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Pell-Lucas Collocation Method to Solve Second-Order Nonlinear Lane-Emden Type Pantograph Differential Equations

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Abstract: In this article, we present a collocation method for second-order nonlinear Lane-Emden type pantograph differential equations under intial conditions. According to the method, the solution of the problem is sought depending on the Pell-Lucas polynomials. The Pell-Lucas polynomials are written in matrix form based on the standard bases. Then, the solution form and its the derivatives are also written in matrix forms. Next, a transformation matrix is constituted for the proportion delay of the solution form. By using the matrix form of the solution, the nonlinear term in the equation is also expressed in matrix form. By using the obtained matrix forms and equally spaced collocation points, the problem is turned into an algebraic system of equations. The solution of this system gives the coefficient matrix in the solution form. In addition, the error estimation and the residual improvement technique are also presented. All presented methods are applied to three examples. The results of applications are presented in tables and graphs. In addition, the results are compared with the results of other methods in the literature.

Keywords: Collocation method, Lane-Emden differential equation, nonlinear differential equations, pantograph differential equations, Pell-Lucas polynomials.

1. Introduction

Many scientific phenomena are modeled with the nonlinear differential equations [10], [11]. It is not always possible to find the analytical solutions in such equations. For this reason, the numerical methods are of great importance. To date, many numerical methods are available in the literature for various types of the nonlinear differential equations [1]-[6], [9], [12]-[17], [19]-[26], [29], [35]-[51]. Yüzbaşı et. al have worked on various numerical methods by using many special polynomials until today [31]-[33], [52]-[62]. In addition, the pantograph-type differential equations have been used to characterize the problems in many fields such as physics, electronic systems,

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electrodynamics, control problems, engineering, biology, population studies, infectious diseases, medicine, and economics. Many studies on the pantograph differential equations are available in the literature [3, 7, 8, 24, 30, 34, 36, 47].

On the other hand, there are studies by using the Pell-Lucas polynomials for many types of the differential equations [18], [49], [63], [64]. Since no study has yet been done by using the Pell-Lucas polynomials for the solutions of the nonlinear Lane-Emden type pantograph differential equation (LETPDE), the Pell-Lucas polynomials are used for the approximate solutions of the nonlinear second-order Lane-Emden type pantograph differential equation (LETPDE) in this study.

In this paper, we consider the second-order nonlinear Lane-Emden type pantograph differential equation (LETPDE)

$$\gamma y^{''}(\gamma t) + \frac{\beta}{t} y^{'}(\gamma t) + y^{k}(t) = g(t), \quad 0 < t \leq L, \tag{1}$$

with the initial conditions

$$y(0) = \lambda, \quad y^{'}(0) = \mu. \tag{2}$$

Here, $k \in \mathbb{N}$, $y(t)$ is the unknown function, $g(t)$ is a continuous function $[0, L]$ and $\gamma, \beta, \lambda, \mu$ are some suitable constants. $L > 0$ and $t = 0$ is the single singular point of the LETPDE. Our aim is to find the approximate solution of the problem (1)-(2) in the form

$$y(t) \cong y_N(t) = \sum_{n=0}^{N} a_n Q_n(t). \tag{3}$$

Here, $a_n \quad (n = 0, 1, ..., N)$ are the Pell Lucas coefficients, N is any positive integer and $Q_n(t)$ are the Pell Lucas polynomials defined by

$$Q_n(t) = \sum_{k=0}^{[\![n/2]\!]} 2^{n-2k} \frac{n}{n-k} \binom{n-k}{k} t^{n-2k}.$$

In this study, we use two important properties of the Pell-Lucas polynomials: the recurrence relation [27], [28]

$$Q_n(s) = 2sQ_{n-1}(s) + Q_{n-2}(s), \quad n \geq 2,$$

where $Q_0(s) = 2$, $Q_1(s) = 2s$, and the recurrence relation for the derivative [27], [28]

$$Q_n^{'}(s) = 2sQ_{n-1}^{'}(s) + Q_{n-2}^{'}(s) + 2Q_{n-1}(s), \quad n \geq 2, \tag{4}$$

where $Q_0^{'}(s) = 0$, $Q_1^{'}(s) = 2$. For more features on the Pell-Lucas polynomials, please see [27], [28].

The rest of this paper is organized as follows: The matrix form of the approximate solution and the required matrix relations are established in Section 2. The Pell-Lucas collocation method is constituted in Section 3. The error estimation are presented in Section 4. The applications of the method are given and the results are discussed in Section 5. In Section 6, the conclusions of the paper are given.

2. Basic Matrix Relations

In this section, we express the matrix forms of the problem (1)-(2) via the Pell-Lucas polynomials.

Lemma 2.1 *The vector* $\mathbf{Q}_N(t)$ *can be written as*

$$\mathbf{Q}_N(t) = \mathbf{T}_N(t)\mathbf{D}_N, \tag{5}$$

where $\mathbf{T}_N(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^N \end{bmatrix}$ *and if* N *is odd*

$$\mathbf{D}_N^T = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2^1 \frac{1}{1}\binom{1}{0} & 0 & \cdots & 0 \\ 2^0 \frac{2}{1}\binom{1}{1} & 0 & 2^2 \frac{2}{2}\binom{2}{0} & \cdots & 0 \\ 0 & 2^1 \frac{3}{2}\binom{1}{1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 2^1 \frac{N}{\frac{N+1}{2}}\binom{\frac{N+1}{2}}{\frac{N-1}{2}} & 0 & \cdots & 2^N \frac{N}{N}\binom{N}{0} \end{bmatrix},$$

and if N *is even*

$$\mathbf{D}_N^T = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2^1 \frac{1}{1}\binom{1}{0} & 0 & \cdots & 0 \\ 2^0 \frac{2}{1}\binom{1}{1} & 0 & 2^2 \frac{2}{2}\binom{2}{0} & \cdots & 0 \\ 0 & 2^1 \frac{3}{2}\binom{1}{1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^0 \frac{N}{\frac{N}{2}}\binom{\frac{N}{2}}{\frac{N}{2}} & 0 & 2^2 \frac{N}{\frac{N+2}{2}}\binom{\frac{N+2}{2}}{\frac{N-2}{2}} & \cdots & 2^N \frac{N}{N}\binom{N}{0} \end{bmatrix}.$$

Proof By multiplying the vector $\mathbf{T}_N(t)$ by the matrix $\mathbf{D}_N$ from the right side, the vector $\mathbf{Q}_N(t) = \mathbf{T}_N(t)\mathbf{D}_N$ is obtained. □

Lemma 2.2 *The approximate solution based on the Pell-Lucas polynomials in (3) can be expressed in the form*

$$y(t) \approxeq y_N(t) = \mathbf{T}_N(t)\mathbf{D}_N\mathbf{A}_N, \tag{6}$$

where

$$\mathbf{A}_N = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T.$$

Proof By multiplying the vector $\mathbf{Q}_N(t) = \mathbf{T}_N(t)\mathbf{D}_N$ by the vector $\mathbf{A}_N$ from the right, the relation (6) is found. □

Lemma 2.3 *The matrix relations for the first and second derivatives of the solution form (6) are respectively as follows*

$$\begin{aligned} y'(t) \cong y_N'(t) = \boldsymbol{T}_N(t)\boldsymbol{B}_N\boldsymbol{D}_N\boldsymbol{A}_N, \\ y''(t) \cong y_N''(t) = \boldsymbol{T}_N(t)\boldsymbol{B}_N^2\boldsymbol{D}_N\boldsymbol{A}_N, \end{aligned} \tag{7}$$

where

$$\mathbf{B}_N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Proof When the first and second derivatives of (6) are taken, we have respectively

$$\begin{aligned} y'(t) \cong y_N'(t) = \mathbf{T}_N'(t)\mathbf{D}_N\mathbf{A}_N, \\ y''(t) \cong y_N''(t) = \mathbf{T}_N''(t)\mathbf{D}_N\mathbf{A}_N. \end{aligned} \tag{8}$$

Next, the first and second derivatives of $\mathbf{T}_N(t)$ are taken to obtain

$$\begin{aligned} \mathbf{T}_N'(t) = \mathbf{T}_N(t)\mathbf{B}_N, \\ \mathbf{T}_N''(t) = \mathbf{T}_N(t)\mathbf{B}_N^2. \end{aligned} \tag{9}$$

Thus, by writing the relations (9) in place of (8) respectively, we obtain the matrix relations for the first and second derivatives of the solution form (6). □

Lemma 2.4 *The matrix relations for the proportional delay of the first and second derivatives of the solution form (6) become respectively as follows*

$$\begin{aligned} y'(\gamma t) \cong y_N'(\gamma t) = \mathbf{T}_N(t)\mathbf{M}_N(\gamma)\mathbf{B}_N\mathbf{D}_N\mathbf{A}_N, \\ y''(\gamma t) \cong y_N''(\gamma t) = \mathbf{T}_N(t)\mathbf{M}_N(\gamma)\mathbf{B}_N^2\mathbf{D}_N\mathbf{A}_N, \end{aligned} \tag{10}$$

where

$$\mathbf{M}_N(\gamma) = \begin{bmatrix} (\gamma)^0 & 0 & \cdots & 0 \\ 0 & (\gamma)^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\gamma)^N \end{bmatrix}.$$

Proof If γt is written instead of t in (7), then it is achieved

$$\begin{aligned} y'(\gamma t) \cong y_N'(\gamma t) = \mathbf{T}_N(\gamma t)\mathbf{B}_N\mathbf{D}_N\mathbf{A}_N, \\ y''(\gamma t) \cong y_N''(\gamma t) = \mathbf{T}_N(\gamma t)\mathbf{B}_N^2\mathbf{D}_N\mathbf{A}_N. \end{aligned} \tag{11}$$

On the other hand, by multiplying the vector $\mathbf{T}_N(t)$ by the vector $\mathbf{M}_N(\gamma)$ from the right, we gain

$$\mathbf{T}_N(\gamma t) = \mathbf{T}_N(t)\mathbf{M}_N(\gamma). \tag{12}$$

Finally, the desired result is obtained by substituting the relation (12) in (11). □

Lemma 2.5 *The matrix representation of the solution (6) for the nonlinear term in (1) is expressed as follows*

$$y^k(t) \cong y_N^k(t) = (\mathbf{T}_N(t)\mathbf{D}_N\mathbf{A}_N)^{k-1}(\mathbf{T}_N(t)\mathbf{D}_N\mathbf{A}_N). \tag{13}$$

Proof We can write $y_N^k(t)$ as

$$y_N^k(t) = y_N^{k-1}(t)y_N(t). \tag{14}$$

Then by substituting (6) in (14), the desired results is obtained. □

Lemma 2.6 *The matrix relations of the initial conditions (2) for the assumed solution form (6) are respectively as follows*

$$\begin{aligned} \mathbf{U}_N\mathbf{A}_N &= \lambda_N, \quad \mathbf{U}_N = \mathbf{T}_N(0)\mathbf{D}_N, \\ \mathbf{V}_N\mathbf{A}_N &= \mu_N, \quad \mathbf{V}_N = \mathbf{T}_N(0)\mathbf{B}_N\mathbf{D}_N. \end{aligned} \tag{15}$$

Proof If 0 is written instead of t in (6) and (7), it becomes $y(0) \cong y_N(0) = \mathbf{T}_N(0)\mathbf{D}_N\mathbf{A}_N$ and $y'(0) \cong y'_N(0) = \mathbf{T}_N(0)\mathbf{B}_N\mathbf{D}_N\mathbf{A}_N$. Hence, we have $\mathbf{U}_N\mathbf{A}_N = \lambda_N$, $\mathbf{U}_N = \mathbf{T}_N(0)\mathbf{D}_N$ and $\mathbf{V}_N\mathbf{A}_N = \mu_N$, $\mathbf{V}_N = \mathbf{T}_N(0)\mathbf{B}_N\mathbf{D}_N$. Consequently, the desired result is found. □

Theorem 2.1 *It is assumed that the solution of the problem (1)-(2) is sought in form (6). In that case, the following matrix relation is obtained*

$$\left\{\gamma\mathbf{T}_N(t)\mathbf{M}_N(\gamma)\mathbf{B}_N^2\mathbf{D}_N + \frac{\beta}{t}\mathbf{T}_N(t)\mathbf{M}_N(\gamma)\mathbf{B}_N\mathbf{D}_N + (\mathbf{T}_N(t)\mathbf{D}_N\mathbf{A}_N)^{k-1}\mathbf{T}_N(t)\mathbf{D}_N\right\}\mathbf{A}_N = g(t). \tag{16}$$

Proof $y''_N(t)$, $y'_N(t)$ and $y_N^k(t)$ in (11) and (13) are substituted in (1) and thus, the proof of the theorem is completed. □

3. Method of the Solution

The purpose of this section is to present the Pell-Lucas collocation method. For this reason, firstly, evenly spaced collocation points are defined. Next, the method is constructed by using these collocation points and the matrix relations in the previous section.

Definition 3.1 *The evenly spaced collocation points are defined by*

$$t_i = a + \frac{b-a}{N}i, \quad i = 0, 1, ..., N. \tag{17}$$

Note here that: According to Theorem 3.1, these collocation points are used in (1). If the point a here becomes 0, a singularity occurs. In Theorem 3.2, the matrix forms obtained for the conditions are written instead of any two lines in the algebraic equation system obtained in Theorem 3.1. Therefore, in order to prevent the singularity that occurs when a point is 0, instead of the row consisting of point that cause this singularity in Theorem 3.2, the matrix form formed for the first condition is written.

Theorem 3.1 *Suppose that the solution of (1) is of form (6). In this instance, (1) by using the collocation points (17) is reduced to a system of nonlinear algebraic equations as follows*

$$\mathbf{W}_N \mathbf{A}_N = \mathbf{G}_N, \tag{18}$$

where

$$\mathbf{W}_N = \begin{bmatrix} \bar{\mathbf{W}}_N(t_0) & \bar{\mathbf{W}}_N(t_1) & \cdots & \bar{\mathbf{W}}_N(t_N) \end{bmatrix}^T, \quad \mathbf{G}_N = \begin{bmatrix} g(t_0) & g(t_1) & \cdots & g(t_N) \end{bmatrix}^T,$$

$$\bar{\mathbf{W}}_N(t_i) = \gamma \mathbf{T}_N(t_i)\mathbf{M}_N(\gamma)\mathbf{B}_N^2\mathbf{D}_N + \frac{\beta}{t_i}\mathbf{T}_N(t_i)\mathbf{M}_N(\gamma)\mathbf{B}_N\mathbf{D}_N + (\mathbf{T}_N(t_i)\mathbf{D}_N\mathbf{A}_N)^{k-1}\mathbf{T}_N(t_i)\mathbf{D}_N.$$

Proof If the collocation points (17) are written in (16), then we get

$$\gamma \mathbf{T}_N(t_i)\mathbf{M}_N(\gamma)\mathbf{B}_N^2\mathbf{D}_N\mathbf{A}_N + \tfrac{\beta}{t_i}\mathbf{T}_N(t_i)\mathbf{M}_N(\gamma)\mathbf{B}_N\mathbf{D}_N\mathbf{A}_N + (\mathbf{T}_N(t_i)\mathbf{D}_N\mathbf{A}_N)^{k-1}\mathbf{T}_N(t_i)\mathbf{D}_N\mathbf{A}_N = g(t_i) \tag{19}$$

or

$$\left\{\gamma \mathbf{T}_N(t_i)\mathbf{M}_N(\gamma)\mathbf{B}_N^2\mathbf{D}_N + \tfrac{\beta}{t_i}\mathbf{T}_N(t_i)\mathbf{M}_N(\gamma)\mathbf{B}_N\mathbf{D}_N + (\mathbf{T}_N(t_i)\mathbf{D}_N\mathbf{A}_N)^{k-1}\mathbf{T}_N(t_i)\mathbf{D}_N\right\}\mathbf{A}_N = g(t_i). \tag{20}$$

We can briefly write (20) as $\mathbf{W}_N\mathbf{A}_N = \mathbf{G}_N$. Here,

$$\mathbf{W}_N = \begin{bmatrix} \bar{\mathbf{W}}_N(t_0) & \bar{\mathbf{W}}_N(t_1) & \cdots & \bar{\mathbf{W}}_N(t_N) \end{bmatrix}^T$$

and

$$\bar{\mathbf{W}}_N(t_i) = \gamma \mathbf{T}_N(t_i)\mathbf{M}_N(\gamma)\mathbf{B}_N^2\mathbf{D}_N + \frac{\beta}{t_i}\mathbf{T}_N(t_i)\mathbf{M}_N(\gamma)\mathbf{B}_N\mathbf{D}_N + (\bar{\mathbf{T}}_N(t_i)\bar{\mathbf{D}}_N\bar{\mathbf{A}}_N)^{k-1}\mathbf{T}_N(t_i)\mathbf{D}_N,$$

which completes the proof of the theorem. □

Theorem 3.2 *It is assumed that the solution of the problem (1)-(2) is sought in type (6). Then, the problem (1)-(2) by using the collocation points (17) is reduced to a system of nonlinear algebraic equations as follows*

$$\widetilde{\mathbf{W}}_N\mathbf{A}_N = \widetilde{\mathbf{G}}_N. \tag{21}$$

Here, $[\widetilde{\mathbf{W}}_N; \widetilde{\mathbf{G}}_N]$ *is obtained by writing the matrix forms formed for the conditions instead of any two rows in* $[\mathbf{W}_N; \mathbf{G}_N]$ *in (18).*

Proof A new matrix system is created by writing two equations obtained for the conditions (2) in Lemma 2.6 instead of any two rows in the system of algebraic equations in Theorem 3.1. This new system is also represented as $\widetilde{\mathbf{W}}_N \mathbf{A}_N = \widetilde{\mathbf{G}}_N$. The important part here is to write the row obtained for the first condition instead of the row with the singularity in the matrix $[\mathbf{W}_N; \mathbf{G}_N]$. Thus, the proof is completed. □

4. Error Estimation

The purpose of this section is to given an error estimation technique with the help of the residual function. In addition, by using this error estimation technique and the Pell-Lucas polynomial solution, the residual improvement technique is also presented.

Theorem 4.1 *(Error Estimation) Let* $y(t)$ *be the exact solution and* $y_N(t)$ *be the Pell-Lucas polynomial solution with* $N-th$ *degree of the problem (1)-(2). Then, the error problem can be obtained as follows*

$$\begin{cases} \gamma e_N''(\gamma t) + \frac{\beta}{t} e_N'(\gamma t) + \sum_{i=1}^{k} \binom{k}{i} e_N^i(t) y_N^{k-i}(t) = -R_N(t), \\ e_N(0) = 0, \quad e_N'(0) = 0, \end{cases} \tag{22}$$

where $e_N(t) = y(t) - y_N(t)$*,* $e_N'(t) = y'(t) - y_N'(t)$*,* $e_N'(\gamma t) = y'(\gamma t) - y_N'(\gamma t)$*,* $e_N''(\gamma t) = y''(\gamma t) - y_N''(\gamma t)$ *and* $R_N(t)$ *is the residual function of the problem (1)-(2).*

Proof Firstly, inasmuch as the Pell-Lucas polynomial solution (3) satisfies the problem (1)-(2), we can write

$$\begin{cases} R_N(t) = \gamma y_N''(\gamma t) + \frac{\beta}{t} y_N'(\gamma t) + y_N^k(t) - g(t), \\ y_N(0) = \lambda, \quad y_N'(0) = \mu \end{cases} \tag{23}$$

or

$$\begin{cases} R_N(t) + g(t) = \gamma y_N''(\gamma t) + \frac{\beta}{t} y_N'(\gamma t) + y_N^k(t), \\ y_N(0) = \lambda, \quad y_N'(0) = \mu. \end{cases} \tag{24}$$

Now, let's subtract the equations in the problem (24) from equations in the problem (1)-(2), respectively, and thus we have

$$\begin{cases} \gamma e_N''(\gamma t) + \frac{\beta}{t} e_N'(\gamma t) + y^k(t) - y_N^k(t) = -R_N(t), \\ e_N(0) = 0, \quad e_N'(0) = 0. \end{cases} \tag{25}$$

Now, since $e(t) = y(t) - y_N(t)$, $y(t)$ can be expressed as $y(t) = e(t) + y_N(t)$. From here, we can write $y^k(t) = (e(t) + y_N(t))^k$. Now, if the binomial expansion is also used in $(e(t) + y_N(t))^k$, it becomes

$$y^k(t) = \sum_{i=0}^{k} \binom{k}{i} e_N^i(t) y_N^{k-i}(t).$$

Here, since this expression for $i = 0$ is $y_N^k(t)$, the error problem is obtained as

$$\begin{cases} \gamma e_N''(\gamma t) + \frac{\beta}{t} e_N'(\gamma t) + \sum_{i=1}^{k} \binom{k}{i} e_N^i(t) y_N^{k-i}(t) = -R_N(t), \\ e_N(0) = 0, \quad e_N'(0) = 0. \end{cases} \tag{26}$$

Hence, proof of theorem is completed. □

Corollary 4.1 *When the error problem (22) is solved according to the method in the Section 3, the estimated error function $e_{N,M}(t)$ is obtained.*

Corollary 4.2 *If the Pell-Lucas polynomial solution $y_N(t)$ is summed with the estimated error function $e_{N,M}(t)$, the improved approximate solution $y_{N,M}(t)$ is achieved.*

Corollary 4.3 *The improved error function is calculated by*

$$E_{N,M}(t) = y(t) - y_{N,M}(t). \tag{27}$$

5. Numerical Examples

In this section, the three numerical applications are presented. These applications and the graphics are obtained with the help of Matlab program. In this section, $y(t)$, $y_N(t)$ and $y_{N,M}(t)$ represent respectively the exact solution, the Pell-Lucas polynomial solution, the improved approximate solution. Also, $e_N(t)$, $e_{N,M}(t)$ and $E_{N,M}(t)$ represent respectively the actual absolute error function, the estimated absolute error function and the improved absolute error function.

Example 5.1 *Our first example is the nonlinear differential equation [3]*

$$\frac{1}{2} y''\left(\frac{1}{2}t\right) + \frac{3}{t} y'\left(\frac{1}{2}t\right) + y^2(t) = t^8 + 2t^4 + 3t^2 + 1, \quad 0 < t \le 1, \tag{28}$$

and the initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{29}$$

The exact solution of the problem (28)-(29) is $y(t) = 1 + t^4$. The Pell-Lucas polynomial solution of the problem (28)-(29) for $N = 3$ is sought in the form

$$y_3(t) = \sum_{n=0}^{3} a_n Q_n(t). \tag{30}$$

Here, the colocation points for $N = 3$ are $\{t_0 = 0,\ \ t_1 = \frac{1}{3},\ \ t_2 = \frac{2}{3},\ \ t_3 = 1\}$. Notice that the point $t_0 = 0$ creates a singularity in (28). For this reason, one of the lines created for the conditions is used instead of this line, which creates a singularity in the continuation of the method. With the help of the Theorem 3.1, the fundamental matrix equation is written as

$$\mathbf{W}_3\mathbf{A}_3 = \mathbf{G}_3, \tag{31}$$

where

$$\mathbf{W}_3 = \begin{bmatrix} \bar{\mathbf{W}}_3(t_0) & \bar{\mathbf{W}}_3(t_1) & \bar{\mathbf{W}}_3(t_2) & \bar{\mathbf{W}}_3(t_3) \end{bmatrix}^T,$$

$$\bar{\mathbf{W}}_3(t_i) = \frac{1}{2}\mathbf{T}_3(t_i)\mathbf{M}_3(1/2)\mathbf{B}_3^2\mathbf{D}_3 + \frac{3}{t_i}\mathbf{T}_3(t_i)\mathbf{M}_3(1/2)\mathbf{B}_3\mathbf{D}_3 + (\mathbf{T}_3(t_i)\mathbf{D}_3\mathbf{A}_3)^1\mathbf{T}_3(t_i)\mathbf{D}_3,$$

$$\mathbf{D}_3 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \mathbf{G}_3 = \begin{bmatrix} 1 \\ 8911/6561 \\ 18157/6561 \\ 7 \end{bmatrix}, \quad \mathbf{B}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

$$\mathbf{T}_3(t_i) = \begin{bmatrix} 1 & t_i & t_i^2 & t_i^3 \end{bmatrix}, \quad \mathbf{M}_N(1/2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix}.$$

On the other hand, the matrix representations of the condition (29) are

$$[\mathbf{U}_3; \lambda_3] = \begin{bmatrix} 2 & 0 & 2 & 0 & ; & 1 \end{bmatrix}$$

and

$$[\mathbf{V}_3; \mu_3] = \begin{bmatrix} 0 & 2 & 0 & 6; & 0 \end{bmatrix}.$$

Since singularity occurs in the first row of the matrix (31), the matrix $[\mathbf{U}_3; \lambda_3]$ is written instead of this row. On the other hand, the matrix $[\mathbf{V}_3; \mu_3]$ is written instead of the last row of the matrix (31) and thus a new algebraic matrix system called $\left[\widetilde{\mathbf{W}}_3; \widetilde{\mathbf{G}}_3\right]$ is obtained. It should be noted that the second or third line can also be used instead of the last line here. When the nonlinear system $\left[\widetilde{\mathbf{W}}_3; \widetilde{\mathbf{G}}_3\right]$ is solved by using the Matlab program, the Pell-Lucas coefficients matrix is calculated as follows:

$$\mathbf{A}_3 = \begin{bmatrix} 655/999 & -1065/1838 & -450/2891 & 355/1838 \end{bmatrix}^T.$$

Then, the Pell-Lucas polynomial solution becomes

$$y_3(t) = 1.5452e + 00t^3 - 6.2262e - 01t^2 - 5.5511e - 17t + 1. \tag{32}$$

Table 1: Comparison of the results of the problem (28)-(29) for the Pell-Lucas collocation method and the Bernoulli collocation method [3]

t_i	The Actual Error for PM $N=4$	The Estimated Error for PM $(N,M)=(4,6)$	The Improved Error for PM $(N,M)=(4,6)$	The Actual Error for BCM [3] $N=4$
0.1	1.9317e-15	2.1204e-15	1.8866e-16	5.1070e-14
0.2	5.0030e-15	5.2348e-15	2.3175e-16	1.2745e-13
0.3	8.0536e-15	8.7198e-15	6.6618e-16	1.6875e-13
0.4	1.2899e-14	1.5806e-14	2.9072e-15	1.8607e-13
0.5	2.4440e-14	3.8594e-14	1.4153e-14	2.6268e-13
0.6	5.0667e-14	1.1186e-13	6.1191e-14	5.5400e-13
0.7	1.0265e-13	3.1564e-13	2.1299e-13	1.2870e-12
0.8	1.9456e-13	8.0664e-13	6.1208e-13	3.7605e-12
0.9	3.4364e-13	1.8584e-12	1.5147e-12	5.3457e-12
1	5.7022e-13	3.9101e-12	3.3399e-12	9.4853e-12

Thus, the estimated error function for $M=4$ is written as

$$e_{3,4}(t) = t^4 - 1.5452e+00t^3 + 6.2262e-01t^2 + 5.5511e-17t. \tag{33}$$

Figure 1 shows the exact solution and the Pell-Lucas polynomial solutions for $N=4,6,9$ of the model (28)-(29). Figure 2 displays the exact solution and the impoved Pell-Lucas polynomial solutions for $(N,M)=(4,6),(6,7),(9,10)$ of the model (28)-(29). In Figure 3, the actual absolute errors functions of the model (28)-(29) for $N=4$ and $N=7$ are compared. In Figure 4, the actual absolute error function, the estimated absolute error function and the improved absolute error function for $(N,M)=(4,6)$ are compared. Table 1 indicates the actual absolute error for $N=4$, the estimated absolute error for $(N,M)=(4,6)$ and the improved absolute error for $(N,M)=(4,6)$ of the model (28)-(29). In addition, the results of the present method are compared with the results of the Bernoulli collocation method for $N=4$ in Table 1.

According to Figure 3, a better result is obtained with a larger value of N. According to Figure 4, the estimated absolute error function gives similar results to the actual absolute error function, and the improved absolute error function gives better results than the actual absolute error function. According to Table 1, the results of the present method give better results than the results of the Bernoulli collocation method [3].

Example 5.2 As the next example, let's consider the nonlinear differential equation

$$\frac{1}{2}y''\left(\frac{1}{2}t\right) + \frac{1}{t}y'\left(\frac{1}{2}t\right) + y^2(t) = \frac{1}{2}e^{-t/2} - \frac{1}{t}e^{-t/2} + e^{-2t} \tag{34}$$

and the initial conditions

$$y(0)=1, \quad y'(0)=-1. \tag{35}$$

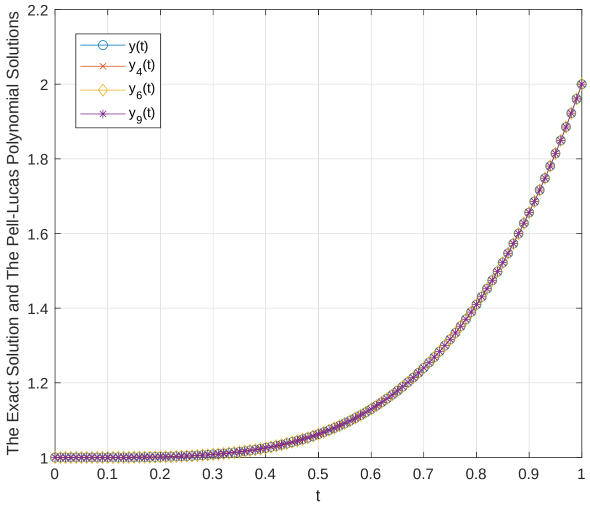

Figure 1: Comparison of the exact solution with the Pell-Lucas polynomial solutions of the problem (28)-(29) for $N = 4, 6, 9$

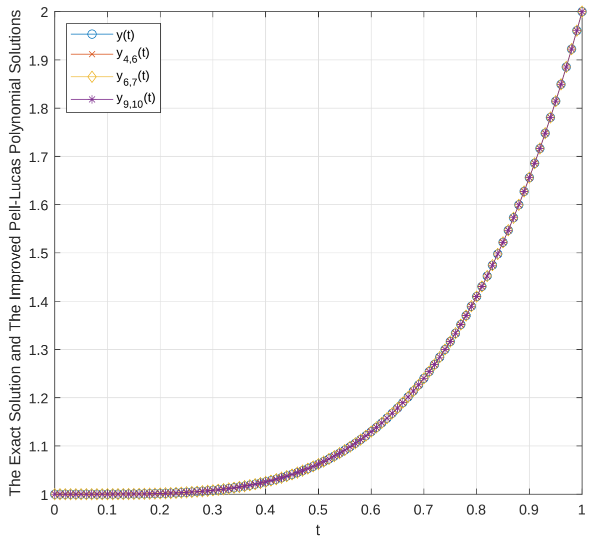

Figure 2: Comparison of the exact solution with the improved Pell-Lucas polynomial solutions of the problem (28)-(29) for $(N, M) = (4, 6)$, $(N, M) = (6, 7)$ and $(N, M) = (9, 10)$

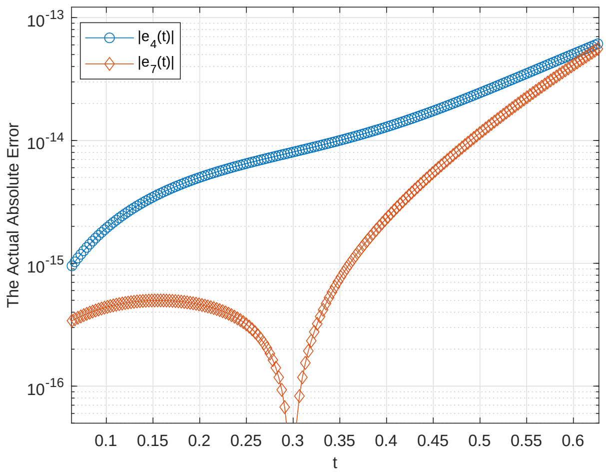

Figure 3: Comparison of the actual absolute errors of the problem (28)-(29) for $N = 4$ and $N = 7$

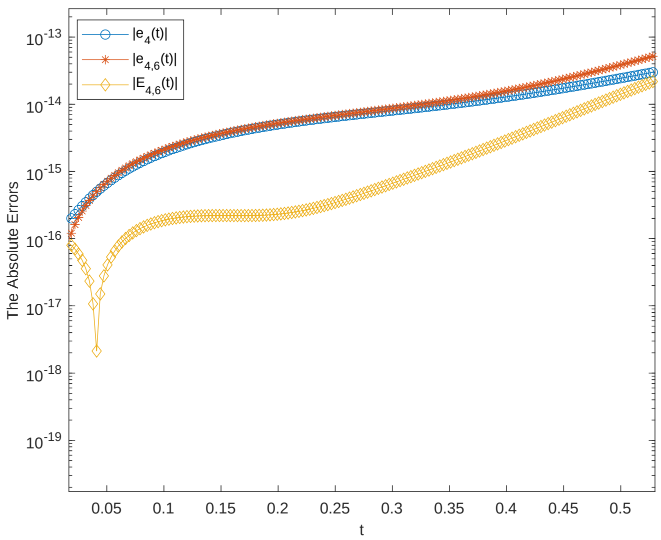

Figure 4: Comparison of the actual absolute error for $N = 4$, the estimated absolute error for $(N, M) = (4, 6)$ and the improved absolute error for $(N, M) = (4, 6)$ of the problem (28)-(29)

The exact solution of the problem (34)-(35) is $y(t) = e^{-t}$. The solution of the problem (34)-(35) for any N is sought in the form

$$y_N(t) = \sum_{n=0}^{N} a_n Q_n(t). \tag{36}$$

By using the Theorem 3.1, the fundamental matrix equation is obtained as

$$\mathbf{W}_N \mathbf{A}_N = \mathbf{G}_N, \tag{37}$$

$$\mathbf{W}_N = \begin{bmatrix} \bar{\mathbf{W}}_N(t_0) & \bar{\mathbf{W}}_N(t_1) & \cdots & \bar{\mathbf{W}}_N(t_N) \end{bmatrix}^T, \quad \mathbf{G}_N = \begin{bmatrix} g(t_0) & g(t_1) & \cdots & g(t_N) \end{bmatrix}^T,$$

$$\bar{\mathbf{W}}_N(t_i) = \frac{1}{2}\mathbf{T}_N(t_i)\mathbf{M}_N(1/2)\mathbf{B}_N^2\mathbf{D}_N + \frac{1}{t_i}\mathbf{T}_N(t_i)\mathbf{M}_N(1/2)\mathbf{B}_N\mathbf{D}_N + (\mathbf{T}_N(t_i)\mathbf{D}_N\mathbf{A}_N)^1\mathbf{T}_N(t_i)\mathbf{D}_N.$$

The actual absolute errors, the estimated absolute errors, and the improved absolute errors of the problem (34)-(35) for various values of N and M are presented in Table 2. The exact solution of the problem (34)-(35) are compared with the Pell-Lucas polynomial solutions for $N = 3, 5, 8$ in Figure 5. Also, the improved Pell-Lucas polynomial solutions are compared for $(N, M) = (3, 4), (5, 6), (8, 9)$ in Figure 6. The actual absolute error functions of the problem (34)-(35) for $N = 3$, $N = 5$ and $N = 8$ are compared in Figure 7. The actual absolute error function of the problem (34)-(35) for $N = 3$ is compared with the estimated absolute error function and the improved absolute error function for $(N, M) = (3, 4)$ in Figure 8.

Figure 7 shows that the errors decrease as the value of N increases. From Figure 8, it can be observed that the estimated absolute error function gives similar results to the actual absolute error function, and the improved absolute error function gives better results than the actual absolute error function. Their results can also be seen from Table 2. According to all tables and graphs, it is concluded that the method gives very successful results.

Example 5.3 The last example is the nonlinear differential equation

$$\frac{1}{2}y''\left(\frac{1}{2}t\right) + \frac{1}{t}y'\left(\frac{1}{2}t\right) + y^2(t) = -\frac{1}{2}sin(t/2) + \frac{1}{t}cos(t/2) + sin^2(t) \tag{38}$$

and the initial conditions

$$y(0) = 0, \quad y'(0) = 1. \tag{39}$$

The exact solution of the problem (38)-(39) is $y(t) = sint$. The solution of the problem (38)-(39) for any N is sought in the form

$$y_N(t) = \sum_{n=0}^{N} a_n Q_n(t). \tag{40}$$

Table 2: Absolute errors of Example 5.2 for $N = 3$, $N = 5$, $N = 8$, $(N, M) = (3, 4)$, $(N, M) = (8, 9)$

	Actual Absolute Error			Estimated Absolute Error		Improved Absolute Error
t_i	$N = 3$	$N = 5$	$N = 8$	$(N, M) = (3, 4)$	$(N, M) = (8, 9)$	$(N, M) = (3, 4)$
0.1	1.7278e-04	8.4002e-7	5.2864e-11	1.7663e-04	4.8429e-11	3.8457e-6
0.2	5.2756e-04	1.8825e-6	8.7766e-11	5.4919e-04	8.3974e-11	2.1638e-05
0.3	8.8554e-04	2.4972e-6	1.0562e-10	9.4038e-04	1.0451e-10	5.4837e-05
0.4	1.1500e-03	2.8774e-6	1.1801e-10	1.2532e-03	1.1906e-10	1.0319e-04
0.5	1.2983e-03	3.1012e-6	1.2039e-10	1.4709e-03	1.3015e-10	1.7261e-04
0.6	1.3751e-03	2.9342e-6	4.4906e-11	1.6574e-03	1.4307e-10	2.8226e-04
0.7	1.4857e-03	2.3052e-6	1.4415e-10	1.9566e-03	2.8488e-10	4.7093e-04
0.8	1.7904e-03	2.3894e-6	2.5977e-11	2.5932e-03	1.3396e-9	8.0281e-04
0.9	2.4993e-03	7.2417e-6	1.5975e-9	3.8721e-03	5.9841e-9	1.3727e-03
1	3.8675e-03	2.5928e-05	5.2919e-9	6.1784e-03	2.0743e-8	2.3109e-03

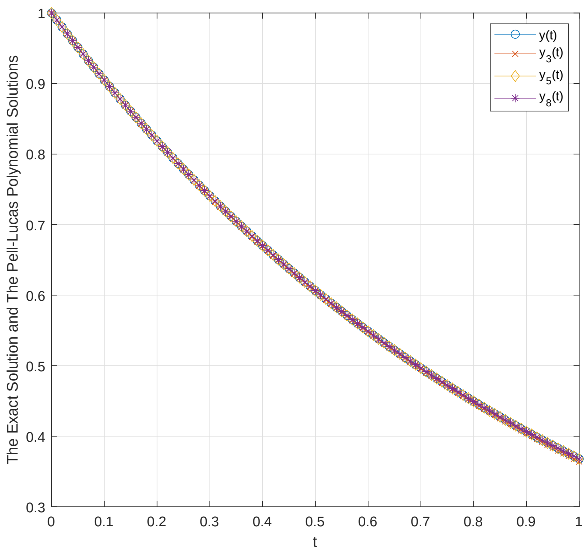

Figure 5: Comparison of the exact solution and the Pell-Lucas polynomial solutions of the problem (34)-(35) for $N = 3, 5, 8$

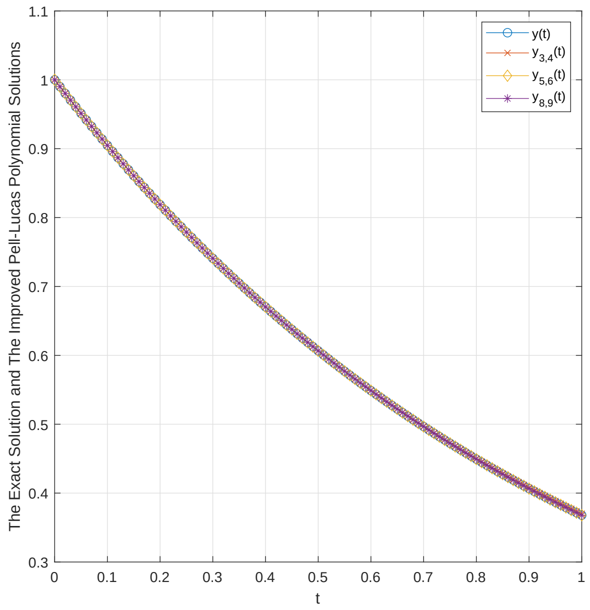

Figure 6: Comparison of the exact solution and the improved Pell-Lucas polynomial solutions of the problem (34)-(35) for $(N, M) = (3, 4)$, $(N, M) = (5, 6)$ and $(N, M) = (8, 9)$

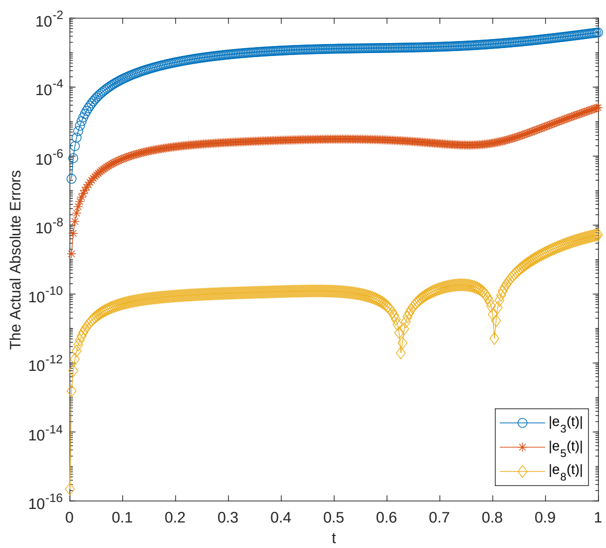

Figure 7: Comparison of the actual absolute errors of the problem (34)-(35) for $N = 3, 5, 8$

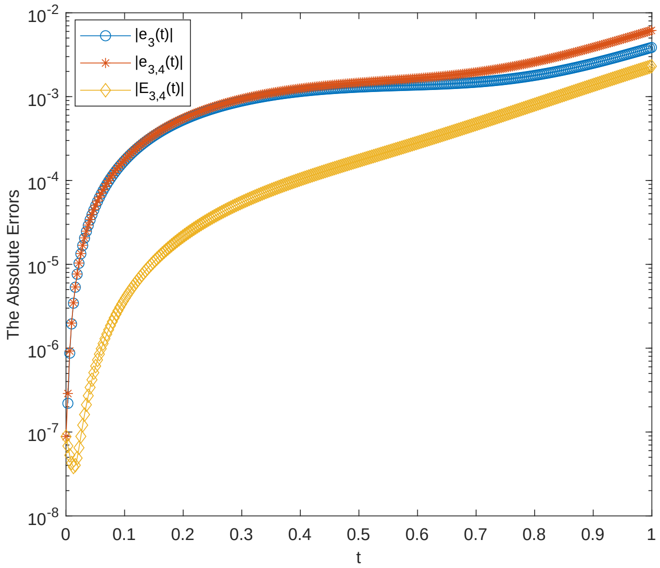

Figure 8: Comparison of the absolute errors of the problem (34)-(35) for $N=3$ and $M=4$

From the Theorem 3.1, the fundamental matrix equation becomes

$$\mathbf{W}_N \mathbf{A}_N = \mathbf{G}_N, \tag{41}$$

where

$$\mathbf{W}_N = \begin{bmatrix} \bar{\mathbf{W}}_N(t_0) & \bar{\mathbf{W}}_N(t_1) & \cdots & \bar{\mathbf{W}}_N(t_N) \end{bmatrix}^T, \mathbf{G}_N = \begin{bmatrix} g(t_0) & g(t_1) & \cdots & g(t_N) \end{bmatrix}^T,$$

$$\bar{\mathbf{W}}_N(t_i) = \frac{1}{2}\mathbf{T}_N(t_i)\mathbf{M}_N(1/2)\mathbf{B}_N^2\mathbf{D}_N + \frac{1}{t_i}\mathbf{T}_N(t_i)\mathbf{M}_N(1/2)\mathbf{B}_N\mathbf{D}_N + (\mathbf{T}_N(t_i)\mathbf{D}_N\mathbf{A}_N)^1\mathbf{T}_N(t_i)\mathbf{D}_N.$$

In Table 3, we give the actual absolute errors, the estimated absolute errors and the improved absolute errors of the problem (38)-(39) for various values of N and M. In Figure 9 and Figure 10, we respectively compare the exact solution of the problem (38)-(39) with the approximate solutions and the improved approximate solutions. In Figure 11, we show the actual absolute error function of the problem (38)-(39) for $N=4$, $N=7$ and $N=9$. In Figure 12, we depict the actual absolute error function of the problem (38)-(39) for $N=3$, the estimated absolute error function and the improved absolute error function for $(N,M)=(3,4)$.

It can be observed from Figure 11 and Table 3 that a more accurate result is obtained with a larger value of N. The interpretation that the estimated absolute errors are very close to the actual absolute errors and that the improved absolute error function gives better results than the actual error function can be made from Figure 12 and Table 3. It is concluded from all tables and graphs that the presented method is a suitable method for the nonlinear second-order Lane-Emden type pantograph differential equation (LETPDE) (1).

Table 3: Absolute errors for of the problem (38)-(39) $N = 3$, $N = 7$, $N = 9$, $(N, M) = (3, 4)$, $(N, M) = (9, 10)$

	The Actual Absolute Error			The Estimated Absolute Error		The Improved Absolute Error
t_i	$N = 3$	$N = 7$	$N = 9$	$(N, M) = (3, 4)$	$(N, M) = (9, 10)$	$(N, M) = (3, 4)$
0.1	7.7249e-05	9.6696e-10	1.0658e-12	9.2330e-05	8.5039e-13	1.5081e-05
0.2	2.4963e-04	1.7706e-9	1.7337e-12	2.8836e-04	1.4459e-12	3.8732e-05
0.3	4.3707e-04	2.2067e-9	2.0464e-12	4.9283e-04	1.7841e-12	5.5762e-05
0.4	5.7931e-04	2.4466e-9	2.2737e-12	6.5394e-04	2.0335e-12	7.4622e-05
0.5	6.4562e-04	2.7323e-9	1.9327e-12	7.6354e-04	2.4017e-12	1.1791e-04
0.6	6.4414e-04	3.6030e-9	9.0949e-13	8.5715e-04	5.2066e-12	2.1301e-04
0.7	6.3085e-04	4.1655e-9	4.3201e-12	1.0139e-03	3.1257e-11	3.8309e-04
0.8	7.1812e-04	1.9963e-10	2.1487e-11	1.3567e-03	1.9426e-10	6.3858e-04
0.9	1.0826e-03	1.4526e-8	4.8317e-11	2.0519e-03	9.3476e-10	9.6929e-04
1	1.9727e-03	2.3543e-8	8.3901e-11	3.3097e-03	3.6115e-9	1.3370e-03

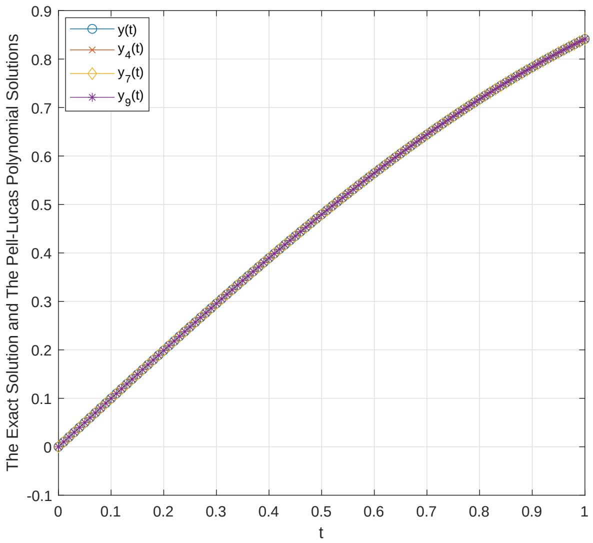

Figure 9: Comparison of the exact solution and the approximate solutions of the problem (38)-(39) for $N = 4, 7, 9$

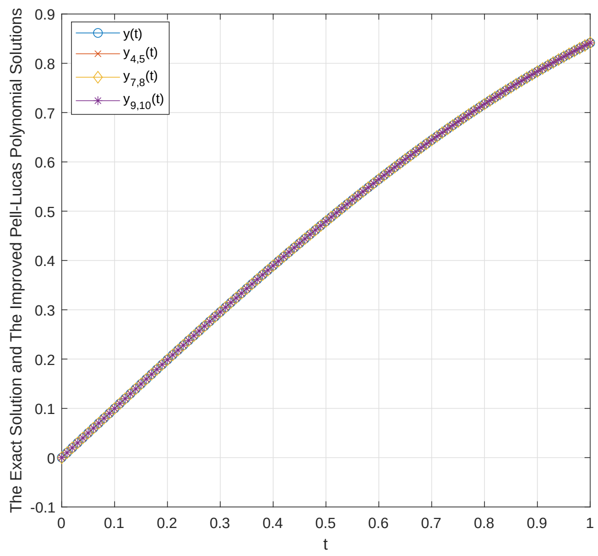

Figure 10: Comparison of the exact solution and the improved approximate solutions of the problem (38)-(39) for $(N, M) = (4, 5)$, $(N, M) = (7, 8)$ and $(N, M) = (9, 10)$

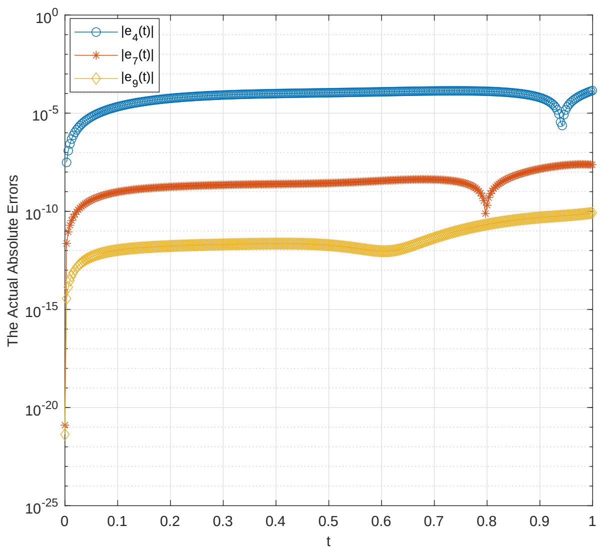

Figure 11: Comparison of the actual absolute errors of the problem (38)-(39) for $N = 4, 7, 9$

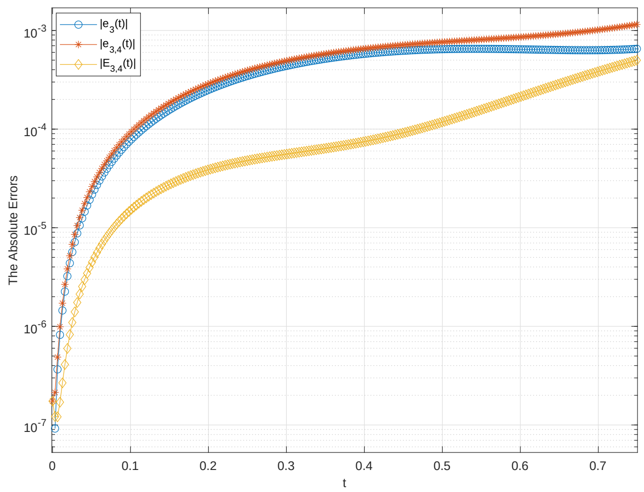

Figure 12: Comparison of the absolute errors of the problem (38)-(39) for $N = 3$ and $M = 4$

6. Conclusions

The aim of this study is to present an effective and a reliable method for a class of the nonlinear differential equations. For this purpose, the Pell-Lucas collocation method is presented in the third part of the article. In addition, in the fourth part of the article, an error estimation method is introduced by using the residual function and with the help of the third part of the article. Moreover, the residual improved technique is also presented. In the fifth section of the article, the methods presented in the previous sections are tested for three examples. These applications are made by using the Matlab program. Application results are tabulated and graphed. According to these results, it is observed that more accurate results are obtained when the value of N in the method is chosen large enough. In addition, it can be said that the error estimation method is quite successful. The importance of the error estimation method is to have information about the results of the method even when the exact solution of the problem is not known. Another result of the study is that the residual improvement technique also yields appropriate results. According to all these results, it can be interpreted that the Pell-Lucas collocation method, the error estimation technique and the residual improvement method for the nonlinear second-order Lane-Emden type pantograph differential equation (LETPDE) (1) are quite effective and reliable. These presented methods can also be applied to other types of the nonlinear differential equations after the necessary adjustments are made.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Şuayip Yüzbaşı]: Thought and designed the research/problem, contributed to construct the suggested method, error estimation and numerical applications (%55).

Author [Gamze Yıldırım]: Collected the data, contributed to research method, wrote a code of the method and of solving of numerical examples and wrote the manuscript (%45).

Conflicts of Interest

The authors declare no conflict of interest.

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