b_2 a_3 - \xi b_4 - a_1 b_2}{b_1 a_2 - a_1 b_3}, \quad C_3 = \frac{-\xi b_2}{b_1 a_2 - a_1 b_3},$$

in which,

$$a_1 = 1 + \frac{[\rho(\lambda+2\mu)][(3\lambda+2\mu)+(3\lambda\alpha_1+2\mu\alpha_2)s]s\alpha_t\alpha_c}{a\rho c^2[\rho(\lambda+2\mu)+\mu_0^2 H_0^2+(\lambda\alpha_1+2\mu\alpha_2)^2]},$$

$$a_2 = \frac{[\rho(\lambda+2\mu)][(3\lambda+2\mu)+(3\lambda\alpha_1+2\mu\alpha_2)s](3\lambda+2\mu)\alpha_t\alpha_c}{a\varepsilon\rho c^2[\rho(\lambda+2\mu)+\mu_0^2 H_0^2+(\lambda\alpha_1+2\mu\alpha_2)^2]},$$

$$a_3 = \frac{[\rho(\lambda+2\mu)][(3\lambda+2\mu)+(3\lambda\alpha_1+2\mu\alpha_2)s][(s+t_0 s^{\alpha+1})((3\lambda+2\mu)^2\alpha_t\alpha_c - a\varepsilon\rho c^2)]}{a\varepsilon\rho c^2[\rho(\lambda+2\mu)+\mu_0^2 H_0^2+(\lambda\alpha_1+2\mu\alpha_2)^2](3\lambda+2\mu)},$$

$$b_1 = 1 + \frac{s(3\lambda\alpha_1+2\mu\alpha_2)(a+b\alpha_t)}{a(3\lambda+2\mu)},$$

$$b_2 = \frac{\varepsilon s \alpha_t (s+t_0 s^{\alpha+1})(3\lambda\alpha_1+2\mu\alpha_2)}{a\varepsilon\eta D(3\lambda+2\mu)},$$

$$b_3 = \frac{b\alpha_t}{a\varepsilon\alpha_c(s+t_0 s^{\alpha+1})},$$

$$b_4 = \frac{a\rho c^2}{\varepsilon\alpha_t\alpha_c(3\lambda+2\mu)^2} + \frac{\alpha_t}{a\varepsilon\alpha_c\eta D} + \frac{b\alpha_t\lambda(1+\alpha_1 s)}{a\varepsilon\alpha_c(s+t_0 s^{\alpha+1})(\lambda+2\mu)},$$

$$\xi = \frac{\rho(\lambda+2\mu)s^2}{\rho(\lambda+2\mu)+\mu_0^2 H_0^2+(\lambda\alpha_1+2\mu\alpha_2)^2}$$

Similarly, we can show that $\bar{e}$ and $\bar{C}$ satisfy the equations

$$(\nabla^6 - C_1\nabla^4 + C_2\nabla^2 - C_3)\{\bar{e}, \bar{C}\} = 0 \tag{50}$$

Introducing k_i, $i = 1,2,3$ into equation (49), one obtained

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)\bar{\theta} = 0 \tag{51}$$

where, k_1, k_2 and k_3 are the positive roots for the characteristic equation

$$k^6 - C_1 k^4 + C_2 k^2 + C_3 = 0 \tag{52}$$

The roots k_1, k_2 and k_3 are

$$k_1 = \sqrt{\frac{1}{3}[2p\ \sin(q) + C_1]},$$

$$k_2 = \sqrt{\frac{-p}{3}[\sqrt{3}\ \cos(q) + \sin(q)] + \frac{C_1}{3}},$$

$$k_3 = \sqrt{\frac{p}{3}[\sqrt{3}\ \cos(q) - \sin(q)] + \frac{C_1}{3}}$$

Where

$$p = \sqrt{C_1^2 - 3C_2}, \quad q = \frac{1}{3}\sin^{-1}(\chi),$$

$$\chi = -\frac{2C_1^3 - 9C_1 C_2 + 27C_3}{2p^3}$$

The solution of equation (52), which is bounded at infinity, is given by

$$\bar{\theta}(r,s) = \frac{1}{\sqrt{r}}\sum_{i=1}^{3} B_i(s) K_{1/2}(k_i r) \tag{53}$$

where B_i are parameters depending on s and $K_{1/2}(.)$ are the half order modified Bessel function of the second kind.

Similarly,

$$\{\bar{e}(r,s), \bar{C}(r,s)\} = \frac{1}{\sqrt{r}}\sum_{i=1}^{3} \{B_i'(s), B_i''(s)\} K_{1/2}(k_i r) \tag{54}$$

Where

$$B_i' = \frac{b_3 k_i^4 - b_4 k_i^2 + b_2}{b_1 k_i^2 - b_2} B_i = E_i B_i$$

$$B_i'' = \frac{(b_1 - a_5 b_3)k_i^4 - (b_1 a_4 + b_4 - b_4 a_5)k_i^2 + a_4 b_2 - a_5 b_2}{w_6(y_1 k_i^2 - y_2)} B_i = G_i B_i$$

in which

$$a_4 = s + t_0 s^{\alpha+1}, \qquad a_5 = \epsilon(s + t_0 s^{\alpha+1}) \frac{(3\lambda\alpha_1 + 2\mu\alpha_2)s}{3\lambda + 2\mu}$$

Substituting equation (40) into equations (30)-(34), one obtains

$$\{\bar{e}(r,s), \bar{C}(r,s)\} = \frac{1}{\sqrt{r}} \sum_{i=1}^{3} \{E_i, G_i\} B_i K_{1/2}(k_i r) \tag{55}$$

Using the relation between $\bar{u}$ and $\bar{e}$, one gets the solution for the dimensionless form of displacement assuming that $\bar{u}$ vanishes at infinity as:

$$\bar{u} = -\frac{1}{\sqrt{r}} \sum_{i=1}^{3} \frac{E_i}{k_i} B_i K_{3/2}(k_i r) \tag{56}$$

Thus, from equations (55) and (56), one obtains

$$\bar{\sigma}_{rr}(r,s) = \frac{1}{\sqrt{r}} \sum_{i=1}^{3} \left((\beta_9 E_i - \beta_8 - \beta_8 G_i) K_{\frac{1}{2}}(k_i r) + \frac{4\mu(1+\alpha_2 s)}{\lambda+2\mu} \frac{E_i}{k_i r} K_{3/2}(k_i r) \right) B_i(s) \tag{57}$$

$$\bar{\sigma}_{\theta\theta}(r,s) = \frac{1}{\sqrt{r}} \sum_{i=1}^{3} \left(\left(\left(1 + \frac{2\mu}{\lambda+2\mu}\right)(1 + \alpha_1 s)E_i - \beta_8 - \beta_8 G_i \right) K_{\frac{1}{2}}(k_i r) - \frac{2\mu(1+\alpha_2 s)}{\lambda+2\mu} \frac{E_i}{k_i r} K_{3/2}(k_i r) \right) B_i(s) \tag{58}$$

$$\bar{P}(r,s) = \frac{1}{\sqrt{r}} \sum_{i=1}^{3} (\beta_3 G_i - \beta_2 - \beta_8 E_i) K_{1/2}(k_i r) B_i(s) \tag{59}$$

The transformed boundary conditions become

$$\bar{\sigma}_{rr} = 0, \; \bar{\theta} = \frac{\bar{\theta}_0}{s}, \; \bar{P} = \frac{\bar{P}_0}{s}, \quad \text{at } r = R \tag{60}$$

Apply the boundary conditions given in equation (60) together with equations (55) and (57-59) is used. we obtains:

$$\sum_{i=1}^{3} B_i(s) K_{1/2}(k_i R) = \frac{\theta_0 \sqrt{R}}{s} \tag{61}$$

$$\sum_{i=1}^{3} \left((\beta_9 E_i - \beta_8 - \beta_8 G_i) K_{\frac{1}{2}}(k_i R) + \frac{4\mu(1+\alpha_2 s)}{\lambda+2\mu} \frac{E_i}{k_i R} K_{3/2}(k_i R) \right) B_i(s) = 0 \tag{62}$$

$$\sum_{i=1}^{3} (\beta_3 G_i - \beta_2 - \beta_8 E_i) K_{1/2}(k_i R) B_i(s) = \frac{P_0 \sqrt{R}}{s} \tag{63}$$

Equations (61)-(63) is a system of linear equations with $B_i(s)$ as unknown parameters. On solving these equations, we get the complete solution of the problem in the Laplace transform domain.

4. Numerical Inversion of the Laplace Transforms

Laplace transformation of the continuous f(t) function is presented

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt \tag{64}$$

for $t > 0$ and $s = x + iy$.

The inversion integral is utilized to identify the actual function f(t) when the solution is provided in the Laplace domain.

$$f(t) = \int_{\gamma-i\infty}^{\gamma+i\infty} e^{-st} \bar{f}(s) ds \tag{65}$$

Where, contour should be placed to the right of all $\bar{f}(s)$ singularities. The direct Equation (65) integration is usually challenging and sometimes not feasible analytically. We use a numerical inverse approach based on the Stehfest for ultimate solution of the stress distribution, displacement temperature in the time domain [29]. In the given approach, the inverse f(t) of Laplace $\bar{f}(s)$ is estimated by the relationship.

$$f(t) = \frac{\ln 2}{t} \sum_{j=1}^{N} V_j \; F\left(\frac{\ln 2}{t} j\right) \tag{66}$$

Where the following equation is presented V_j:

$$V_j = (-1)^{((N/2)+1)} \sum_{k=(i+1)/2}^{\min(i,N/2)} \frac{k^{((N/2)+1)}(2k)!}{(N/2-k)! \; k! \; (i-k)! \; (2k-1)!} \tag{67}$$

The N parameter is the summation number (63) of terms and must be maximized by trial and error. Rising N improves the result accuracy to a point and subsequently decreases accuracy due to increased round-off errors. All parameters' solutions in the space time domain are therefore provided with

$$\theta(r,t) = \frac{\ln 2}{t} \sum_{j=1}^{N} V_j \; \bar{\theta}\left(r, \frac{\ln 2}{t} j\right) \tag{68}$$

$$u(r,t) = \frac{\ln 2}{t} \sum_{j=1}^{N} V_j \; \bar{u}\left(r, \frac{\ln 2}{t} j\right) \tag{69}$$

$$\sigma_{rr}(r,t) = \frac{\ln 2}{t} \sum_{j=1}^{N} V_j \; \bar{\sigma}_{rr}\left(r, \frac{\ln 2}{t} j\right) \tag{70}$$

$$\sigma_{\theta\theta}(r,t) = \frac{\ln 2}{t} \sum_{j=1}^{N} V_j \; \bar{\sigma}_{\theta\theta}\left(r, \frac{\ln 2}{t} j\right) \tag{71}$$

$$P = \frac{\ln 2}{t} \sum_{j=1}^{N} V_j \; \bar{P}\left(r, \frac{\ln 2}{t} j\right) \tag{72}$$

5. Numerical Results and Discussion

The copper material was chosen for purposes of numerical evaluations and the constants of the problem were taken as following Table 1.

The numerical calculation and graphs are carried out with the help of computational mathematical software PTC Mathcad Prime-7.0.0.0

Figure 2-6 shows the variation of the temperature field, displacement and stresses vary with different values of times, t = 0.25, 0.50, 0.75, 1 with fractional-order parameter $\alpha = 1$. Figure 2 has been plotted to illustrate the variation of temperature field in radial direction with different time parameters. The temperature filed start with the maximum value (in magnitude) and then gradually decreases with increase the radius. Figure 3 shows that the displacement increases as time t increases for $r \leq 0.2$ and its remains constant for $r \geq 0.2$. Figure 4 shows that variation of radial stress in radial direction, it is clear that initially radial stresses decreases within region $0 \leq r \leq 0.1$ and increases within the region $0.1 \leq r \leq 1$ with increases time. Figure 5 shows that the value of angular stress increases with an increase in time t along the radial direction.

Table 1. Material constants.

Physical constants	Value
Reference uniform temperature (T_0)	293 K
Thermal diffusivity (c)	84.18 m^2/s
Thermal conductivity (k')	386 W/(m. K)
Density (ρ)	8954 kg/m^3
Lame's constants (μ)	3.86×10^{10} $kg/(m.\ s^2)$
Lame's constants (λ)	7.76×10^{10} $kg/(m.\ s^2)$
Coefficients of linear thermal expansion (α_t)	1.78×10^{-5} K^{-1}
Coefficients of linear diffusion expansion(α_c)	1.98×10^{-4} m^3/kg
Specific heat at constant strain (C_E)	383.1 J/(kg. K)
Magnetic permeability (μ_0)	$4\pi \times 10^{-7}$ H/m
Applied Magnetic field (H_0)	$10^7/\ 4\pi$ H/m
Coefficient describing the measure of thermoelastic diffusion effects (a)	1.2×10^4 $m^2/(K.\ s^2)$
Coefficient describing the measure of thermoelastic diffusive effects (b)	0.9×10^6 $m^5/(kg.\ s^2)$
Diffusion coefficient (D)	0.85×10^{-8} $kg/m^3.s$
Thermal relaxation time (t_0)	0.2 s
Diffusion relaxation time (t_1)	0.02 s
Component of thermoviscoelastic relaxation time (α_1)	0.06 s
Component of thermoviscoelastic relaxation time (α_2)	0.09 s

Figure 6-9 shows the variation of the temperature, displacement and stress with different values of fractional-order parameter α at time t = 0.5. Figure 6 depicts the variation of temperature distribution along radial direction with t = 0.5 for different values of parameter α. Also, it be seen that, the fractional parameter has an increasing effects on the magnitude of this field. The profile of displacement distribution at t = 0.5 for different values of fractional-order parameter α is displaced in figure 7. The fractional-order parameter is found to have decreasing effects on this distribution. The radial and angular stresses σ_rr and $\sigma_{\theta\theta}$ are presented in figure 8 and 9 respectively, to investigate the effects of fractional parameter α. It is noticed that the stresses σ_{rr} and $\sigma_{\theta\theta}$ increases with decreasing the fractional-order parameter α.

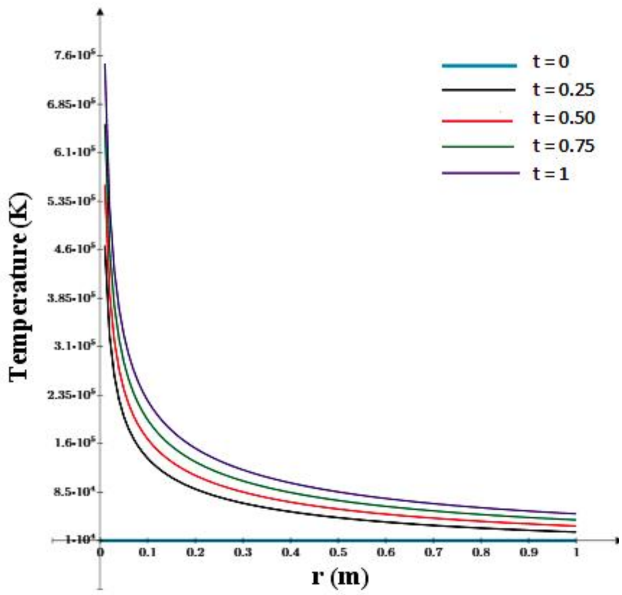

Figure 2. Temperature distribution at α = 0.5 and different values of t.

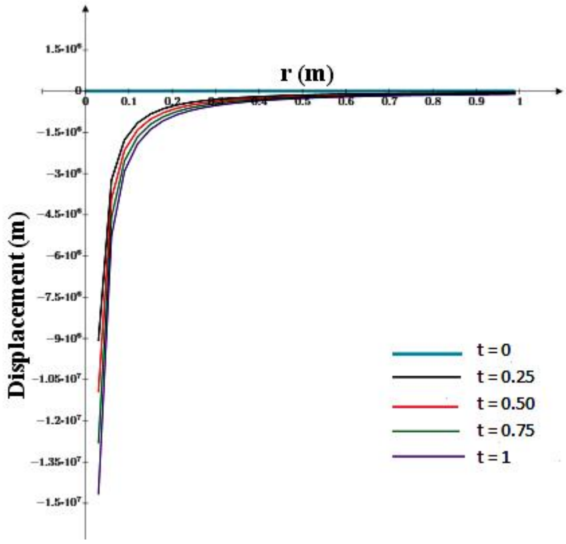

Figure 3. Displacement distribution at α = 0.5 and different values of t.

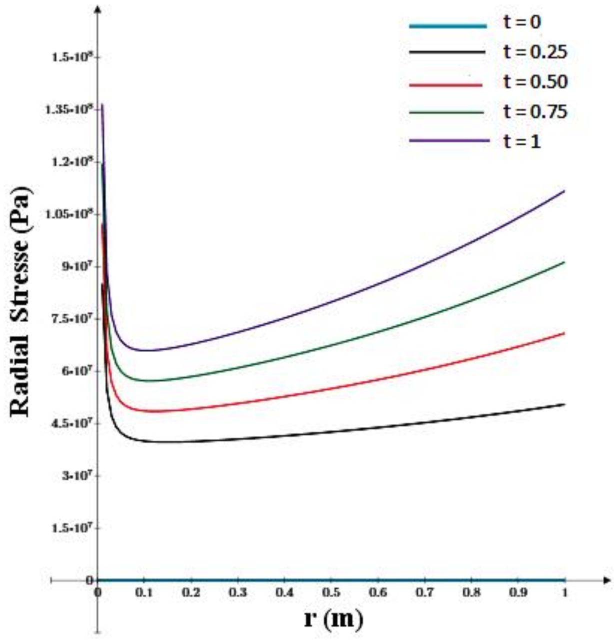

Figure 4. Radial stress distribution at α = 0.5 and different values of t.

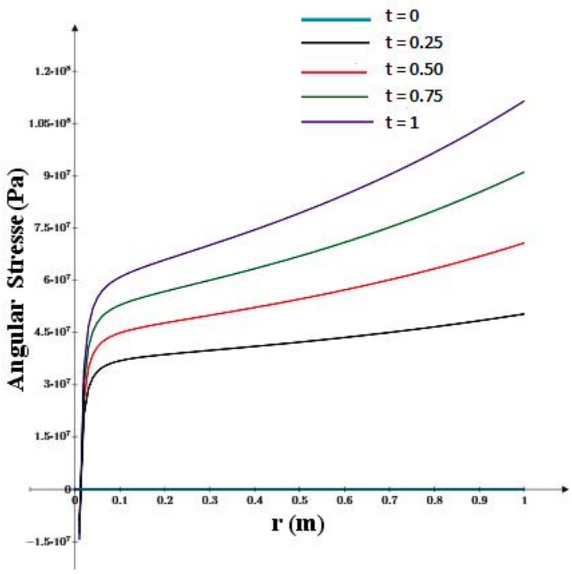

Figure 5. Angular stress distribution at α = 1 and different values of t.

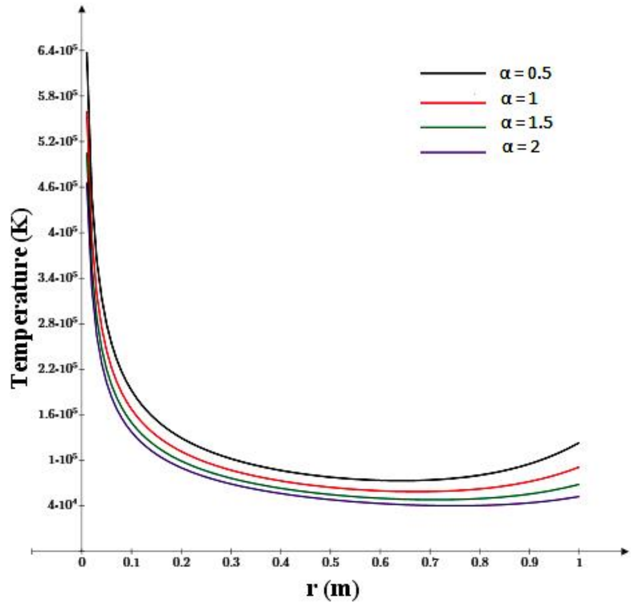

Figure 6. Temperature distribution at t =0.5 and different values of α.

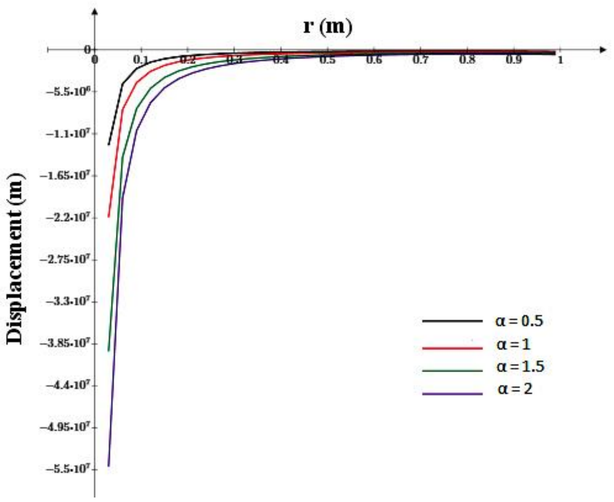

Figure 7. Displacement distribution at t =0.5 and different values of α.

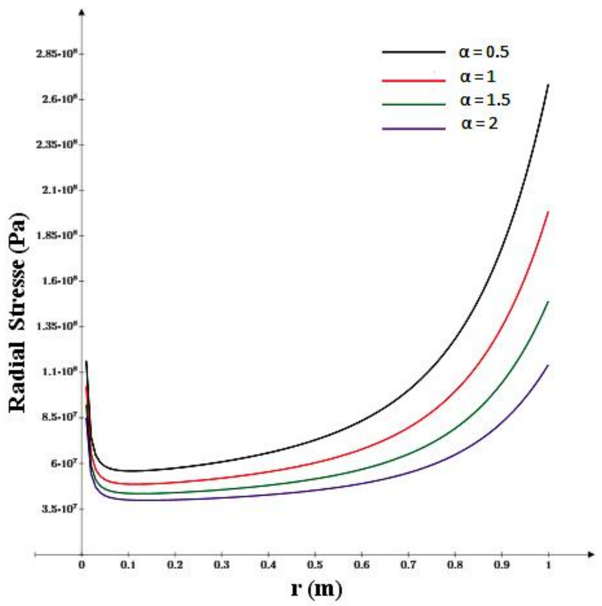

Figure 8. Radial stress distribution at t =0.5 and different values of α.

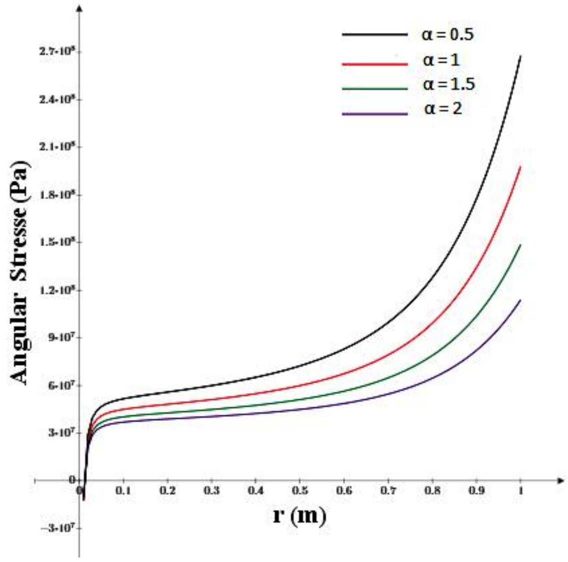

Figure 9. Angular stress distribution at t = 0.5 and different values of α.

5. Conclusion

A two-dimensional boundary value problem based on the theory of generalised magneto-thermo-viscoelasticity for a spherical cavity with one relaxation time based on a fractional order model is solved. The spherical cavity of a solid surface is taken to be traction free with subject to both heating and an external magnetic field. Theoretical and numerical results reveal, that all the fractional-order parameters and time have a salient effect on the considered physical variables. The following concluding remakes can be considered according to the results of the present study.

1. In figure 2–5, the effect of time is quite pertinent on all the fields and can easily be noticed from the figures. The increase in the values of time results in increases in the numerical values of the physical variables. Hence, it has a increasing effect.
2. In figure 6–9, we observe that the, the fractional-order parameter strongly affects the physical quantities. It has a decreasing effect (in terms of magnitude) a profile of temperature, displacement and stresses.
3. The fractional order parameter $0 < \alpha < 1$, $1 < \alpha < 2$ and $\alpha = 1$ indicates the weak, strong and normal conductivity respectively. For a normal conductivity $\alpha = 1$ the results coincide with all the previous of application that are taken in the context of the generalised thermoelasticity with one relaxation time in the various field.
4. The results presented in this paper will be very helpful for researchers concerned with martial science, and designers of new materials, etc.

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Nomenclature (List of Symbols):

J current density vector (A/m^2)

E induced electric field (V/m)

H_0 applied Magnetic field (N.s/C.m)

h the perturbation occurred in the total magnetic field by induction (Tesla)

u_i components of displacement vector (m)

T absolute temperature (K)

T_0 reference uniform temperature (K)

C concentration of the diffusive material in the elastic body (m^2/s)

α_t coefficients of linear thermal expansion (K^{-1})

α_c coefficients of linear diffusion expansion (K^{-1})

k' thermal conductivity (W/m.K)

C_E specific heat at constant strain (J/Kg.K)

e cubical dilation (K^{-1})

t_0 thermal relaxation time (s)

t_1 diffusion relaxation time (s)

F_i component of Lorentz force (Tesla)

P chemical potential (J/kg)

D diffusion coefficient (kg/m^3.s)

c speed of propagation of isothermal elastic waves

q_r heat flux in the radial direction(W/m^2)

a coefficient describing the measure of thermoelastic diffusion effects

b coefficient describing the measure of thermoelastic diffusive effects

α_1, component of thermoviscoelastic relaxation time(s)

α_2 component of thermoviscoelastic relaxation time(s)

$\theta=T-T_0$ temperature increment such that $|\theta/T_0|$= 1 (K)

Greek symbols

λ , μ Lame's constants (GPa)

μ_0 magnetic permeability (H/m)

ρ density (kg/m^3)

δ_{ij} Kronecker's delta tensor

σ_{ij} components of stress tensor

Abbreviations

1D one-dimensional (m)

2D two-dimensional (m)

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