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TAYLOR SERIES SOLUTION OF MULTI OBJECTIVE LINEAR FRACTIONAL PROGRAMMING PROBLEM

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Abstract: In this paper, we have proposed a solution to Multi Objective Linear Fractional Programming Problem (MOLFPP) by expanding the order 1*st* Taylor polynomial series these objective functions at optimal points of each linear fractional objective functions in feasible region. MOLFPP reduces to an equivalent Multi Objective Linear Programming Problem (MOLPP). The resulting MOLPP is solved assuming that weights of these linear objective functions are equal and considering the sum of the these linear objective functions. The proposed solution to MOLFPP always yields efficient solution, even a strong-efficient solution. Therefore, the complexity in solving MOLFPP has reduced easy computational.

To show the ability the proposed solution, three different numerical examples have been presented. The given examples are solved using optimization software WINQSB.(Chang, 2001)

Keywords: Multi Objective Linear Fractional Programming Problem (MOLFPP), Multi Objective Linear Programming Problem (MOLPP), Taylor series, WinQSB

Çok Amaçlı Doğrusal Kesirli Programlama Probleminin Taylor Serisiyle Çözümü

Özet: Bu makalede, Çok Amaçlı Doğrusal Kesirli Programlama Probleminin uygun bölgesindeki, her doğrusal kesirli amaç fonksiyonunu optimal yapan noktalarda, kesirli lineer amaç fonksiyonları Taylor serisine açılarak, Çok Amaçlı Doğrusal Kesirli Programlama Problemi, Çok Amaçlı Doğrusal Programlama Problemine dönüştürül-müştür. Daha sonra da, doğrusal amaç fonksiyonlarının ağırlıkları dikkate alınarak, ağırlıklı toplamı bulunmuştur. Ardından, tek amaçlı doğrusal programlama problemi elde edilmiştir. Bu doğrusal programlama probleminin optimal çözümü, çok amaçlı doğrusal kesirli programlama probleminin etkin, hatta, kuvvetli etkin çözümlerini belirlemektedir.

Önerilen çözümün etkinliğini göstermek için, örnek uygulamalar yapılmış olup, örneklerin çözümünde WinQSB bilgisayar paket programı kullanılmıştır.

Anahtar Kelimeler: Çok Amaçlı Doğrusal Kesirli Programlama Problemi, Çok Amaçlı Doğrusal Programlama Problemi, Taylor serisi, WinQSB

Introduction

In the modelling of the real word problems like financial and corporate planning, production planning, marketing and media selection, university planning and student admissions, health care and hospital planning, air force maintenance units, bank branches, etc. frequently may be faced up with decision to optimise dept/equity ratio, profit/cost, inventory/sales, actual cost/standard cost, output/employee, student/cost, nurse/patient ratio etc. respect to some constraints (Lai and Hwang, 1996).

In the literature, different approaches appear to solve different models of Linear Fractional Programming Problem (LFPP). Because, fractional programming solves more efficiently the above problems respect to Linear Programming Problem (LPPP).

When some of the studies have achieved solution methods (Lai and Hwang, 1996), (Charnes and Cooper, 1962), (Zionts, 1968), (Chakraborty and Gupta, 2002), others have concentrated on applications (Lai and Hwang, 1996), (Munteanu and Rado, 1960), (Gilmore and Gomory, 1963), (Sengupta et all, 2001), (Chakraborty and Gupta, 2002). In these papers are discussed in details LFPP. It is showed that LFPP can be optimised easily. But, in the great scale decision problems, there is more than one objective, which must be satisfied at the same time as possible. However, most of these are linear fractional objectives. It is difficult to talk about the optimal solutions of these problems. The solutions searched for these problems are weak efficient or strong efficient. If required, one compromise solution can be reached by the affection of the models with the decision makers (DMs).

There exist several methodologies to solve multi objective linear fractional programming problem (MOLFPP) in the literature. Most of these methodologies are computationally burdensome (Chakraborty and Gupta, 2002). Kornbluth and Steuer (1981), Y.J.Lai and C.L.Hwang (1996) have developed an algorithm for solving the MOLFPP for all weak-efficient vertices of the feasible region. Nykowski, Z.Zolkiewski (1978) and Dutta et all (1992) have proposed a compromise procedure for MOLFPP. Choo and Atkins (1982) have given an analysis of the bicriteria LFPP.

M.K.Luhandjula (1984) solved MOLFPP using a fuzzy approach (Lai and Hwang, 1996). He used linguistic approach to solve MOLFPP by introducing linguistic variables to represent linguistic aspirations of the DM. Dutta et all (1992) modified the linguistic approach of Luhandjula to solve MOLFPP.

In this paper, we have proposed a solution to MOLFPP using the 1*st* order Taylor polynomial series at optimal point of each linear fractional objective function in feasible region. LFPP has been transformed into a more simplified structure MOLPP by using Taylor polynomial series and an efficient, even a strong-efficient solution has been proposed by using the approach.

Linear Fractional Programming Problem (LFPP)

The general format of Linear Fractional Programming Problem may be written as:

$$Max\ \frac{c^T x+\alpha}{d^T x+\beta}$$

$$\text{s.t}\quad Ax=b,$$
$$x\geq 0,\ x,\ c^T,\ d^T \in R^n$$
$$A\in R^{mxn}, \alpha,\ \beta\in R. \tag{1}$$

Definition:

The two Mathematical programming problems

(i) Max $F(x)$ subject to $x\in\Delta$,

(ii) Max $G(x)$, subject to $x\in\Delta$, will be said to be equivalent if only if is a one one map q(.) of the feasible set of (i), onto the feasible set of (ii), such that *F(x)=G(q(x))* for all $x\in\Delta$ (Chakraborty and Gupta, 2002).

Multiple Objective Linear Fractional Programming Problem (MOLFPP)

A MOLFPP may be written following as:

$Max\ Z(x) = \{Z_1(x), Z_2(x), \dots, Z_k(x)\}$

s.t.

$$x \in X = \{x \in R^n, Ax \le b, x \ge 0\}, \tag{2}$$

with $b \in R^n$, $A \in R^{mxn}$.

$$Z_i(x) = \frac{c_i x + \alpha_i}{d_i x + \beta_i} = \frac{N_i(x)}{D_i(x)}, c_i, d_i \in R^n \ and\ \alpha_i, \beta_i \in R.$$

Let be $Max \quad Z_i(x) = \frac{c_i x + \alpha_i}{d_i x + \beta_i} = \frac{N_i(x)}{D_i(x)} = Z_i^*$ and $Min\, Z_i(x) = \frac{c_i x + \alpha_i}{d_i x + \beta_i} = \frac{N_i(x)}{D_i(x)} = Z_i^{Min}$.

Model Development

In this paper, we consider the Multi Objective Linear Fractional Programming Problem (MOLFPP)

$$\begin{aligned} &MaxZ_i(x) = Max\{Z_1(x), Z_2(x), \dots, Z_k(x)\} \\ &s.t. \\ &x \in X = \{x \in R^n, Ax \le b, x \ge 0\} \\ &with\ b \in R^n, A \in R^{mxn} \end{aligned} \tag{3}$$

and $Z_i(x) = \frac{c_i x + \alpha_i}{d_i x + \beta_i} = \frac{N_i(x)}{D_i(x)}$ where $c_i, d_i \in R^n\ and\ \alpha_i, \beta_i \in R,\ D_i(x) > 0,\ \forall i.$

Let be maximum value i*th* objective function $Max\, Z_i(x) = Z_i^*, \forall i,$ on the feasible region it occurs when $x_i^* = (x_{i1}^*, x_{i2}^*, \dots, x_{in}^*)$, for i=1,...,k.

Suppose that $Z(x)$ and all of its partial derivates of order less than or equal n+1 are continuous on the feasible region X, $x^* \in X$. By expanding the 1*st* order Taylor polynomial series for objective function $Z_i(x)$ about x_i^*, objective function $Z_i(x)$ is obtained from $Z_i(x) = P_{i1}(x) + R_{i1}(x)$ where function $P_{i1}(x)$ is called the 1*st* Taylor polynomial in n variables about x_i^* and $R_{i1}(x)$ is the remainder term associated with $P_{i1}(x)$. We have

$$\begin{aligned} Z_i(x) \cong P_{i1}(x) = Z_i(x_i^*) + [(x_1 - x_{i1}^*)\frac{\partial Z_i(x_i^*)}{\partial x_1} + (x_2 - x_{i2}^*)\frac{\partial Z_i(x_i^*)}{\partial x_2} + \\ \dots + (x_n - x_{in}^*)\frac{\partial Z_i(x_i^*)}{\partial x_1}\ ] + O(h^2) \end{aligned}$$

$$Z_i(x) \cong P_{i1}(x) = Z_i(x_i^*) + \sum_{j=1}^{n}[(x_j - x_{ij}^*)\frac{\partial Z_i(x_i^*)}{\partial x_j} + O(h^2) \text{ for i=1,...,k.} \tag{4}$$

Where $o(h^2)$ is order of the maximum error. This polynomial gives an accurate approximation to $Z_i\ (x)$ when x is close to x^*.

By replacing $Z_i(x) = \dfrac{c_i x + \alpha_i}{d_i x + \beta_i} = \dfrac{N_i(x)}{D_i(x)}$ by $Z_i\ (x) \cong P_{i1}(x) = Z_i\ (x_i^*) + \sum_{j=1}^{n}[(x_j - x_{ij}^*)\dfrac{\partial Z_i\ (x_i^*)}{\partial x_j}]$ in MOLFPP (3), all of the objective functions $Z_i\ (x)$, i=1,2,...,k, become the 1*st* order linear functions as $e_i + \sum_{j=1}^{n} a_j x_j,\ e_i,\ a_j \in R,\ i = 1,2,...,k.$ So, MOLFPP (3) reduces the following MOLPP (5):

$$Max\left\{ Z_1(x) = e_1 + \sum_{j=1}^{n} a_{1j} x_j, Z_2(x) = e_2 + \sum_{j=1}^{n} a_{2j} x_j, ..., Z_k(x) = e_k + \sum_{j=1}^{n} a_{kj} x_j \right\}$$

$$x \in X = \{x \in R^n, Ax \le b, x \ge 0\}, \tag{5}$$

with $b \in R^n$, $A \in R^{mxn}$.

If we assume that the weights of objective functions in problem (5) are equal, then problem (5) is written as follows:

$$Max\left\{ \sum_{j=1}^{n} a_{1j} x_j + \sum_{j=1}^{n} a_{2j} x_j + ... + \sum_{j=1}^{n} a_{kj} x_j + \sum_{i=1}^{k} e_i \right\}$$

$$x \in X = \{x \in R^n, Ax \le b, x \ge 0\}, \tag{6}$$

with $b \in R^n$, $A \in R^{mxn}$.

In problem (6), set X is non-empty convex set having feasible points. The optimal solution of problem (6) gives the efficient solution of MOLFPP (5). Because, weights of objective functions that are expanded Taylor series are equal and in (6) is considered the weighted objective function.

Numerical Examples

Example 1. Let us consider a MOLFPP with two objectives as follows:

$$Max\left\{ Z_1(x) = \frac{-3x_1 + 2x_2}{x_1 + x_2 + 3},\ Z_2(x) = \frac{7x_1 + x_2}{5x_1 + 2x_2 + 1} \right\}$$

s.t.

$$x_1 - x_2 \ge 1$$
$$2x_1 + 3x_2 \le 15 \tag{7}$$
$$x_1 \ge 3$$
$$x_1, x_2 \ge 0.$$

It is observed that $Z_1 < 0,\ Z_2 \ge 0,$ for each x in the feasible region. $Z_1^{Max}(3.6;2.6) = \dfrac{-14}{23},\ Z_2^{Max}(7.5;0) = \dfrac{15}{11}.$

By expanding the 1*st* order Taylor polynomial series for objective functions $Z_1(x)\ and\ Z_2(x)$ about points (3.6; 2.6) and (7.5; 0) in feasible region, respectively are obtained from

$$Z_1(x) = \frac{-14}{23} + (x_1 - 3.6)\frac{\partial Z_1(3.6;2.6)}{\partial x_1} + (x_2 - 2.6)\frac{\partial Z_1(3.6;2.6)}{\partial x_2} \tag{8}$$

$$Z_1(x) = -0.2599x_1 + 0.2835x_2 - 0.41$$

$$Z_2(x) = \frac{15}{11} + (x_1 - 7.5)\frac{\partial Z_2(7.5;0)}{\partial x_1} + (x_2 - 0)\frac{\partial Z_2(7.5;0)}{\partial x_2}$$

$$Z_2(x) = 0.004722x_1 - 0.04486x_2 + 1.3282$$

So, all fractional objectives are transformed to the linear objectives. The obtained MOLPP is equivalent to the following LPP when weights of objective functions are equal:

$$Max\{Z_1(x) + Z_2(x)\} = -0.255178x_1 + 0.23864x_2 + 0.9182$$

s.t.

$$\begin{aligned} & x_1 - x_2 \geq 1 \\ & 2x_1 + 3x_2 \leq 15 \\ & x_1 \geq 3 \\ & x_1, x_2 \geq 0. \end{aligned} \tag{9}$$

Optimal solution of problem (9) is at point (3,2) and maximum value is 0.6299. The point (3,2) is efficient solution of the given original problem in the feasible region.

The solution for original problem is obtained as $x_1 = 3,\ x_2 = 2,\ Z_1 = \frac{-5}{8},\ Z_2 = \frac{23}{20}$.

Example 2. Let us consider a MOLFPP with three objectives as follows:

$$\text{Max } \left\{ Z_1(x) = \frac{-3x_1 + 2x_2}{x_1 + x_2 + 3},\ Z_2(x) = \frac{7x_1 + x_2}{5x_1 + 2x_2 + 1}, Z_3(x) = \frac{x_1 + 4x_2}{2x_1 + 3x_2 + 2} \right\}$$

s.t.

$$\begin{aligned} & x_1 - x_2 \geq 1 \\ & 2x_1 + 3x_2 \leq 15 \\ & x_1 + 9x_2 \geq 9 \\ & x_1 \geq 3 \\ & x_1, x_2 \geq 0. \end{aligned} \tag{10}$$

It is observed that $Z_1 < 0,\ Z_2 \geq 0,\ Z_3 \geq 0$ for each x in the feasible region. $Z_1^{Max}(3.6;2.6) = \frac{-14}{23},\ Z_2^{Max}(7.2;0.2) = \frac{23}{17},\ Z_3^{Max}(3.6;2.6) = \frac{14}{17}$

By expanding the 1*st* order Taylor polynomial series for objective functions $Z_1(x),\ Z_2(x)\ and\ Z_3(x)$ about points (3.6; 2.6), (7.5;0) and (3.6; 2.6) in feasible region, respectively are obtained the following linear objective functions:

$$Z_1(x) = -0.2599x_1 + 0.2835x_2 - 0.41,$$
$$Z_2(x) = 0.00629x_1 - 0.0456x_2 - 1.3167, \quad (11)$$
$$Z_3(x) = -0.03806x_1 + 0.08996x_2 + 0.72662.$$

So, the obtained MOLPP is equivalent to the following LPP when weights of objective functions are equal:

$$Max\{Z_1(x) + Z_2(x) + Z_3(x)\} = -0.29167x_1 + 0.32786x_2 + 1.63332$$

s.t.

$$x_1 - x_2 \geq 1$$
$$2x_1 + 3x_2 \leq 15 \quad (12)$$
$$x_1 + 9x_2 \geq 9$$
$$x_1 \geq 3$$
$$x_1, x_2 \geq 0.$$

The solution to the above LPP is obtained as $Max\{Z_1(x) + Z_2(x) + Z_3(x)\}(3.6; 2.6) = 1.4372$. The solution for original problem is obtained as ;

$$x_1 = 3.6,\ x_2 = 2.6,\ Z_1 = \frac{-14}{23}, Z_2 = \frac{139}{121},\ Z_3 = \frac{14}{17}.$$

Example 3. Let us consider a MOLFPP with three objectives as follows:

$$\text{Max}\left\{Z_1(x) = \frac{-x_1 + x_2 - 4}{6x_1 + x_2 + 3},\ Z_2(x) = \frac{x_1 - x_2 - 5}{x_2 + 1}, Z_3(x) = \frac{3x_1 + x_2 - 17}{-3x_1 + 16}\right\}$$

s.t.

$$x_1 \leq 4$$
$$x_2 \leq 4 \quad (13)$$
$$x_1 + x_2 \leq 7$$
$$-x_1 + x_2 \leq 3$$
$$x_1 - x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

It is observed that $Z_1 < 0,\ Z_2 < 0,\ Z_3 < 0$ for each x in the feasible region.

$$Z_1^{Max}(1,4) = \frac{-1}{13},\ Z_2^{Max}(4,1) = -1,\ Z_3^{Max}(4,3) = -\frac{1}{2}$$

At these corresponding points, expanding Taylor series, we have

$$Z_1 + Z_2 + Z_3 = .58358x_1 + .33284x_2 - 3.6167$$

The given MOLFPP problem is written as:

$$Max\{Z_1 + Z_2 + Z_3\} = .58358x_1 + .33284x_2 - 3.6167$$

s.t

$$\begin{aligned} & x_1 \leq 4 \\ & x_2 \leq 4 \\ & x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_1 - x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{aligned} \quad (14)$$

The solution to the above LP is obtained as Max $\{Z_1 + Z_2 + Z_3\}(4,3) = -.2867$. The solution for original problem is obtained as

$$x_1 = 4,\ \ x_2 = 3,\ \ Z_1 = \frac{-1}{6},\ \ Z_2 = -1,\ \ Z_3 = -\frac{1}{2}.$$

Conclusion

In this paper, we have proposed a solution to Multi Objective Linear Fractional Programming Problem (MOLFPP) using Taylor polynomial series. With the help of the order 1*st* Taylor polynomial series at optimal points of each linear fractional objective function in feasible region, Multi Objective Linear Fractional Programming Problem reduces to an equivalent Multi Objective Linear Programming Problem (MOLPP). The obtained MOLPP is solved assuming that weights of these linear objective functions are equal and considering the sum of the linear objective functions. The proposed solution to MOLFPP always yields efficient solution, even a strong-efficient solution. Therefore, the complexity in solving MOLFPP has reduced easy computational.

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