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CHAOTIC SYNCHRONIZATION METHODS BASED ON STABILITY ANALYSIS OF LINEAR SYSTEMS

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Abstract: *In this paper three methods for chaotic synchronization, based on the known linear-nonlinear decomposition method, are proposed. The main advantage of this kind of decomposition is that the stability analysis of the synchronization scheme can be done by a linear error system, so there is no need to calculate the conditional Lyapunov exponents or to design Lyapunov functions. The new aspect of the proposed approaches is, that in contrast to the standard linear-nonlinear decomposition method, strict rules to design the system couplings with many different combinations of additional decomposition of the linear part of the system or with additional feedback coupling are defined.*

Keywords: *Chaotic synchronization, Feedback coupling, Linear-Nonlinear decomposition.*

INTRODUCTION

One particular task in the field of the nonlinear dynamics - the chaotic synchronization, received increasing attention since 1990. Chaotic systems were long considered as non-usable in real-world applications due to their high sensitivity to initial conditions and pseudo-random behaviour. However, soon after Pecora and Carroll first proposed a method for synchronizing the dynamics of two identical chaotic systems, started from different initial conditions [1,2], it was found that this phenomenon can be of significant advantage when used in some applications. Generally, the main tendency is to use chaotic synchronization schemes in different types of secure communication systems, where the specific properties of a chaotic signal are used to hide information signals. This on the other hand motivates the further work on developing new synchronization methods with better properties or to modify and improve the existing ones. The fact that no common and uniform nonlinear systems theory is formulated also stimulates the further work on chaotic synchronization problems. Many different kinds of synchronization methods are proposed after the Pecora&Carroll method. Some of them are modifications and improvements of the basic Pecora&Carroll method [3-5], which implies a simple decomposition of a given chaotic system in two parts and building a copy of one of them, driven by the variables of the other. Another common group of synchronization methods are the feedback-based ones [4,6,7]. Some not so popular methods are those with occasional [8] or nonlinear coupling [9].

Generally, two chaotic systems are called synchronized, if a functional relation exists between their states. The most common case is when the distance between the system states, called error function, converges to zero for $t \to \infty$. To achieve any kind of synchronization behavior for two or more chaotic systems, one has to design a proper coupling between them. Usually the coupling is unidirectional. In this case the system providing the coupling signal is called Master and the one receiving the synchronization signal – Slave [4,10,11]. The main difficulty is to find a coupling which guarantees stable synchronization. The stability analysis is done by the error system, obtained by subtracting the equations of the Slave system from those of the Master system. In the majority of the known synchronization methods, the error system however is a nonlinear system, which makes the stability analysis a complex task. Some approaches for examining the synchronization stability from the error system are proposed so far – the conditional Lyapunov exponents can be calculated, but they give only a necessary condition for stability; or a Lypunov function approach can be used, but no strict rules for designing such functions for any chaotic system are known. On the other hand, if the error system is a wholly linear system, the stability analysis can be easily made, since the linear systems' theory is well elaborated.

Back in 2003, Yu and Liu [12] proposed a synchronization approach, by which the error system becomes linear and the system's eigenvalues determine the stability of the synchronization scheme. Basically, they suggested the decomposition of the Master system on a linear and a nonlinear part and building a copy of the linear part to act as a Slave system, which is driven by the nonlinear part of the Master system. However, no strict rules for such decomposition are given in [12], moreover the "nonlinear" part of the decomposed system contains also some linear elements.

This synchronization approach is further developed in this paper. Three approaches for designing synchronization schemes with linear error systems are proposed. Generally, they are based on the Yu&Liu linear-nonlinear decomposition method. Their method is further extended, assuming that one can make additional decomposition of the linear part of the system or apply additional linear feedback coupling. The aim is to define strict rules for designing synchronization schemes based on the linear-nonlinear decomposition principle, such that for a particular synchronization problem one can have the possibility to test consistently the greatest possible number of synchronization schemes with linear error systems and to find the best one in terms of synchronization speed or other criteria.

The first of the proposed methods implies additional decomposition of Master system's linear part in two parts, one of which is used as a second coupling signal along with the nonlinear part. The second method implies the introduction of a linear feedback-type coupling in the Slave system as a second coupling signal. The third method combines the principles of the first two methods.

Thus, if for a given chaotic system the simple linear-nonlinear decomposition does not provide stable synchronization or there is such one, but with long transient before the systems synchronize, following the straight rules of the proposed methods there is still the possibility to test scores of other couplings with linear error systems and choose the best solution for the particular problem.

To illustrate the proposed approaches, the Shimizu-Morioka third-order chaotic system is used.

MODIFIED LINEAR-NONLINEAR DECOMPOSITION METHODS

The basic Yu&Liu method [12] states, that a continuous chaotic system can be decomposed in two parts:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) = \mathbf{g}(\mathbf{x}(t)) + \mathbf{h}(\mathbf{x}(t), t), \quad (1)$$

where $\mathbf{x} \in \Re^n$, the function $\mathbf{f} : \Re^n \to \Re^n$ defines a vector field in *n*-dimensional space, $\mathbf{g}(\mathbf{x}(t)) = \mathbf{A}\mathbf{x}(t)$ is the linear part of the system and $\mathbf{h}(\mathbf{x}(t), t)$ is the nonlinear part.

If the chaotic system, defined with Eq. (1) is considered to act as a Master system for a given synchronization scheme, the Slave system is then designed as a copy of the linear part of Eq. (1), driven by the nonlinear part:

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, \mathbf{x}, t) = \mathbf{g}(\tilde{\mathbf{x}}(t)) + \mathbf{h}(\mathbf{x}(t), t), \quad (2)$$

where $\tilde{\mathbf{x}} \in \Re^n$ is the state vector and $\mathbf{g}(\tilde{\mathbf{x}}(t)) = \mathbf{A}\tilde{\mathbf{x}}(t)$ is the linear part of the Slave system.

The error system is obtained by subtracting Eq. (2) from Eq. (1):

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)) = \mathbf{A}\mathbf{e}(t). \quad (3)$$

where $\mathbf{e}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t)$ is the error function or the distance between the states of the two systems.

Obviously the error system is linear. If identical synchronization with $\lim_{t\to\infty} \mathbf{e}(t) = 0$ is aimed, then to prove the synchronization stability one has only to find the eigenvalues of the constant **A** matrix, because the zero point of $\mathbf{e}(t)$ is the equilibrium point of Eq. (3). If all eigenvalues have negative real parts, according to the stability criterion for linear systems the zero point of $\mathbf{e}(t)$ is stable and $\lim_{t\to\infty} \mathbf{e}(t) = 0$. For conciseness the basic linear-nonlinear decomposition method will be referred to as the *LN* method.

However, the exact principles of system decomposition in the form of Eq. (1) are not given by Yu&Liu, as the nonlinear function $\mathbf{h}(\mathbf{x}(t), t)$, used in the examples in [12], contains not only nonlinearities, but also some linear elements, which according to the linear-nonlinear decomposition principle should be in the linear part $\mathbf{g}(\mathbf{x}(t))$.

In the modified methods proposed here, it will be assumed that the $\mathbf{h}$ function contains only the nonlinearities of $\mathbf{f}$ and all linear elements are included in the **A** matrix.

Modified linear-nonlinear decomposition method with one-way feedback coupling (LNF method)

By this modification the addition of a second coupling, proportional to the error function (as with the feedback-based synchronization methods), is suggested. The Master system is again decomposed in the form of Eq. (1). The Slave system is then constructed in the following way:

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{g}(\tilde{\mathbf{x}}(t)) + \mathbf{h}(\mathbf{x}(t),t) + \alpha\,\mathbf{E}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)), \quad (4)$$

where $\mathbf{g}(\tilde{\mathbf{x}}(t)) = \mathbf{A}\tilde{\mathbf{x}}(t)$ and α and **E** are the coupling gain and the coupling matrix of the feedback coupling.

The error system:

$$\dot{\mathbf{e}}(t) = \mathbf{A}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)) - \alpha\,\mathbf{E}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)) = (\mathbf{A} - \alpha\,\mathbf{E})\mathbf{e}(t) \quad (5)$$

remains linear and the synchronization stability is verified by the eigenvalues of the constant $(\mathbf{A} - \alpha\,\mathbf{E})$ matrix.

Modified linear-nonlinear decomposition method with partial replacement (LNP method)

The linear part of the Master system can be decomposed in two parts in the following way:

$$\mathbf{g}(\mathbf{x}(t)) = \mathbf{A}\mathbf{x}(t) = \mathbf{A}_1\mathbf{x}(t) + \mathbf{A}_2\mathbf{x}(t), \quad (6)$$

and the $\mathbf{A}_2\mathbf{x}(t)$ part can be used for the coupling along with the nonlinear part $\mathbf{h}(\mathbf{x}(t),t)$:

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}_1\tilde{\mathbf{x}}(t) + \mathbf{A}_2\mathbf{x}(t) + \mathbf{h}(\mathbf{x}(t),t). \quad (7)$$

If the linear decomposition is such that the $\mathbf{A}_2$ matrix contains only one non-zero element, the additional coupling is close to the principle of the partial replacement synchronization method [3], where one Master system variable drives the Slave system by replacing its corresponding variable at only one place in the Slave system's model. The driving variable and the place of substitution are in fact specified in the $\mathbf{A}_2$ matrix. The error system is:

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}_1(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)) = \mathbf{A}_1\mathbf{e}(t). \quad (8)$$

As with the *LNF* method, the error system is linear and the stability of synchronization is again verified with the eigenvalues of the $\mathbf{A}_1$ matrix.

Modified linear-nonlinear decomposition method with feedback coupling and partial replacement (LNFP method)

The principles of the *LNF* and *LNP* methods can be applied together, when designing the coupling. Then the Master, the Slave and the error systems are:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t) + \mathbf{A}_2\mathbf{x}(t) + \mathbf{h}(\mathbf{x}(t),t), \quad (9)$$

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}_1\tilde{\mathbf{x}}(t) + \mathbf{A}_2\mathbf{x}(t) + \mathbf{h}(\mathbf{x}(t),t) + \alpha\,\mathbf{E}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)), \quad (10)$$

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A}_1 - \alpha\,\mathbf{E})\mathbf{e}(t). \quad (11)$$

The error system is linear and the stability analysis is made by the eigenvalues of the $(\mathbf{A}_1 - \alpha\,\mathbf{E})$ matrix. The combined method is called *LNFP* method (*L*inear-*N*onlinear decomposition with *F*eedback coupling and *P*artial replacement).

The main advantage of the *LNF*, *LNP* and *LNFP* methods over the basic linear-nonlinear decomposition method is that for every pair of chaotic systems, subjected to synchronization, these methods provide a great variety of possible coupling schemes. If the standard *LN* method doesn't yield synchronization for a particular chaotic system, it is highly possible that from the vast number of possible couplings, provided by the modified methods, one can find the best possible solution for the given problem – a scheme with fast synchronization, a scheme with marginal synchronization, etc. At the same time the proposed methods retain the main advantage of the *LN* method – the stability analysis is based on a linear error system. However, one should mention that not all of the known chaotic models can be decomposed in such form. Also, with the linear-nonlinear decomposition approaches usually there is need to send not one, but in some cases even all of the state variables of the Master system to the Slave system, so the coupling isn't "economic" and in some systems not all system variables are accessible.

NUMERICAL EXPERIMENTS

To test the proposed synchronization methods, one of the simplest third-order chaotic systems, proposed by Shimizu and Morioka [13], is used. The system is described by:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1 - ax_2 - x_1x_3, \\ \dot{x}_3 &= -bx_3 + x_1^2, \end{aligned} \tag{12}$$

where for $a = 0.85$ and $b = 0.5$ the system is chaotic.

***Example* 1**. First, the basic linear-nonlinear decomposition is examined. The system is decomposed in the form of Eq. (1) with linear part:

$$\mathbf{Ax}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -a & 0 \\ 0 & 0 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{13}$$

and nonlinear part:

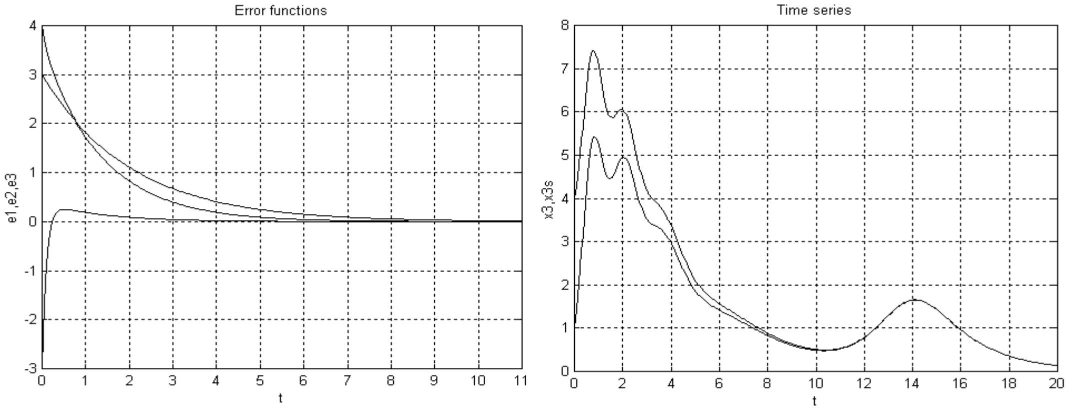
$$\mathbf{h}(\mathbf{x}(t), t) = \begin{bmatrix} 0 & -x_1x_3 & x_1^2 \end{bmatrix}^T. \tag{14}$$

To achieve stable synchronization between two Shimizu-Morioka systems, according to the *LN* method all eigenvalues of the **A** matrix must have negative real part. However the eigenvalues are: $\lambda_1 = -1.5,\ \lambda_2 = -0.5,\ \lambda_3 = 0.7$. Obviously, due to the positive value of λ_3, no synchronization is possible if Eq. (14) is used as a coupling between the systems.

***Example* 2.** To try to achieve synchronization, a second coupling according to the *LNF* method can be introduced. One of the many possible variants of this method is to introduce $\alpha(x_1 - \tilde{x}_1)$ as an additional coupling in the first equation of the Slave system:

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 + \alpha(x_1 - \tilde{x}_1), \\ \dot{\tilde{x}}_2 &= \tilde{x}_1 - a\tilde{x}_2 - x_1x_3, \\ \dot{\tilde{x}}_3 &= -b\tilde{x}_3 + x_1^2. \end{aligned} \tag{15}$$

The error system, according to Eq. (5), is described with the matrix $(\mathbf{A} - \alpha\mathbf{E}) = [-\alpha, 1, 0; 1, -a, 0; 0, 0, -b]$, whose eigenvalues for $\alpha = 10$ are $\lambda_1 = -0.5,\ \lambda_2 = -0.7$ and $\lambda_3 = -10.1$, i.e. the synchronization scheme is stable. The error functions $e_i(t) = x_i(t) - \tilde{x}_i(t)$, obtained by simulation with initial conditions $\mathbf{x}_0 = \begin{bmatrix} 2 & 5 & 4 \end{bmatrix}^T$ and $\tilde{\mathbf{x}}_0 = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T$ are shown on Fig. 1a. Apparently the two systems' states converge rapidly to each other. Fig. 1b shows the time evolutions of x_3 and $\tilde{x}_3$ (denoted as 'x3s' on the figure).



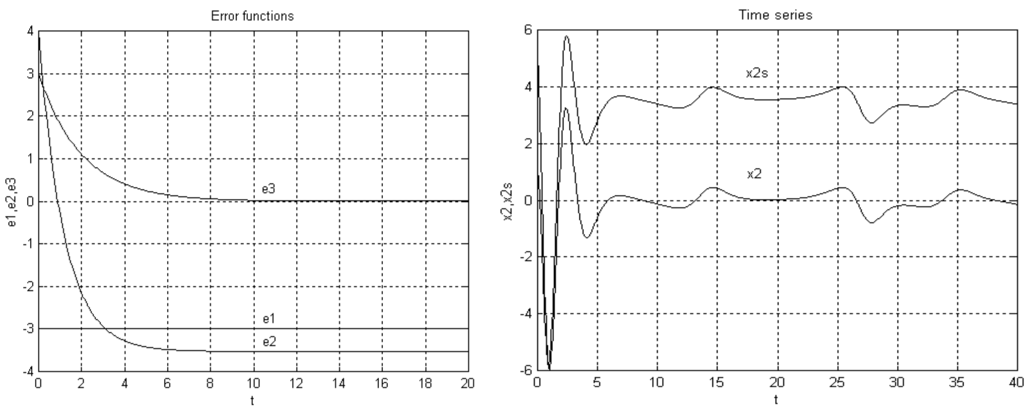
a. b.

Fig. 1. Identical synchronization with *LNF* coupling: a – error dynamics; b – time evolution of x_3 and $\tilde{x}_3$

***Example* 3.** One of the possible variants of the *LNP* principle is to introduce x_2 as an additional driving signal in the first equation of the Slave system:

$$\begin{aligned} \dot{\tilde{x}}_1 &= x_2, \\ \dot{\tilde{x}}_2 &= \tilde{x}_1 - a\tilde{x}_2 - x_1 x_3, \\ \dot{\tilde{x}}_3 &= -b\tilde{x}_3 + x_1^2. \end{aligned} \tag{16}$$

The linear error system is described with the matrix $\mathbf{A}_1 = [0,0,0;1,-a,0;0,0,-b]$ with eigenvalues $\lambda_1 = 0$, $\lambda_2 = -0.5$ and $\lambda_3 = -0.85$. The zero maximum eigenvalue is an indication of marginal synchronization [14], which is characterized by $\lim_{t\to\infty} \mathbf{e}(t) = c$, where *c* is a constant, depending on the initial conditions. The error functions and the time series of x_2 and $\tilde{x}_2$ are shown on Fig. 2. Both figures give clear view of the typical marginal synchronization behaviour. The initial conditions are same as in the previous example.



a. b.

Fig. 2. Marginal synchronization with *LNP* coupling: a – error dynamics; b – time evolution of x_2 and $\tilde{x}_2$

***Example* 4.** The *LNFP* method offers the greatest number of possible couplings. One of them is to design the Slave system, driven by the nonlinear part of the Master system, the x_2 variable in the first equation and the feedback coupling $\alpha(x_3 - \tilde{x}_3)$ in the third equation:

$$\begin{aligned} \dot{\tilde{x}}_1 &= x_2, \\ \dot{\tilde{x}}_2 &= \tilde{x}_1 - a\tilde{x}_2 - x_1 x_3, \\ \dot{\tilde{x}}_3 &= -b\tilde{x}_3 + x_1^2 + \alpha(x_3 - \tilde{x}_3). \end{aligned} \tag{17}$$

The error system is described by the matrix $(\mathbf{A}_1 - \alpha\mathbf{E}) = [0,0,0; 1,-a,0; 0,0,-b-\alpha]$ with eigenvalues $\lambda_1 = 0$, $\lambda_2 = -0.85$ and $\lambda_3 = -10.5$ for $\alpha = 10$. Again, marginal synchronization occurs, but as λ_2 and λ_3 are more "distant" from the zero point than those of the previous example, the systems will synchronize more rapidly. Fig. 3 shows the error functions for this case.

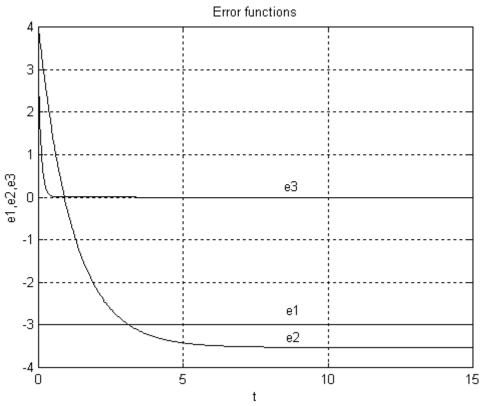


Fig. 3. Marginal synchronization with *LNFP* coupling

***Example* 5.** If the feedback coupling in Example 4 is changed with $\alpha(x_1 - \tilde{x}_1)$, added to the first equation of the Slave system, the error system will be described by the matrix $(\mathbf{A}_1 - \alpha\mathbf{E}) = [-\alpha, 0, 0; 1, -a, 0; 0, 0, -b]$ with eigenvalues $\lambda_1 = -0.5$, $\lambda_2 = -0.85$ and $\lambda_3 = -10$ for $\alpha = 10$. The error dynamics for this case is exactly the same as in Example 2. Thus, the additional coupling with x_2 does not provide any benefit in terms of synchronization speed. This is due to the fact that the closest eigenvalue to zero in both cases is the same - $\lambda_1 = -0.5$.

Then, by examining the eigenvalues of the possible synchronization schemes, provided by the *LN*, *LNF*, *LNP* and *LNFP* methods, one can not only prove the synchronization stability and determine the synchronization type, but also compare the synchronization speeds of different variants. The coupling gain α in the feedback-based variants can serve as an additional instrument to tune the synchronization scheme. However, large values of α are not advisable in general, as the coupling gain increases the eventual noise, associated with the driving signal from the Master system.

Another problem, concerning chaotic synchronization schemes in general, is their sensitivity to parameter changes. One of the necessary conditions identical or marginal synchronization to occur is the coincidence of all parameters of the two chaotic systems. However, this is hard to achieve in real-world applications. On the other hand, minor parameter differences do not affect stable chaotic synchronization schemes notwithstanding the used synchronization method. To test the parameter sensitivity of the proposed synchronization schemes, Examples 2-5 are simulated with 1% difference between the parameter vectors. The parameters of the Master system keep their nominal values

$a = 0.85$ and $b = 0.5$, while those of the Slave system are changed to $\tilde{a} = 0.8585$ and $\tilde{b} = 0.505$. The simulations do not give any visible deviations from the synchronization processes depicted on Figs. 1-3.
The proposed chaotic synchronization methods are also tested on other typical low-order chaotic systems (Chua, Rossler, Hide systems, etc.). The results are similar to those obtained with the Shimizu-Morioka system.

CONCLUSION

Three new modifications of the linear-nonlinear decomposition chaotic synchronization method are proposed in this letter. The *LNF*, *LNP* and *LNFP* methods retain the main advantage of the basic *LN* method, providing a linear error system to facilitate the stability analysis. The proposed approaches provide rules to consistent design and test of a great number of possible couplings between the Master and the Slave system in order to find the best solution for each particular synchronization problem.

REFERENCES

[1] Pecora, L., T. Carroll. Driving systems with chaotic signals. Physical Review A, Vol.44, No.4, 1991, pp.2374-2384.
[2] Pecora, L., T. Carroll. Synchronization in chaotic systems. Physical Review Letters, Vol.64, No.8, 1990, pp.821-824.
[3] Guemez, J., M. Matias. Modified method for synchronizing and cascading chaotic systems, Physical Review E 52, 1995, pp.2145-2148.
[4] Pecora, L., T. Carroll, G. Johnson, D. Mar, J. Heagy. Fundamentals of synchronization in chaotic systems, concepts, and applications. Chaos 7(4), 1997, pp.520-543.
[5] Kocarev, L., U. Parlitz. General approach for chaotic synchronization with applications to communication. Physical Review Letters, Vol.74, No.25, 1995, pp.5028-5031.
[6] Ogorzalek, M. Taming chaos – part I: Synchronization. IEEE Transactions on Circuits and Systems-I, Vol.40, No.10, 1993, pp.693-699.
[7] Boccaletti, S., J. Kurths, G. Osipov, D. Valladares, C. Zhou. The synchronization of chaotic systems. Physics Reports 366 (2002), pp.1-101.
[8] Morgul, O., M. Feki, Synchronization of chaotic systems by using occasional coupling, Physical Review E, Vol.55, No.5, 1997, pp.5004-5010.
[9] Ali, M., J. Fang. Synchronization of chaos and hyperchaos using linear and nonlinear feedback functions. Physical Review E, Vol.55, No.5, 1997, pp.5285-5290.
[10] Curran, P., J. Suykens, L. Chua. Absolute stability theory and master-slave synchronization. International Journal Bifurcation and Chaos, Vol.7(12), 1997, pp. 2891-2896.
[11] Suykens, J., A. Vanderwalle. Master-Slave synchronization of Lur'e systems. International Journal Bifurcation and Chaos, Vol.7(3), 1997, pp. 665-669.
[12] Yu, H., L. Yanzhu. Chaotic synchronization based on stability criterion of linear systems. Physics Letters A, Vol. 314, Issue 4, 2003, pp.292-298.
[13] Shimizu, T., N. Morioka, On the bifurcation of symmetric limit cycle to an asymmetric one in a simple model, Physics Letters 76A, 1980, pp.201-204.
[14] Guemez, J., C. Martin. On the behaviour of coupled chaotic systems exhibiting marginal synchronization. Physics Letters A Vol.226, 1997, pp.264-268.