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Implementation of Interior-point Methods for $P_*(\kappa) - LCP$ Based on New Search Direction

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Abstract. In this paper, we present an interior-point algorithm for solving $P_*(\kappa)$-linear complementarity problem. The specificity of our method is to compute Newton's step using a modified system of the centrality equation. For this reason, we consider three know function in the literature applied in the centrality equation, so a new Newton's direction are determined. The convergence of these algorithms is accomplished. This search is followed by comparative study between the numerical result obtained through these three functions.

Keywords: Interior-point methods · $P_*(\kappa)$-linear complementarity problem · Full-Newton step · Polynomial complexity

1 Introduction

Given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, the standard linear complementarity problem (LCP) is to find a vector pair $(x, s) \in \mathbb{R}^n \times \mathbb{R}^n$such that

$$s = Mx + q,\ xs = 0,\ (x, s) \geq 0. \tag{1}$$

where xs denotes Hadamard product of vector x and s, i.e., $xs = [x_1 s_1, ..., x_n s_n]^T$. We shall also use the notation $\frac{x}{s} = \left[\frac{x_1}{s_1}, ..., \frac{x_n}{s_n}\right]$, where $s_i \neq 0$ for all $1 \leq i \leq n$.

$LCPs$ arises in many areas such as variational inequalities, economic equilibri a problems and bimatrix games. It is known that this problem trivially includes the two important domains in optimization: the linear programming $(LP)$and the convex quadratic programming (CQP) in their usual formulations, then this problem became the subject of many research interest.

In this paper, we consider problem (1) with M being a $P_*(\kappa)$-matrix. The class

of $P_*(\kappa)$-matrices was introduced by Kojima et al. [6]. Let κ be a nonnegative number. A matrix $M \in \mathbb{R}^{n\times n}$ is called a $P_*(\kappa)$-matrix if and only if

$$(1 + 4\kappa) \sum_{i\in I_{+(x)}} x_i (Mx)_i + \sum_{i\in I_{-(x)}} x_i (Mx)_i \geq 0, \forall x \in \mathbb{R}^n$$

where

$$I_+(x) = \{i \in I : x_i (Mx)_i \geq 0\}, \qquad I_-(x) = \{i \in I : x_i (Mx)_i < 0\}.$$

The class of all $P_*(\kappa)$-matrices is denoted by $P_*(\kappa)$, and the class P_* is defined by $P_* = \bigcup_{\kappa\geq 0} P_*(\kappa)$, i.e., M is a P_* -matrix if $M \in P_*(\kappa)$ for some $\kappa \geq 0$. Obviously, $P_*(0)$ is the class of positive semidefinite matrices.

2 Problem statement

The basic idea of IPMs for $P_*(\kappa)$-LCP is to replace the second equation in $P_*(\kappa)$-LCP by the parameterized equation $xs = \mu e$, with parameter $\mu > 0$ and $e = (1, 1, ..., 1)^T$. The system (1) becomes :

$$s = Mx + q,\ xs = \mu e,\ (x, s) \geq 0. \tag{2}$$

Without loss of generality, we assume that (1) satisfies the interior point condition (IPC), i.e., there exists $(x_0, s_0) > 0$ such that $s_0 = Mx_0 + q$. Since M is a $P_*(\kappa)$-matrix and IPC holds, the parameterized system (2) has a unique solution for any $\mu > 0$ (Lemma 4.3 of [6]) and it is denoted as $((x_\mu, s_\mu))$. We call it μ-center for $\mu > 0$ and the solution set $\{(x_\mu, s_\mu) \setminus \mu > 0\}$ is called the central path of (1). As goes to zero, the limit of the central path exists and it naturally yields the optimal solution for (1) (Theorem 4.4 of [6]).

3 A new search directions

The basic idea behind this approach is to replace the non linear equation : $xs = \mu e$ in (2) by an equivalent algebraic transformation $\psi(\frac{xs}{\mu}) = \psi(e)$, where ψ, is a real valued function on $[0, +\infty[$such that $\psi(0) = 0$ and differentiable on$(0, +\infty)$such that $\psi'(t) > 0$, for all $t > 0$. Then the system (2) can be written as the following equivalent form:

$$s = Mx + q, \psi(\frac{xs}{\mu}) = \psi(e), (\ x, s) \geq 0. \tag{3}$$

For any strictly feasible point x and s, we find displacements Δx and Δs such that

$$\begin{cases} -M\,(x + \Delta x) + (s + \Delta s) = q \\ \psi(\frac{xs}{\mu} + \frac{x\Delta s + s\Delta x + \Delta s\Delta x}{\mu}) = \psi(e) \end{cases} \tag{4}$$

Neglecting the quadratic term $\Delta s\Delta x$ in the above equation and using Taylor's development we get

$$\begin{cases} -M \triangle x + \triangle s = 0 \\ x\Delta s + s\Delta x = \mu(\psi'(\frac{xs}{\mu}))^{-1}\,(\psi(e) - \psi(\frac{xs}{\mu})) \end{cases} \tag{5}$$

Now to simplify the matters, we define the vectors

$$v = \sqrt{\frac{xs}{\mu}},\ d_x := \frac{v\Delta x}{x} \text{ and } d_s := \frac{v\Delta s}{s}. \tag{6}$$

Then we have the scaled Newton system as follows

$$\begin{cases} -\overline{M}\ \ d_x + \ d_s = 0 \\ d_x + \ d_s = p_v \end{cases}, \tag{7}$$

where $\bar{M} = DMD$ with $D = X^{\frac{1}{2}}S^{-\frac{1}{2}}$, $X := diag(x)$, $S := diag(s)$ and $p_v = \dfrac{\psi(e) - \psi(v^2)}{v\psi'(v^2)}$

By choosing function $\psi(t)$ appropriately, the system (7) can be used to define a class of search directions. For example:

$$\begin{cases} \psi(t) = t \ \text{ yields } p_v = v^{-1} - v \text{ which gives the classical search directions } [8] \\ \psi(t) = \sqrt{t} \ \text{ yields } p_v = 2\,(e - v) \ \text{ which gives Darvay search directions } [3] \\ \psi(t) = \dfrac{\sqrt{t}}{2\,(1 + \sqrt{t})} \text{ yields } p_v = e - v^2 \text{ the new search directions } [5] \end{cases}$$

The new iterate is obtained by taking a full-Newton step according to

$$x^+ := x + \Delta x,\ s^+ := s + \Delta s \tag{8}$$

For the analysis of the algorithm, we de ne a norm-based proximity measure as follows:

$$\delta(v) = \delta(x, s; \mu) := \|p_v\| \tag{9}$$

Generic feasible IPM for $P_*(\kappa)$-LCP

Input:
Accuracy parameter $\epsilon > 0$;
a barrier update parameter θ, $0 < \theta < 1$;
threshold parameter $0 < \tau < 1$;
a strictly feasible point (x_0, s_0) and $\mu_0 > 0$ such that $\delta(x_0, s_0; \mu) \leq \tau$.
begin
$x := x_0$, $s := s_0$, $\mu = \mu_0$;
while $n\mu > \epsilon$ do
begin
solve (7) to obtain (d_x, d_s) and then use (6) to compute $(\Delta x, \Delta s)$
$x := x + \alpha\Delta x$; $s := s + \alpha\Delta s$; $\mu := (1 - \theta)\mu$
end
end.

Analysis of the algorithm

For the choice of $\psi(t) = \dfrac{\sqrt{t}}{2\left(1 + \sqrt{t}\right)}$, we have the following results:

1. $\delta(v) = \left\|e - v^2\right\|$ and $\delta(v) = 0 \Leftrightarrow v = e \Leftrightarrow xs = \mu e$.
2. Let $\delta := \delta(x, s; \mu) < \dfrac{2}{1 + \sqrt{1 + 4\kappa}}$. Then the full-Newton step is strictly feasible and $\delta(x^+, s^+, \mu) \leq (1 + 2\kappa)\,\delta^2$, (the quadratic convergence of the newton step).
3. Let (x, s) such that $\delta := \delta(x, s; \mu) < \dfrac{2}{1 + \sqrt{1 + 4\kappa}}$ and $\mu^+ = (1 - \theta)\,\mu$ where $\theta \in (0, 1)$ then

$$\delta(x^+, s^+, \mu^+) \leq \frac{1}{(1 - \theta)} + (\theta\sqrt{n} + \delta(x^+, s^+, \mu)).$$

4. Let

$$\delta := \delta(x, s; \mu) < \frac{1}{2(1 + 2\kappa)} < \frac{2}{1 + \sqrt{1 + 4\kappa}}, \theta = \frac{1}{2(4 + 7\kappa)\sqrt{n}}$$

and $n \geq 4$. Then $\delta(x^+, s^+, \mu^+) \leq \dfrac{1}{2(1 + 2\kappa)}$.

Let $\tau = \dfrac{1}{2(1+2\kappa)}$ and $\theta = \dfrac{1}{(4+7\kappa)\sqrt{n}}$. Then the algorithm requires at most

$$\mathcal{O}\left((4+7k)\sqrt{n}\log\frac{(x_0)^T s_0}{\epsilon}\right).$$

Lemma 1 : Assume that (x_0, s_0) is a strictly feasible solution of (2), $\mu_0 = \dfrac{(x_0)^T s_0}{n}$ and $\delta(x_0, s_0; \mu_0) \leq \dfrac{1}{2(1+2\kappa)}$. Moreover, let $\left(x^k, s^k\right)$ be the point obtained after k iterations. Then the inequality $\left(x^k\right)^T s^k \leq \epsilon$ is satisfied for

$$k \geq \frac{1}{\theta}\log\frac{(x_0)^T s_0}{n}$$

4 Numerical results

To illustrate the numerical behavior of the proposed function compared to other functions of the literature, we implemented an algorithm with these functions and we took the results in Table 1 and Table 2. The results of the application of the algorithm on this function with different values of θ ($\theta_1 = \dfrac{1}{(4+7\kappa)\sqrt{n}}$, $\theta_2 = \dfrac{1}{2(1+4\kappa)\sqrt{n}}$ and $\theta = 0.05$) are given in Table 3 : 7.

Example 1 : Let be a $P_*(0) - LCP$ or

$$M = \begin{pmatrix} 4 & -1 & 0 & 0 & \dots & 0 \\ -1 & 4 & -1 & 0 & \dots & 0 \\ 0 & -1 & 4 & -1 & \ddots & \vdots \\ 0 & 0 & -1 & 4 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & \dots & 0 & -1 & 4 \end{pmatrix}, q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

and

$$x^0 = \begin{pmatrix} 0.65 \\ 0.65 \\ 0.65 \\ 0.65 \\ \vdots \\ 0.65 \end{pmatrix}$$

Table 1. $\theta = 0.01, n = 7$

p_v	Iter	gap	time
$v^{-1} - v$	1112	1.0004×10^{-4}	0.660534
$2(e - v)$	1112	1.0004×10^{-4}	0.659813
$e - v^2$	1112	1.0003×10^{-4}	0.795040

Table 2. $\theta = 0.05, n = 7$

p_v	Iter	gap	time
$v^{-1} - v$	219	1.0259×10^{-4}	0.211269
$2(e - v)$	219	1.0253×10^{-4}	0.184771
$e - v^2$	219	1.0246×10^{-4}	0.720793

Table 3. $\theta = \theta_1$

n	Iter	gap	time
10	141	1.0639×10^{-4}	0.330995
20	214	1.0092×10^{-4}	0.583387
30	271	1.0443×10^{-4}	1.264589
40	321	1.0334×10^{-4}	2.587953
50	366	$1.0196 \times 10 - 4$	4.544553
100	547	1.0171×10^{-4}	27.322898
200	815	1.008×10^{-4}	164.913919
300	926	1.0117×10^{-4}	544.87272
500	1373	1.0104×10^{-4}	2572.057799

Table 4. $\theta = \theta_2$

n	Iter	gap	time
10	68	1.1497×10^{-4}	0.938725
20	104	1.1117×10^{-4}	0.400232
30	133	1.0692×10^{-4}	0.659304
40	158	1.0494×10^{-4}	1.259045
50	180	$1.0683 \times 10 - 4$	2.329963
100	271	$1.0164 \times 10 - 4$	10.943022
200	405	1.0184×10^{-4}	251.86287
300	511	1.0039×10^{-4}	277.172326
500	684	1.0016×10^{-4}	1342.633206

Example2: Consider a PCL with $P_*(\kappa)-$matrix such that $\kappa \geq 0$.

$$M = \begin{pmatrix} Q_2 & & & & \\ & Q_3 & & & \\ & & \ddots & & \\ & & & Q_2 & \\ & & & & Q_3 \end{pmatrix}$$

$$\text{or } Q_2 = \begin{pmatrix} 0 & 1+4\kappa_1 \\ 1 & 0 \end{pmatrix}, \quad Q_3 \begin{pmatrix} 0 & 1+4\kappa_2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We choose the starting point $x_0 = s_0 = e$ and $q = s_0 - Mx_0$. In this example, we take $M \in \mathbb{R}^{50\times 50}$ and $\kappa = \kappa_1 = \kappa_2 \in \{1, 2, 3, 10, 100, 1000\}$ The results obtained are summarized in the three tables below

Table 5. $P_*(\kappa) - LCP(\theta = \theta_1)$

κ	Iter	gap	time
1	1016	$1.001284409536842 \times 10^{-4}$	22.419961
2	1665	$1.004988263694473 \times 10^{-4}$	34.141740
3	2315	$1.000897276312732 \times 10^{-4}$	47.832904
10	6861	$1.01590604004929 \times 10^{-4}$	116.245207
100	65318	$1.000196883795912 \times 10^{-4}$	892.014489
1000	649890	$1.000008971770453 \times 10^{-4}$	9061.023120

Table 6. $P_*(\kappa) - LCP(\theta = \theta_2)$

κ	Iter	gap	Time
1	923	1.004310×10^{-4}	20.212444
2	1665	1.00499×10^{-4}	36.185753
3	2407	1.00518×10^{-4}	42.901943
10	7604	1.00025×10^{-4}	90.887090
100	74412	1.00006×10^{-4}	843.481973
1000	742493	1.00002×10^{-4}	8137.608236

Table 7. $P_*(\kappa) - LCP(\theta = 0.05)$

κ	Iter	gap	Time
1	257	1.04227×10^{-4}	12.439228
10	257	1.04227×10^{-4}	13.807880
100	257	1.04227×10^{-4}	14.901724
1000	257	1.04227×10^{-4}	16.691180

Remark: Through the numerical examples, we notice that:

- The number of iterations and the computation time recorded when the choice of $\theta = 0.05$ are much better results
- It is clear from Tables that the iteration number of the algorithm depends on the update parameter θ and the dimension of the problem. It is quite surprising that the larger θ gives the lowest iteration count in all cases. The

iteration number of the algorithm increases as the dimension of the problem is increased.
- The algorithms based on the three functions are the same results
- In the last table: the number of iterations and the computation time are almost the same.

Therefore, this numerical results show that our algorithm is competitive and reliable.

5 Conclusion

In this paper, we have proposed an implementation of *interior point algorithm for solving $P_*(\kappa)-$linear* based on a new search direction given by $\psi(t)$. Some preliminary numerical results are provided to reveal the influence of the update parameters θ and the dimension of the problem on the number of iterations.

The determination of a new Newton class is done either by introducing a new functionality which satisfies the above hypotheses, or by introducing other functions based on the properties of the kernel functions. These algorithms can be extended for other semidefinite and semidefinite quadratic programs.

References

1. M. ACHACHE , *"Complexity analysis and numerical implementation of a shortstep primal-dual algorithm for linear complementarity problems"*; Applied Mathematics and Computation, 216(7), 1889-1895,(2010).
2. R. COTTLE, J.S. PANG, AND R.E. STONE , *"The Linear Complementarity Problem"*; Academic Press, Boston, (1992).
3. ZS. DARVAY , *"New interior-point algorithms in linear programming"*; Advanced-Modeling and Optimization, 5(1), 51-92, (2003).
4. ZS. DARVAY, L.-M. PAPP, AND P.-R. TAKACS , *"An infeasible full-Newton stepalgorithm for linear optimization with one centering step in major iteration"*; Studia Universitatis Babes-Bolyai, Series Informatica, 59(1), 28-45, (2014).
5. B. KHEIRFAM, AND M. HAGHIGHI , *"A full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-LCPbased on a new search direction"*; Croatian Operational Research Review, CRORR 7, 277-290, (2016).
6. M. KOJIMA, N. MEGIDDO, AND A. YOSHISE , *" Unified Approach to Interior Point Algorithms for Linear Complementarity Problems"*; Lecture Notes in Computer Science, vol. 538. Springer-Verlag, Berlin, (1991).
7. G.Q. WANG, C.J. YU, AND K.L. TEO , *"A full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-linear complementarity problem"*; Journal of Global Optimization, 59(1), 81-99, (2014).
8. C. ROOS, T. TERLAKY AND J-PH.VIAL, *"Theory and Algorithms for Linear Optimization.An Interior-Point Approach"*; John Wiley and Sons, Chichester, UK, (1997).