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Research Article

On the boundary crossing problem in memoryless models

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Abstract

The joint Laplace transform of the two sided boundary crossing stopping rule is known for the negative exponential model only under certain conditions. In this paper we eliminate the need for such conditions. Our results also apply to the boundary crossing problem for the geometric models. We further illustrate how the results can be used to obtain the distribution for the multidimensional boundary crossing stopping rules under the memoryless models.

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1. Introduction

While monitoring an information stream, $X_1, X_2, \cdots$, of independent random variables with a common distribution, the following well known two sided boundary crossing stopping rule has been studied in detail, [4, 8, 16, 17, 20], in various branches of statistics and probability,

$$\tau := \tau_{a,h} := \inf\{n \geq 1 : S_n \notin (a, h)\}, \tag{1.1}$$

where $S_n = Y_1 + Y_2 + \cdots + Y_n$, $Y_j = X_j - E(X_j) - \delta$ (here $E(X_1)$ is the initial mean and δ is taken as a reference constant) and two specified boundary constants $a \leq 0 < h$. The optimality properties for appropriately chosen constants, a, h, are also well known, [4, 8, 16]. While implementing such stopping rules the main problem lies with the lack of exact closed form expressions for the joint distribution and/or moments of τ and the stopped process S_τ. The analogous continuous time stopping rules for Markov processes have been studied in quite detail and their links to partial differential equations are well known, [13]. Although Brownian motion approximations, [17, 20], sometimes can be quite useful, such approximations of τ become useless while investigating the sizes of overshoot and undershoot in discrete time models.

Closed form expressions for the joint distribution of τ, S_τ are known only for some very specific models [12]. One of the earliest open problems has been to obtain the joint distribution of τ, S_τ when the underlying distribution is negative exponential, $P(X_i \leq$

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$x) := F(x,\lambda) = 1 - \exp\{-\lambda x\}$, $x > 0$, going as far back as 1954 when Anscombe and Page [2] proposed it in the context of the Sequential Probability Ratio Test (SPRT). The average run length (ARL) of τ, namely $E(\tau)$, was studied by several authors, Chow et al. [5], Cox and Roseberry [6] and Hoeffding [10]. Chow et al. [5] provided moments of the stopped random walk, S_τ, in terms of the moments of the the SPRT τ. Using Monte Carlo simulations, Cox and Roseberry [6] suspected that the standard deviation of τ may be proportional to the ARL. Hoeffding [10] was able to derive lower bounds for the ARL. Stadje [18] was the first to find an exact expression for $E(\tau)$ by solving some related integral equations. De and Zacks [7] obtained the distribution of a truncated sequential probability ratio test (SPRT) for the negative exponential model. Recently Starvaggi and Khan [19] provided the joint Laplace transform of τ and S_τ under the assumption that the roots of a certain characteristic polynomial are real, distinct and lied in the interval $(0, 1)$.

The goal of this paper is to show that the need for the existence of distinct real roots of a certain characteristic polynomial can be avoided altogether and we provide the joint Laplace transform of τ, S_τ in general. The technique of [19] leads to a similar type of root finding problem for the geometric model. Additionally we show how one may drop this condition for the geometric model as well. This is expected since both of these models obey the memoryless property, $P(X_i \geq t + s | X_i \geq s) = P(X_i \geq t)$, for $t, s \geq 0$, while for the geometric case t, s are integers. The study of memoryless models is not only historically interesting, they may also help in understanding the worst case scenarios, see Example 4.2 for more details. Furthermore, due to the fact that the inter-arrival times of a Poisson process are negative exponential random variables, one may use the results of the present paper to investigate stopping rules involving Poisson process.

Section 2 presents the main result. Its proof is presented in Section 3. Section 4 contains some examples where we also provide the exact distribution of a first passage time stopping rule of a multidimensional random walk when its independent components come from memoryless models. It seems to be the first result of this type for any model in discrete time multivariate settings as far as we know, and suggests that such problems need not be hopelessly complicated.

2. The main result

Let $X_1, X_2, \cdots$ be independent and identically distributed with a negative exponential (with mean $1/\lambda$) or geometric distribution (with mean q/p, $q = 1 - p$). Let $Y_i = X_i - c$ for $i = 1, 2, \cdots$ where $c > 0$, and let τ be the boundary crossing stopping time as defined in (1.1). Note that under the fixed sample theory of testing hypothesis when one uses the likelihood ratio,

$$Y_i := \ln\left(\frac{f_1(X_i)}{f_0(X_i)}\right),$$

where f_0, f_1 are the densities of X_i under the null and the alternative hypotheses, one obtains $Y_i = X_i - c$ where the constant c depends on the choices provided by H_0 and H_1. For the more general boundary crossing problem considered here, $c > 0$, can be any constant.

Introduce the unique natural number k such that $(k-1)c < h - a \leq kc$. Let $\mathcal{F}_n = \sigma(Y_1, \cdots, Y_n)$ be the sigma field generated by $Y_1, Y_2, \cdots, Y_n$. Throughout $I(A)$ will stand for the indicator function of the event A and $I_n := I(A_n)$ with $A_n := \{\tau > n\} = \{S_j \in (a, h) \text{ for } j = 1, ..., n\}$. Let $\varphi(\theta) := E(e^{\theta Y_i})$ be the moment generating function of Y_i. We will use the notation $f_n := E\left(e^{\lambda S_n} I_n\right)$, for both the negative exponential and the geometric models, by taking $q = e^{-\lambda}$ in the latter case. In the geometric model the constants a, h, c will be taken to be integers without loss of any generality. Also, $f_0 = 1$

and $S_0 = 0$, and for the fixed k, let

$$c_{i,k} = c_i = \begin{cases} (-1)^{i+1}\dfrac{(pq^c)^i}{i!}\displaystyle\prod_{j=0}^{i-1}\{h-a-(i-1)c-1-j\}, & 1 \le i \le k, \quad \text{(geometric)} \\ (-1)^{i+1}\dfrac{(\lambda e^{-\lambda c})^i}{i!}[h-a-(i-1)c]^i, & 1 \le i \le k, \quad \text{(exponential).} \end{cases}$$

For the geometric models we will use the notation $b_i(\theta)$ for the following expressions.

$$\begin{aligned} b_1(\theta) &= -\varphi(\theta)(e^\theta q)^{a+c+1} \\ b_{k+1}(\theta) &= -\varphi(\theta)^{k+1}[(e^\theta q)^{h+c} \\ &\quad -(e^\theta q)^{a+kc+1}(1+\sum_{j=1}^{k-1}\prod_{l=0}^{j-1}(h-a-(k-1)c-1-l)\frac{(1-e^\theta q)^j}{(-1)^j j!})] \\ b_i(\theta) &= \varphi(\theta)^i(e^\theta q)^{a+(i-1)c+1}(1+\sum_{j=1}^{i-2}\prod_{l=0}^{j-1}(h-a-(i-2)c-1-l)\frac{(1-e^\theta q)^j}{(-1)^j j!}) \\ &\quad -\varphi(\theta)^i(e^\theta q)^{a+ic+1}(1+\sum_{j=1}^{i-2}\prod_{l=0}^{j-1}(h-a-(i-1)c-1-l)\frac{(1-e^\theta q)^j}{(-1)^j j!}) \\ &\quad +(-1)^i\varphi(\theta)e^{(a+c+1)\theta}\frac{(pq^c)^{i-1}}{(i-1)!}\prod_{j=0}^{i-2}[h-a-(i-1)c-1-j], \quad 2 \le i \le k. \end{aligned}$$

For the negative exponential models $b_i(\theta)$ stands for the following expressions.

$$\begin{aligned} b_1(\theta) &= -\varphi(\theta)e^{(\theta-\lambda)(a+c)}, \\ b_{k+1}(\theta) &= \varphi(\theta)\{-\varphi(\theta)^k e^{(\theta-\lambda)(h+c)} \\ &\quad +\sum_{j=0}^{k-1}(-1)^j\frac{(\lambda e^{-\lambda c})^j}{j!}\varphi(\theta)^{k-j}e^{(\theta-\lambda)(a+(k-j)c)}(h-a-(k-1)c)^j\}, \\ b_i(\theta) &= \sum_{j=0}^{i-2}(-1)^j\frac{(\lambda e^{-\lambda c})^j}{j!}\varphi(\theta)^{i-j}e^{(\theta-\lambda)(a+(i-1-j)c)}(h-a-(i-2)c)^j \\ &\quad -\sum_{j=0}^{i-2}(-1)^j\frac{(\lambda e^{-\lambda c})^j}{j!}\varphi(\theta)^{i-j}e^{(\theta-\lambda)(a+(i-j)c)}(h-a-(i-1)c)^j \\ &\quad +(-1)^i\varphi(\theta)e^{(\theta-\lambda)(a+c)}\frac{(\lambda e^{-\lambda c})^{i-1}}{(i-1)!}(h-a-(i-1)c)^{i-1}, \qquad 2 \le i \le k. \end{aligned}$$

The following is the main result of the paper which completely solves the discrete time boundary crossing problem in memoryless models.

Theorem 2.1. *For the memory less models, the joint Laplace transform of τ, S_τ is*

$$\begin{aligned} E(e^{\theta S_\tau - r\tau}) &= \left(\varphi(\theta)e^{(\theta-\lambda)(h+c)-r}+\sum_{i=1}^{k+1}b_i(\theta)e^{-ri}\right)G(r)+T(\theta,r), \qquad \textit{where} \\ G(r) &= \frac{\sum_{n=0}^{k-1}e^{-rn}f_n-\sum_{i=1}^{k-1}c_{i,k}e^{-ri}\sum_{n=0}^{k-i-1}e^{-rn}f_n}{1-\sum_{i=1}^{k}c_{i,k}e^{-ri}}, \\ T(\theta,r) &= \sum_{n=1}^{k}E(e^{\theta S_n - rn}\mathbb{I}(\tau=n,S_n\le a))-\sum_{i=1}^{k}b_i(\theta)e^{-ir}\sum_{n=0}^{k-i}e^{-rn}f_n, \end{aligned}$$

and $f_n = \sum_{i=1}^{k} c_i f_{n-i}$, for $n \ge k$.

3. The proofs

For the proof of the main theorem the following result is needed.

Lemma 3.1. *For both of the memoryless models the following holds for any d and n so that $1 \le d \le k,\ d \le n$,*

$$\begin{aligned} E(e^{\theta S_n} I_n \cdot \mathbb{I}(S_n > a + (k-d)c)) &= \sum_{i=1}^{d} Q_{i,d}(\theta) f_{n-i}, \\ E(q^{-S_n} I_n \cdot \mathbb{I}(S_n > a + (k-d)c)) &= \sum_{i=1}^{d} c_{i,d} f_{n-i}, \end{aligned}$$

where, for the geometric case,

$$\begin{aligned} c_{i,d} &= (-1)^{i+1} \frac{(pq^c)^i}{i!} \prod_{j=0}^{i-1} [h - a - (k-d+i-1)c - 1 - j], \\ Q_{i,d}(\theta) &= -\varphi(\theta)^i (e^\theta q)^{a+(k-d+i)c+1} [(e^\theta q)^{h-a-(k-d+i-1)c-1} \\ &- \sum_{j=1}^{i-1} (\frac{(1-e^\theta q)^j}{(-1)^j j!} \prod_{l=0}^{j-1} (h - a - (k-d+i-1)c - 1 - l)) - 1], \end{aligned}$$

where the last sum stands for zero when $i = 1$. For the negative exponential case,

$$\begin{aligned} c_{i,d} =& (-1)^{i+1} \frac{(\lambda e^{-\lambda c})^i}{i!} [h - a - (k-d+i-1)c]^i, \\ Q_{i,d}(\theta) =& \sum_{j=0}^{i-1} (-1)^j \frac{(\lambda e^{-\lambda c})^j}{j!} \varphi(\theta)^{i-j} e^{(\theta-\lambda)(a+(k-d+i-j)c)} (h - a - (k-d+i-1)c)^j \\ &- \varphi(\theta)^i e^{(\theta-\lambda)(h+c)}. \end{aligned}$$

Proof: (of Lemma 3.1) We will prove this lemma by induction for the geometric model. The corresponding result for the negative exponential model is known, cf. [19]. Let $\mathcal{G}_n$ be the $\sigma-$field generated by $X_1, ..., X_n$, Let $d = 1$, and $n \ge d$,

$$\begin{aligned} &E(e^{\theta S_n} I_n \cdot \mathbb{I}(S_n > a + (k-1)c)) \\ &= E(e^{\theta S_{n-1}} I_{n-1} \cdot E(e^{\theta(X_n - c)} \mathbb{I}(a + (k-1)c - S_{n-1} + c < X_n < h - S_{n-1} + c) | \mathcal{G}_{n-1})) \\ &= pe^{-\theta c} E \left\{ e^{\theta S_{n-1}} I_{n-1} \frac{(e^\theta q)^{a+kc-S_{n-1}+1} - (e^\theta q)^{h+c-S_{n-1}}}{1 - e^\theta q} \right\} \\ &= -\varphi(\theta)(e^\theta q)^{a+kc+1}((e^\theta q)^{h-a-(k-1)c-1} - 1) f_{n-1} \ = \ Q_{1,1} f_{n-1}. \end{aligned}$$

This gives that

$$\begin{aligned} E(q^{-S_n} I_n \cdot \mathbb{I}(S_n > a + (k-1)c)) &= \lim_{\theta \to -\ln q} E(e^{\theta S_n} I_n \cdot \mathbb{I}(S_n > a + (k-1)c)) \\ &= pq^c (h - a - (k-1)c - 1) f_{n-1} \ = \ c_{1,1} f_{n-1}. \end{aligned}$$

Then for $m = d$, and all $n \ge d,\ k \ge d$,

$$\begin{aligned} &E(e^{\theta S_n} I_n \cdot \mathbb{I}(S_n > a + (k-d)c)) \\ &= e^{-\theta c} E(e^{\theta S_{n-1}} I_{n-1} \cdot E(e^{\theta X_n} \mathbb{I}(a + (k-(d-1))c - S_{n-1} < X_n < h + c - S_{n-1}) | \mathcal{G}_{n-1})) \\ &= e^{-\theta c} E(e^{\theta S_{n-1}} I_{n-1} (\mathbb{I}(S_{n-1} \le a + (k-(d-1))c) + \mathbb{I}(S_{n-1} > a + (k-(d-1))c)) \\ &\quad \cdot E(e^{\theta X_n} \mathbb{I}(a + (k-(d-1))c - S_{n-1} < X_n < h + c - S_{n-1}) | \mathcal{G}_{n-1})) \end{aligned}$$

$$
\begin{aligned}
&= e^{-\theta c}E\left\{e^{\theta S_{n-1}}I_{n-1}\mathbb{I}(S_{n-1}\le a+(k-(d-1))c)\cdot\sum_{j=a+(k-(d-1))c-S_{n-1}+1}^{h+c-S_{n-1}-1}pq^je^{\theta j}\right\}\\
&\quad +e^{-\theta c}E(e^{\theta S_{n-1}}I_{n-1}\mathbb{I}(S_{n-1}> a+(k-(d-1))c)\cdot\sum_{j=0}^{h+c-S_{n-1}-1}pq^je^{\theta j})\\
&= \varphi(\theta)E(q^{-S_{n-1}}I_{n-1}\mathbb{I}(S_{n-1}\le a+(k-(d-1))c)\cdot((e^\theta q)^{a+(k-(d-1))c+1}-(e^\theta q)^{h+c}))\\
&\quad +\varphi(\theta)E((e^{\theta S_{n-1}}-q^{-S_{n-1}}(e^\theta q)^{h+c})I_{n-1}\mathbb{I}(S_{n-1}>a+(k-(d-1))c))\\
&= \varphi(\theta)\{E(e^{\theta S_{n-1}}I_{n-1}\mathbb{I}(S_{n-1}>a+(k-(d-1))c))+((e^\theta q)^{a+(k-(d-1))c+1}\\
&\quad -(e^\theta q)^{h+c})f_{n-1}-(e^\theta q)^{a+(k-(d-1))c+1}E(q^{-S_{n-1}}I_{n-1}\mathbb{I}(S_{n-1}>a+(k-(d-1))c))\}.
\end{aligned}
$$

Using the induction hypothesis,

$$
\begin{aligned}
&E(e^{\theta S_n}I_n\cdot\mathbb{I}(S_n>a+(k-d)c))\\
&\quad= \varphi(\theta)\sum_{i=1}^{d-1}Q_{i,d-1}(\theta)f_{n-1-i}+\varphi(\theta)((e^\theta q)^{a+(k-(d-1))c+1}-(e^\theta q)^{h+c})f_{n-1}\\
&\qquad -\varphi(\theta)(e^\theta q)^{a+(k-(d-1))c+1}\sum_{i=1}^{d-1}c_{i,d-1}f_{n-1-i}\\
&\quad =: \sum_{i=1}^{d}\widehat{Q}_{i,d}(\theta)f_{n-i},\qquad \text{where,}
\end{aligned}
$$

$\widehat{Q}_{1,d}=\varphi(\theta)((e^\theta q)^{a+(k-(d-1))c+1}-(e^\theta q)^{h+c})=\varphi(\theta)(e^\theta q)^{a+(k-d+1)c+1}(-(e^\theta q)^{h-a-(k-d)c-1}+1)$ and when $2\le i\le d$, $\widehat{Q}_{i,d}(\theta)=\varphi(\theta)(Q_{i-1,d-1}(\theta)-(e^\theta q)^{a+(k-(d-1))c+1}c_{i-1,d-1})$. Clearly, $\widehat{Q}_{1,\ d}=Q_{1,\ d}$ when $i=1$. When $2\le i\le d$,

$$
\begin{aligned}
\widehat{Q}_{i,d}(\theta) &= \varphi(\theta)\{-\varphi(\theta)^{i-1}(e^\theta q)^{a+(k-(d-1)+i-1)c+1}[(e^\theta q)^{h-a-(k-(d-1)+i-1-1)c-1}\\
&-\sum_{j=1}^{i-1-1}\frac{(1-e^\theta q)^j}{(-1)^jj!}\prod_{l=0}^{j-1}(h-a-(k-(d-1)+i-1-1)c-1-l)-1]-\\
&(e^\theta q)^{a+(k-(d-1))c+1}(-1)^{i-1+1}\frac{(pq^c)^{i-1}}{(i-1)!}\prod_{j=0}^{i-1-1}[h-a-(k-(d-1)+i-1-1)c-1-j]\}.
\end{aligned}
$$

A simple algebra shows that this equals $Q_{i,d}(\theta)$. To obtain $c_{i,d}$ as the limit of $Q_{i,d}(\theta)$, it is convenient to take $t=e^\theta q$ and note that

$$
\lim_{\theta\to-\ln q}Q_{i,d}(\theta)=-\lim_{\theta\to-\ln q}((pe^{-\theta c})^i(e^\theta q)^{a+(k-d+i)c+1})\cdot\lim_{t\to1}\frac{N(t)}{D(t)},
$$

where $N(t)=t^{h-a-(k-d+i-1)c-1}-\sum_{j=1}^{i-1}\prod_{l=0}^{j-1}(h-a-(k-d+i-1)c-1-l)\frac{(1-t)^j}{(-1)^jj!}-1$ and $D(t)=(1-t)^i$. Now a trite calculation shows that the right hand side is indeed $c_{i,d}$. This proves Lemma 3.1. □

Proof: (of Theorem 2.1) We first derive the generating function of the sequence f_n. Choose $d=k$ in the second equation of Lemma 3.1 to get,

$$
f_n=E(q^{-S_n}I_n)=\sum_{i=1}^{k}c_{i,k}f_{n-i},\qquad n\ge k.
$$

Note $c_{i,k} = c_i$, $i = 1, 2, \cdots, k$. Using this result, and interchanging the order of summations gives that,

$$\sum_{n=0}^{+\infty} e^{-rn} f_n = \sum_{n=0}^{k-1} e^{-rn} f_n + \sum_{n=0}^{+\infty} e^{-rn} f_n \cdot \sum_{i=1}^{k} c_{i,k} e^{-ri} - \sum_{i=1}^{k} (c_{i,k} e^{-ri} \sum_{n=0}^{k-i-1} e^{-rn} f_n).$$

Hence the generating function of f_n is

$$\sum_{n=0}^{+\infty} e^{-rn} f_n = \frac{\sum_{n=0}^{k-1} e^{-rn} f_n - \sum_{i=1}^{k-1} c_{i,k} e^{-ri} \sum_{n=0}^{k-i-1} e^{-rn} f_n}{1 - \sum_{i=1}^{k} c_{i,k} e^{-ri}} = G(r).$$

Now first consider the Laplace transform of the stopped process at the upper exit. For $0 < \theta < -\ln q$ and $n \geq 1$,

$$\begin{aligned} E(e^{\theta S_\tau} \cdot \mathbb{I}(\tau = n, S_n \geq h)) &= E[e^{\theta S_{n-1}} I_{n-1} E(e^{\theta(X_n - c)} \cdot \mathbb{I}(X_n \geq h - S_{n-1} + c) | \mathcal{G}_{n-1})] \\ &= e^{-\theta c} E\left\{ e^{\theta S_{n-1}} I_{n-1} \sum_{i=h-S_{n-1}+c}^{+\infty} e^{\theta i} p q^i \right\} \\ &= \frac{e^{-\theta c} p (e^\theta q)^{h+c}}{1 - e^\theta q} E(q^{-S_{n-1}} I_{n-1}) \ = \ \varphi(\theta)(e^\theta q)^{h+c} \cdot f_{n-1}. \end{aligned}$$

Therefore the joint one-sided Laplace transform of τ and S_τ is

$$\begin{aligned} E(e^{\theta S_\tau - r\tau} \cdot \mathbb{I}(S_\tau \geq h)) &= \sum_{n=1}^{+\infty} e^{-rn} E(e^{\theta S_\tau} \cdot \mathbb{I}(\tau = n, S_n \geq h)) \\ &= \varphi(\theta)(e^\theta q)^{h+c} e^{-r} G(r). \qquad (*) \end{aligned}$$

Second, consider the Laplace transform of stopped process at the lower exit.

$$\begin{aligned} &E(e^{\theta S_n} \cdot \mathbb{I}(\tau = n,\ S_n \leq a)) \\ &= e^{-c\theta} E[e^{\theta S_{n-1}} I_{n-1} \mathbb{I}(S_{n-1} > a + c) E(e^{\theta X_n} \mathbb{I}(X_n \leq a + c - S_{n-1}) | \mathcal{G}_{n-1})] \\ &\quad + e^{-c\theta} E[e^{\theta S_{n-1}} I_{n-1} \mathbb{I}(S_{n-1} \leq a + c) E(e^{\theta X_n} \mathbb{I}(X_n \leq a + c - S_{n-1}) | \mathcal{G}_{n-1})] \\ &= 0 + e^{-c\theta} E(e^{\theta S_{n-1}} I_{n-1} \mathbb{I}(S_{n-1} \leq a + c) E(e^{\theta X_n} \mathbb{I}(X_n \leq a + c - S_{n-1}) | \mathcal{G}_{n-1})) \\ &= \varphi(\theta) E(e^{\theta S_{n-1}} I_{n-1} \mathbb{I}(S_{n-1} \leq a + c)) - \varphi(\theta)(e^\theta q)^{a+c+1} E(q^{-S_{n-1}} I_{n-1} \mathbb{I}(S_{n-1} \leq a + c)) \\ &= \varphi(\theta)[E(e^{\theta S_{n-1}} I_{n-1}) - E(e^{\theta S_{n-1}} I_{n-1} \mathbb{I}(S_{n-1} > a + c))] \\ &\quad - \varphi(\theta)(e^\theta q)^{a+c+1} [E(q^{-S_{n-1}} I_{n-1}) - E(q^{-S_{n-1}} I_{n-1} \mathbb{I}(S_{n-1} > a + c))]. \end{aligned}$$

Choose $d = k$, $k - 1$ respectively in Lemma 3.1. When $n \geq k + 1$,

$$\begin{aligned} &E(e^{\theta S_n} \cdot \mathbb{I}(\tau = n,\ S_n \leq a)) \\ &= \varphi(\theta)[\sum_{i=1}^{k} Q_{i,k}(\theta) f_{n-1-i} - \sum_{i=1}^{k-1} Q_{i,k-1}(\theta) f_{n-1-i}] - \varphi(\theta)(e^\theta q)^{a+c+1} [f_{n-1} \\ &\quad - \sum_{i=1}^{k-1} c_{i,k-1} f_{n-1-i}] \\ &= \varphi(\theta) \sum_{i=2}^{k} (Q_{i-1,k}(\theta) - Q_{i-1,k-1}(\theta) + (e^\theta q)^{a+c+1} c_{i-1,k-1}) f_{n-i} + \varphi(\theta) Q_{k,k}(\theta) f_{n-1-k} \\ &\quad - \varphi(\theta)(e^\theta q)^{a+c+1} f_{n-1} \\ &= \sum_{i=1}^{k+1} b_i(\theta) f_{n-i}, \end{aligned}$$

where $b_1(\theta) = -\varphi(\theta)(e^\theta q)^{a+c+1}$, $b_{k+1}(\theta) = \varphi(\theta) Q_{k,k}(\theta)$ and $b_i(\theta) = \varphi(\theta)(Q_{i-1,k}(\theta) - Q_{i-1,k-1}(\theta) + e^{(a+c+1)\theta} c_{i-1,k-1})$, $2 \leq i \leq k$. Substituting the expressions of $Q_{i,j}(\theta)$ and

$c_{i,d}$ from Lemma 3.1 gives the $b_i(\theta)$ as used in the theorem. Therefore the joint lower-sided-exit Laplace transform of τ and S_τ is

$$\begin{aligned}
&E(e^{\theta S_\tau - r\tau}\cdot \mathbb{I}(S_\tau \le a))\\
&= \sum_{n=1}^{k} E(e^{\theta S_n - rn}\mathbb{I}(\tau = n, S_n \le a)) + \sum_{n=k+1}^{\infty} e^{-rn}\sum_{i=1}^{k+1} b_i(\theta) f_{n-i}\\
&= \sum_{n=1}^{k} E(e^{\theta S_n - rn}\mathbb{I}(\tau = n, S_n \le a)) + \sum_{i=1}^{k+1} b_i(\theta)e^{-ir}(G(r) - \sum_{n=0}^{k-i} e^{-rn} f_n)\\
&= \sum_{i=1}^{k+1} b_i(\theta)e^{-ir}G(r) + \sum_{n=1}^{k} E(e^{\theta S_n - rn}\mathbb{I}(\tau = n, S_n \le a)) - \sum_{i=1}^{k} b_i(\theta)e^{-ir}\sum_{n=0}^{k-i} e^{-rn} f_n\\
&= B(\theta, r)G(r) + T(\theta, r),
\end{aligned}$$

where $B(\theta, r) = \sum_{i=1}^{k+1} b_i(\theta)e^{-ri}$, and $T(\theta, r)$ is as defined in the theorem. Collecting the two one-sided expressions for the joint Laplace transforms finishes the proof. □

4. Examples & discussion

Boundary crossing stopping rules have a wide range of applications. For instance, it is known that as the probabilities of type I and type II errors become small, the ARL of the SPRT can be substantially smaller than the corresponding fixed sample size obtained by the Neyman-Pearson lemma for a simple versus simple hypothesis. This is illustrated in the first example below. Another application of the theory of boundary crossing arises in financial engineering where pricing a barrier option leads to finding the average run length of τ as defined in (1.1). In quality control such stopping rules provide monitoring algorithms that are more sensitive than the traditional Shewhart or exponentially weighted moving average control charts. The second example below illustrates how the theory of boundary crossing is related to the cusum stopping rules. Example three shows how one may extend the results to multidimensional stopping rules.

Example 4.1. (SPRT vs. NP test) Let $X_1,\ X_2,\ \ldots$ be *i.i.d.* negative exponential random variables with density function $f(x) = \lambda e^{-\lambda x}$ and $\tau = \inf\{n \ge 1,\ S_n = \sum_{i=1}^{n}(X_i - c) \notin (a,\ h)\}$. We choose $a = -c$ and $h = c$ so $k = 2$, and furthermore we choose $c = 1$. This gives $c_{1,k} = c_1 = 2\lambda e^{-\lambda} = f_1$ and $c_{2,k} = c_2 = -\frac{1}{2}(\lambda e^{-\lambda})^2$. In the one sided joint Laplace transform (*) of τ and S_τ, and taking $\theta = r = 0$, gives the following expression for the probability of upper exit. Next plugging $\theta = 0$ in $E(e^{\theta S_\tau - r\tau})$ and the first derivative with respect to r gives the following expression from the ARL. Here $b_1(0) = -1$, $b_2(0) = 1 - e^{-\lambda} + \lambda e^{-\lambda}$ and $b_3(0) = e^{-\lambda}(1 - \lambda - e^{-\lambda})$. Also $P(\tau = 2, S_\tau \le a) = 1 - e^{-\lambda} - \lambda e^{-\lambda}$ and $P(\tau = 1, S_\tau \le a) = 0$ since $a = -c$ and X_i are nonnegative random variables.

$$E(\tau) = \frac{6\lambda e^{\lambda} - \lambda^2 - 2e^{\lambda} - 4e^{2\lambda} + 4}{4\lambda e^{\lambda} - \lambda^2 - 2e^{2\lambda}}$$

$$P(S_\tau \ge h) = \frac{e^{-2\lambda}}{1 - 2\lambda e^{-\lambda} + \frac{1}{2}\lambda^2 e^{-2\lambda}}$$

Figure 1 shows the ARL and the standard deviation of the SPRT. The figure shows that the ARL and the standard deviation seem to be approximately proportional as conjectured by [6] by observing some simulations.

Continuing with the above model we may compare the performance of the most powerful Neyman-Pearson (N.P.) test and the sequential probability ratio test (SPRT) for the same probabilities of type I and type II errors. To test $H_0 : \lambda = \lambda_0$ vs $H_1 : \lambda = \lambda_1 > \lambda_0$, we take Neyman-Pearson constant $c = \frac{\log \lambda_1 - \log \lambda_0}{\lambda_1 - \lambda_0}$ as the reference constant. Without loss of generality we can take $\lambda_0 = 1$ since $\lambda_0 X_i \sim f_1(x)$. Picking $a = -c$ and $h = c$, so $k = 2$, the SPRT will reject H_0 if $S_\tau \le a$ and accept H_0 when $S_\tau \ge h$. Figure 2 shows the comparison between N-P test and SPRT.

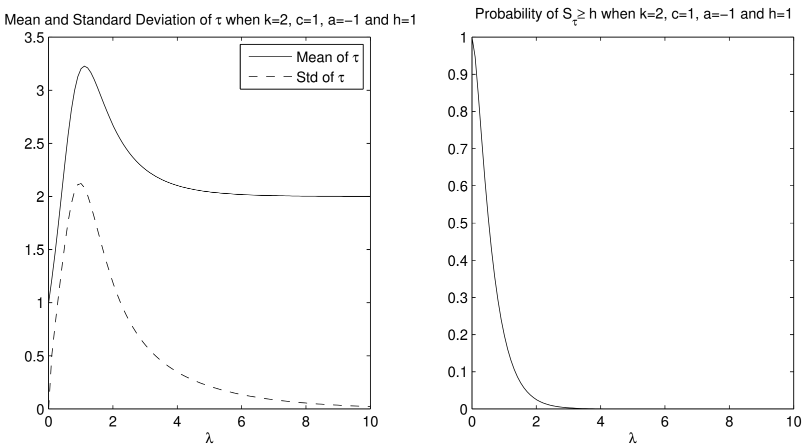

Figure 1. The ARL and standard deviation of the SPRT.

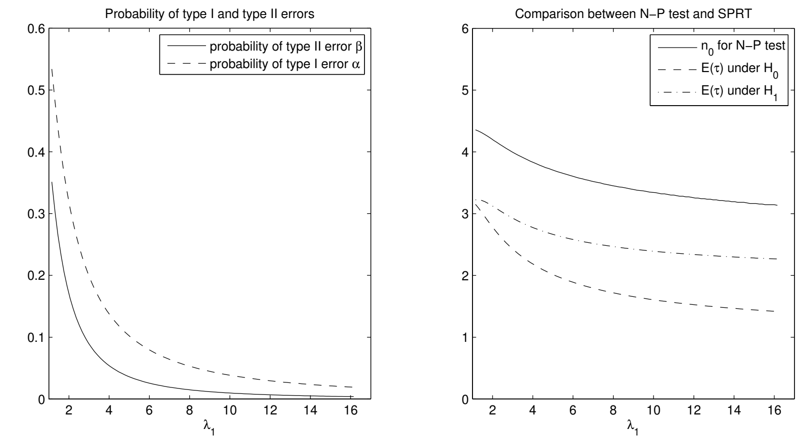

Figure 2. Comparison of N.P. test & SPRT.

Under both hypotheses the ARL is lower than the corresponding N.P. sample size.

Example 4.2. (Cusum) The cumulative sum (cusum) procedure is a process monitoring stopping rule,

$$N_h \;:=\; \inf\{n \ge 1 : D_n \ge h\}, \qquad D_n := S_n - \min_{0 \le k \le n} S_k, \quad S_0 = 0.$$

This stopping rule has a close link with the boundary crossing problem, cf. [12, 14],

$$N_h \;=\; \tau_{0,h} + N^* I(S_{\tau_{0,h}} \le 0),$$

where $\tau_{0,h}$ is the boundary crossing stopping rule (1.1) with $a = 0$, and N^* is another identically distributed cusum process as N_h which is independent of $S_{\tau_{0,h}}$ given $\tau_{0,h}$ (see [12] for details.). Using this link and Theorem 2.1, one may derive the distribution and all the moments of the cusum stopping rule N_h. In particular we may obtain the ARL of the cusum procedure in the negative exponential model which was first obtained by [9] by direct calculations.

Cusum procedure have also found a large collection of applications. For instance, the classic trading the line strategy used for fast financial trading platforms can be analyzed by using the boundary crossing stopping rule τ [1]. The monitoring process of the cusum stopping rule, D_n, is related to the ladder index concept of queuing theory [15], as well as some other related fields such as insurance risk, dams, and data communication. The MAC layer of communication systems contains the back off protocol, should two clients approach the server exactly at the same time and cause a collision. There is a potential for client misbehavior. Cardenas et al. [3] showed that the geometric model, if followed by the misbehaving client, leads to the most difficult detection case.

Example 4.3. (Multivariate SPRT for memoryless models) When dealing with multidimensional boundary crossing and/or cusum stopping times, relatively few results are known, cf. [11] which are primarily asymptotic as the threshold constants get large and the component processes are independent. Recently, Yao and Khan [21], obtained a closed form result for the average run length for the multinomial cusum procedure. In this example we illustrate how exact expressions for the multidimensional boundary crossing stopping can be obtained from the univariate results of Theorem 2.1 for memoryless models.

Consider m independent memoryless processes as defined in Theorem 2.1 with their respective constants, $p^{(i)}, a^{(i)}, h^{(i)}$ in the geometric case or $\lambda^{(i)}, a^{(i)}, h^{(i)}$ for the negative exponential case, with reference constants $c^{(i)} > 0$, $i = 1, 2, \cdots, m$. Consider the case when $h^{(i)} \leq a^{(i)} + c^{(i)}$, $i = 1, 2, \cdots, m$. Define $\tau = \min\{\tau_1, \tau_2, \cdots, \tau_m\}$, then we claim

$$E(e^{-r\tau}) \quad = \quad \frac{1 - A + (A+B)\cdot e^{-r}}{e^r - B}, \qquad \text{where,}$$

$A = \prod_{i=1}^{m}((q^{(i)})^{a^{(i)}+c^{(i)}+1} - q_i^{h^{(i)}+c^{(i)}})$, $B = \prod_{i=1}^{m}((p^{(i)}q^{(i)})^{c^{(i)}}(h^{(i)} - a^{(i)} - 1))$ for the geometric case, and $A = \prod_{i=1}^{m}(e^{-\lambda^{(i)}(a^{(i)}+c^{(i)})} - e^{-\lambda^{(i)}(h^{(i)}+c^{(i)})})$, $B = \prod_{i=1}^{m}(\lambda_i(h^{(i)} - a^{(i)})e^{-\lambda^{(i)}c^{(i)}})$ for the negative exponential case. We illustrate the derivation for the geometric case since the negative exponential case is quite similar.

With $d = k = 1$ in Lemma 3.1 and recalling $f_0 = 1$ gives for $n \geq 1$ and dropping any fixed superscript, $i = 1, 2, \cdots, m$, for a moment for convenience,

$$\begin{aligned}
&E(e^{\theta S_n} I_n) = \varphi(\theta)((e^\theta q)^{a+c+1} - (e^\theta q)^{h+c}) f_{n-1},\\
&f_n = E(q^{-S_n} I_n) = (pq)^c(h-a-1)f_{n-1} = ((pq)^c(h-a-1))^n
\end{aligned}$$

Letting $\theta \to 0$ gives

$$\begin{aligned}
P(\tau_i > n) =& E(e^{0 S_n} I_n) = \varphi(0)((e^0 q)^{a+c+1} - (e^0 q)^{h+c}) f_{n-1}\\
=& (q^{a+c+1} - q^{h+c})[(pq)^c(h-a-1)]^{n-1}.
\end{aligned}$$

The above results hold for each $i = 1, 2, \cdots, m$. Therefore for $\tau = \min\{\tau_1, \tau_2, \cdots, \tau_m\}$,

$$\begin{aligned}
P(\tau > n) &= \prod_{i=1}^{m}((q^{(i)})^{a^{(i)}+c^{(i)}+1} - (q^{(i)})^{h^{(i)}+c^{(i)}}) \left\{\prod_{i=1}^{m}(p^{(i)}q^{(i)})^{c^{(i)}}(h^{(i)} - a^{(i)} - 1)\right\}^{n-1}\\
&= A \cdot B^{n-1}.
\end{aligned}$$

Therefore for $n > 1$, the probability mass function is

$$P(\tau = n) = P(\tau > n-1) - P(\tau > n) \;=\; AB^{n-2}(1-B),$$

and $P(\tau = 1) = 1 - P(\tau > 1) = 1 - A$. This gives that

$$E(e^{-r\tau}) = \sum_{n=1}^{+\infty} e^{-rn} P(\tau = n) = \frac{1 - A + (A+B) \cdot e^{-r}}{e^r - B}.$$

In particular, $E(\tau) = 1 + \frac{A}{1-B}$, and $Var(\tau) = \frac{A\cdot(1+B-A)}{(1-B)^2}$.

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