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Screen Semi-invariant Lightlike Submanifolds of a Golden Semi-Riemannian Manifold

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Abstract
The geometry of golden semi-Riemannian manifold screen semi-invariant lightlike submanifolds is investigated in this study. The intergrability conditions of distributions $S(TN)$ and $RadTN$ are discovered on screen semi-invariant lightlike submanifolds of a golden semi-Riemannian manifold. The necessary and sufficient conditions for the aforementioned distributions to be entirely geodesic foliations are also derived.

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1. INTRODUCTION

Lightlike submanifolds are one of the most exciting topics in differential geometry. In the semi-Riemannian case, the tools used to study the geometry of submanifolds in the Riemannian case are not as useful, hence the classical theory cannot be utilised to define an induced object on a lightlike submanifold. The fact that the intersection of a lightlike submanifold's normal and tangent bundles is not zero poses the biggest problem. The geometry of lightlike submanifolds of semi-Riemannian manifolds was studied by Duggal and Bejancu [9].

The properties of Golden structure (i.e., a polynomial structure with the structure polynomial $Q(x) = x^2 - x - I)$) are studied in Golden differential geometry. Spinadel [8] developed the Metallic means family in 2002. Equation has a positive solution.

$$x^2 - px - q = 0,$$

for some positive integers p and q, is called a (p, q)-metallic number, which is the form

$$\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}.$$

For $p = q = 1$, we have the Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

The Golden proportion was also introduced by Crasmareanu and Hretcanu [4]. In the structure of musical compositions, harmonic sound frequency ratios, and human body dimensions, the Golden proportion can be found. According to [4, 5], the Golden ratio and the Golden rectangle (a rectangle with two sides in the Golden ratio) have been discovered in the harmonious proportions of temples, cathedrals, sculptures, paintings, and pictures.

In recent years, the Golden proportion has become increasingly important in modern physical study, and it is especially important in atomic physics [14]. In special relativity, the Golden rectangle has been used to drive time dilation and Lorentz contraction of lengths, and the Golden ratio has been used to drive the shift from Newton's physics to relativistic mechanics. The Golden ratio has intriguing aspects in topology of four manifolds, conformal field theory, mathematical probability theory, and Cantorian spacetime [7], as well as El Naschie's Golden field theory [6].

Crasmareanu and Hretcanu [3] investigated the Golden Riemannian manifold's invariant submanifolds. Gezer et al. investigated the intergrability of such Golden structures in [13]. Semi-invariant and totally umbilical semi-invariant submanifolds of Golden Riemannian manifolds have been studied by Erdogan and Yildirim [12]. A semi-Riemannian manifold having a Golden structure is known as a Golden semi-Riemannian manifold. Remarks on Metallic maps between Metallic Riemannian manifolds and Constancy of certain maps was investigated by Akyol in [2]. Şahin and Akyol [15] investigated Golden maps between Golden Riemannian manifolds and Constancy of certain maps. The following is the format of the paper:

This discovery generates lightlike submanifolds of metallic semi-Riemannian manifolds [1]. Section 2, which is necessary for this work, contains basic information on lightlike geometry. In section 3, the concept of screen semi-invariant lightlike submanifolds is presented. Intergrability criteria are discovered for the distributions $S(TN)$ and $RadTN$. In section 4, we find the necessary and sufficient criteria for the aforementioned distributions to have entirely geodesic foliation.

2. PRELIMINARIES

A lightlike submanifold [9] is a submanifold N^m immersed in a semi-Riemannian manifold $(\overline{N}^{m+n}, \overline{g})$ that admits a degenerate metric g generated from $\overline{g}$ on N. When g is degenerate on N's tangent bundle TN, N is referred to as a lightlike submanifold.

$TN^{\perp}$ is a degenerate n-dimensional subspace of $T_x\overline{N}$ for a degenerate metric g on N. As a result, both T_xN and $T_xN^{\perp}$ are degenerate orthogonal subspaces, but they are not complimentary. As a result, a subspace known as Radical subspace exists: $Rad(TN) = T_xN \cap T_xN^{\perp}$. If the mapping $Rad(TN): N \longrightarrow TN,$ defines a smooth distribution with rank $r > 0$ on N, then N is a r-lightlike submanifold and the distribution $Rad(TN)$ is a radical distribution on N. Screen distribution in TN and screen transversal distribution in $TN^{\perp}$ are the non-degenerate complementary subbundles of $Rad(TN)$, respectively.

$$TN = Rad(TN) \perp S(TN) \;\; \& \;\; TN^{\perp} = Rad(TN) \perp S(TN^{\perp}). \tag{1}$$

Let $ltr(TN)$(lightlike transversal bundle) and $tr(TN)$(transversal bundle) be complementary but not orthogonal vector bundles to $Rad(TN)$ in $S(TN^{\perp})^{\perp}$ and TN in $T\overline{N}|_N$.

After that, [10] provides the transversal vector bundle $tr(TN)$.

$$tr(TN) = ltr(TN) \perp S(TN^{\perp}). \tag{2}$$

From (1) and (2), we get

$$T\overline{N}|_N = TN \oplus tr(TN) = (Rad(TN) \oplus ltr(TN)) \perp S(TN) \perp S(TN^{\perp}). \quad (3)$$

Theorem 2.1 *[9] An r-lightlike submanifold of a semi-Riemannian manifold (N', g') is defined as $(N, g, S(TN), S(TN^{\perp}))$. Then there is a complementary vector bundle $ltr(TN)$ of $Rad(TN)$ in $S(TN^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TN)|u)$consisting of a smooth section $\{N_i\}$ of $S(TN^{\perp})^{\perp}|_u$, where u is a coordinate neighbourhood of N such that*

$$\bar{g}_{ij}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}_{ij}(N_i, N_j) = 0, \quad (4)$$

for any i, j $\in \{1,2,\dots,r\}$.

The linear connections on N', N, and vector bundle $tr(TN)$ are denoted by $\overline{\nabla}, \nabla$ and ∇^t, respectively. The Gauss and Weingarten formulae are then provided.

$$\overline{\nabla}_W U = \nabla_W U + h(W, U), \forall\, W, U \in \Gamma(TN), \quad (5)$$

$$\overline{\nabla}_W N = -A_N W + \nabla^t_W N, \forall\, W \in \Gamma(tr(TN)), N \in \Gamma(tr(TN)) \quad (6)$$

where $\{\nabla_W U, A_N U\}$ and $\{h(W, U), \nabla^t_W N\}$ belong to $Gamma(TN)$ and $Gamma(tr(TN))$, the linear connections ∇ and ∇^t are on N and the vector bundle $tr(TN)$, and the second fundamental form h is a symmetric $F(N)$-bilinear form (TN).

From (5) and (6) , for any $W, U \in \Gamma(tr(TN))$, $N \in \Gamma(ltr(TN))$ and $W \in \Gamma(S(TN^{\perp}))$, we have

$$\overline{\nabla}_W U = \nabla_W U + h^l(W, U) + h^s(W, U), \quad (7)$$

$$\overline{\nabla}_W N = -A_N W + \nabla^l_W(N) + D^s(W, N), \quad (8)$$

$$\overline{\nabla}_W V = -A_V W + \nabla^s_W(V) + D^l(W, V), \quad (9)$$

where $D^l(W, V), D^s(W, N)$ denote the projections of ∇^t on $\Gamma(ltr(TN))$ and $\Gamma(S(TN^{\perp}))$, respectively, and ∇^l, ∇^s denote linear connections on $\Gamma(ltr(TN))$ and $\Gamma(S(TN^{\perp}))$, respectively. We acquire using (5) and (7) -(9),

$$\overline{g}(h^s(W, U), V) + \overline{g}(U, D^l(W, V)) = g(A_V W, U), \quad (10)$$

$$\overline{g}(D^s(W, N), V) = g(N, A_V W). \quad (11)$$

for $W, U \in \Gamma(TN)$, $\xi \in \Gamma(Rad(TN))$, $V \in \Gamma(S(TN^{\perp})$ and $N \in \Gamma(ltr(TN))$.

The induced connection ∇ and the transversal connection ∇^t_X, in particular, are not metric connections. The following formulae indicate induced and transversal connections, respectively, for $W, U, Z \in \Gamma(TN)$ and $X', U' \in \Gamma(tr(TM))$.

$$(\nabla_W g)(U, Z) = \bar{g}(h^l(W, U), Z) + \bar{g}(h^l(W, Z), U) \quad (12)$$

$$(\nabla^t_W \bar{g})(X', U') = -\{\bar{g}(A_{X'} W, U') + \bar{g}(A_{U'} W, X')\}. \quad (13)$$

Let $\overline{P}$ denote the projection of TN on $S(TN)$ and let ∇^*, ∇^{*t} denote the linear connections on $S(TN)$ and $Rad(TN)$, respectively. Then from the decomposition of tangent bundle of lightlike submanifold, we have

$$\nabla_W \overline{P}U = \nabla^*_W \overline{P}U + h^*(W, \overline{P}U), \tag{14}$$

$$\nabla_W \xi = -A^*_\xi W + \nabla^{*t}_W(\xi), \tag{15}$$

for $W, U \in \Gamma(TN)$ and $\xi \in \Gamma(RadTN)$, where h^*, A^* are the second fundamental form and shape operator of distributions $S(TN)$ and $Rad(TN)$, respectively.

From (11) and (12), we get

$$\bar{g}(h^l(W, \overline{P}U), \xi) = g(A^*_\xi W, \overline{P}U), \tag{16}$$

$$\bar{g}(h^*(W, \overline{P}U), N) = g(A_N W, \overline{P}U), \tag{17}$$

$$\bar{g}(h^*(W, \xi), \xi) = 0, A^*_\xi \xi = 0. \tag{18}$$

If (N', g') is a semi-Riemannian manifold, N' is referred to as a golden semi-Riemannian manifold if $(1,1)$ tensor field P' exists on N'.

$$P'^2 = P' + I, \tag{19}$$

where I is the identity map on N'. Also

$$g'(P'W, U) = g'(W, PU). \tag{20}$$

The semi-Riemannian metric is known as P'-compatible, and the golden semi-Riemannian manifold is known as (N', g', P'). We also have

$$\nabla'_W P'U = P' \nabla'_W U. \tag{21}$$

If P' is a golden structure, then (21) is equivalent to

$$g'(P'W, P'U) = g'(P'W, U) + g'(W, U), \tag{22}$$

for any $W, U \in \Gamma(TN')$.

Throughout this paper, we use

$$\overline{\nabla} P' = 0. \tag{23}$$

3. SCREEN SEMI-INVARIANT LIGHTLIKE SUBMANIFOLDS

This section introduces the concept of semi-invariant lightlike submanifolds of Golden semi-Riemannian manifolds.

Definition 3.1 *Let (N', g', P') be a golden semi-Riemannian manifold and (N, g) be a lightlike N' submanifold. Then, if the following requirements are satisfied, we can declare that N is a screen semi-invariant lightlike submanifold of N':*

$$P'(Rad(TN)) \subseteq S(TN), \tag{24}$$

$$P'(ltr(TN)) \subseteq S(TN). \tag{25}$$

We can define a non-degenerate distribution M_0 for a screen semi-invariant lightlike submanifold of a Golden semi-Riemannian manifold based on the preceding definition, such that $S(TN)$ is decomposed as:

$$S(TN) = M_0 \perp M_1 \oplus M_2, \tag{26}$$
where $M_1 = P'(Rad(TN))$ and $M_2 = P'(ltr(TN))$.

Example 3.1 *Let* $N' = \mathbb{R}^5_2$ *be a golden semi-Riemannian manifold of signature* $(-,+,-,+,+)$ *and golden structure* P' *is defined as*

$$P'(y_1, y_2, y_3, y_4, y_5) = ((1-\sigma)y_1, \sigma y_2, \sigma y_3, \sigma y_4, (1-\sigma)y_1).$$

Let N be a submanifold of $(\mathbb{R}^5_2, P', g')$ is given by

$$y_5 = y_1 + \sigma y_2 + \sigma y_3$$

Then we get

$$U_1 = \partial y_1 + \partial y_5$$
$$U_2 = \partial y_2 + \sigma \partial y_5$$
$$U_3 = \partial y_3 + \sigma \partial y_5$$
$$U_4 = \partial y_4.$$

If we have $\xi = \sigma U_1 + U_2 - U_3 = \sigma \partial y_1 + \partial y_2 - \partial y_3 - \sigma \partial y_5$ and

$$V_1 = U_4, V_2 = U_1 + U_2 - U_3.$$

Now we have $S(TN) = spV_1, V_2$ and $Rad(TN) = sp\xi$, by direct calculation, we get

$$N = 1/2\{(\sigma \partial y_1 - \partial y_2 + \partial y_3 + \sigma \partial y_5)\}.$$

Thus N is a screen semi-invariant lightlike submanifold of N'.

Proposition 3.2 *Let* (N, g) *be a golden semi-Riemannian manifold* (N', g', P') *with a screen semi-invariant lightlike submanifold* (N', g', P')*. Then, with regard to* P', $P'M_0$ *is invariant.*

Proof. Let W be any vector field of $\Gamma(M_0)$.
We are driven by (20) and (26),

$$g'(P'W, \xi) = 0, \; g'(P'W, N) = 0, \tag{27}$$
for $\xi \in \Gamma(Rad(TN))$ and $N \in \Gamma(ltr(TN))$. That is, $P'W \notin \Gamma(ltr(TN)) \cup Rad(TN)$.

Similarly, using (19) and (21), we have

$$g'(P'W, P'\xi) = 0, \; g'(P'U, P'N) = 0, \tag{28}$$

That is, $P'U \notin \Gamma(P'(ltr(TN)) \cup P'Rad(TN))$. This completes the proof.

Thus, TN can be written as:

$$TN = M_1 \oplus M_2 \perp M_0 \perp Rad(TN), \tag{29}$$
If we use M to represent the invariant distribution of TN, we get

$$M = M_0 \perp Rad(TN) \perp P'(Rad(TN)), \tag{30}$$
(29) is reduced to

$$TN = M \oplus M_2. \tag{31}$$

The projection morphisms on M and M_2 are denoted by B and R, respectively. Then we may write $W \in \Gamma(TN)$ as:

$$W = BW + RW, \tag{32}$$
where $BW \in \Gamma(M)$ and $RW \in \Gamma(M_2)$.

Applying P' on (32), we get

$$P'W = P'BW + P'RW, \quad (33)$$

we denote $P'BW$ and $P'RW$ by S_1W and R_1W respectively, then we can rewrite (33) as

$$P'W = S_1W + R_1W, \quad (34)$$

where $S_1W \in \Gamma(M)$ and $R_1W \in \Gamma(ltr(TN))$.

Let N be a golden semi-Riemannian manifold N' with a screen semi-invariant lightlike submanifold.

By using $(7), (7)(14)$ and (23) $\forall\ W, U \in \Gamma(TN)$, we obtain

$$P' \nabla_W U + P'h'(W, U) + P'h^s(W, U) = \nabla^*_W P'U + h^*(W, P'U) + h^l(W, P'U) + h^s(W, P'U). \quad (35)$$

We can deduce the following from the tangential, lightlike transversal, and screen transversal components of (35):

$$P' \nabla_W U = \nabla^*_W P'U + h^*(W, P'U) - P'h^l(W, U), \quad (36)$$

$$h^l(W, P'U) = 0, \quad (37)$$

$$h^s(W, P'U) = P'h^s(W, U), \quad (38)$$

where $\forall\, W, U \in \Gamma(TN)$.

Proposition 3.3 *[1] Let (N, g) be a golden semi-Riemannian manifold (N', g', P') with a screen semi-invariant lightlike submanifold (N', g', P'). Then, with regard to P', $S(TN^\perp)$ is invariant. As a result of (3) and (31), we get the following decomposition:*

$$TN' = M_1 \oplus M_2 \perp M_0 \perp (Rad(TN) \oplus ltr(TN)) \perp S(TN^\perp). \quad (39)$$

Theorem 3.4 *Let N be a golden semi-Riemannian manifold N' with a screen semi-invariant lightlike submanifold. The invariant distribution M is integrable for any $W, U \in \Gamma(M)$, if and only if $h^l(P'U, P'W) = h^l(W, P'U) + h^l(W, U)$.*

Proof. We know that L is integrable if and only if $[W, U] \in \Gamma(M), \forall\ W, U \in \Gamma(M)$. That is, $g'([P'U, W], P'\xi) = 0,\ \forall\ \ W, U \in \Gamma(M), \xi \in \Gamma(Rad(TN))$.

By using (20), (22), we get

$$g'(\overline{\nabla}_{P'U} P'W, \xi) - g'(\overline{\nabla}_W U, P'\xi) - g'(\overline{\nabla}_W U, \xi) = 0,$$

finally, by using (7), we get

$$g'(h^l(P'U, P'W), \xi) - g'(h^l(W, P'U), \xi) - g'(h^l(W, U), \xi) = 0,$$

$$h^l(P'U, P'W) - h^l(W, P'U) - h^l(W, U) = 0,$$

$$h^l(P'U, P'W) = h^l(W, P'U) + h^l(W, U) = 0.$$

Thus, proof is completed.

Theorem 3.5 *Let N be a golden semi-Riemannian manifold N' with a screen semi-invariant lightlike submanifold. The radical distribution $Rad(TN)$ is integrable for an $W, U \in \Gamma(Rad(TN))$, if and only if*

$$\nabla^*_W P'U - \nabla^*_U P'W = (A^*_W - A^*_U), \tag{40}$$

or

$$P' \nabla^*_W P'U - P' \nabla^*_U P'W = (\nabla^*_W P'U - \nabla^*_U P'W). \tag{41}$$

Proof. We assume that $Rad(TN)$ is integrable. Then, $g'([W,U],Z) = 0,\ \forall\ W,U \in \Gamma(Rad(TN)),\ Z \in \Gamma(S(TN))$.

Using (22), we get

$$0 = g'(\overline{\nabla}_W P'U, P'Z) - g'(\overline{\nabla}_W P'U, P'Z) - g'(\overline{\nabla}_U P'W, P'Z) + (\overline{\nabla}_U P'W, P'Z). \tag{42}$$

Using (7) and (14), we obtain

$$g'(\nabla^*_W P'U + A^*_U W - \nabla^*_U P'W - A^*_W U, P'Z) = 0,$$

which satisfies (40).

On the other hand, if we use (7),(14) and (20) in (42), we get

$$g'(\nabla^*_W P'U - \nabla^*_U P'W, P'Z) + g'(\nabla^*_U P'W + h^*(U,P'W) - P'h^l(U,W) - \nabla^*_W P'U + h^*(W,P'U) - P'h^l(W,U), Z) = 0.$$

Since h^l is symmetric, using (36), we have

$$g'(P' \nabla^*_W P'U - P' \nabla^*_U P'W + \nabla^*_U P'W - \nabla^*_W P'U, Z) = 0,$$

which satisfies (41) and proof is completed.

Theorem 3.6 *Let N be a screen semi-invariant lightlike submanifold of a golden semi-Riemannian manifold N'. Then, for any $W, U \in \Gamma(S(TN))$, the screen distribution $S(TN)$ is integrable if and only if*

$$\nabla^*_W P'U - \nabla^*_U P'W = (\nabla^*_W U - \nabla^*_U W), \tag{43}$$

or

$$\nabla^*_W P'U = \nabla^*_U P'W. \tag{44}$$

Proof. We know that $S(TN)$ is integrable. Then, $g'([W,U],N) = 0,\ \forall W,U \in \Gamma(S(TN)),\ N \in \Gamma(ltr(TN))$.

Using (22) and (23), we get

$$g'(\overline{\nabla}_W P'U - \overline{\nabla}_U P'W, P'N) - g'(\overline{\nabla}_W U - \overline{\nabla}_U W, P'N) = 0, \tag{45}$$

by using (7) and (14) in (45), we have

$$g'(\nabla^*_W P'U - \nabla^*_U P'W - \nabla^*_W U + \nabla^*_U W, P'N) = 0,$$

$$\nabla^*_W P'U - \nabla^*_U P'W - \nabla^*_W U + \nabla^*_U W = 0,$$

$$\nabla^*_W P'U - \nabla^*_U P'W = \nabla^*_W U - \nabla^*_U W,$$

is obtained and (43) is satisfies.

On other hand, using (7), (14),(20) in (45), we get

$$g'(\nabla^*_W P'U - \nabla^*_U P'W, P'N) = 0,$$

$$\nabla^*_W P'U - \nabla^*_U P'W = 0,$$

$$\nabla^*_W P'U = \nabla^*_U P'W,$$

which satifies (44) and proof is completed.

Theorem 3.7 *Let N be a screen semi-invariant lightlike submanifold of a golden semi-Riemannian manifold N'. Then, for any $W \in \Gamma(TN)$ and $\xi \in \Gamma(Rad(TN))$, induced connection ∇ on N is a metric connection if and only if one of the followings is satisfied:*

$$\nabla^*_W P'\xi = -A^*_\xi W \tag{46}$$

or

$$A^*_\xi W = 0 \tag{47}$$

Proof. Since ∇ is a metric connection if and only if $\nabla_W \xi \in \Gamma(Rad(TN))$, that is, $g(\nabla_W \xi, Z) = 0$, $\forall\, W \in \Gamma(TN), \xi \in \Gamma(Rad(TM))$ and $Z \in \Gamma(S(TN))$.

Using (7), we have

$$g'(\overline{\nabla}_W \xi, Z) = 0. \tag{48}$$

Using (7), (14), (22) and (23), we obtain

$$g'(\nabla^*_W P'\xi, P'Z) + g'(A^*_\xi W, P'Z) = 0,$$

$$g'(\nabla^*_W P'\xi + A^*_\xi W, P'Z) = 0,$$

$$\nabla^*_W P'\xi + A^*_\xi W = 0,$$

$$\nabla^*_W P'\xi = -A^*_\xi W.$$

Using (22), in (48), we get

$$g'(P'\,\overline{\nabla}_W P'\xi - P'\,\overline{\nabla}_W \xi, Z) = 0,$$

from (7), (14) and (15), we get

$$g'(P'\,\nabla^*_W P'\xi + P'h^*(W, P'\xi) + P'h^l(W, P'\xi) + P'h^s(W, P'\xi) + P'A^*_\xi W - P'\,\nabla^*_W \xi - P'h^l(W, \xi) - P'h^s(W, \xi), Z). \tag{49}$$

Finally, using (22), (37) and (38) in (49), we get

$$g'(-A^*_\xi W + \nabla^{*t}_U \xi, Z) = 0,$$

$$-A^*_{\xi W} = 0,$$

$$A^*_{\xi W} = 0.$$

This completes the proof.

4. DISTRIBUTIONS DETERMINE FOLIATION

We show in this section that foliations formed by distributions on a golden semi-Riemannian manifold's screen semi-invariant lightlike submanifold are fully geodesic.

Theorem 4.1 *Let N be a golden semi-Riemannian manifold N' with a screen semi-invariant lightlike submanifold. Then, for each $\xi \in \Gamma(RadTN)$ and $U \in \Gamma(S(TN))$, $RadTN$ defines a completely geodesic foliation on N if and only if $\nabla^*_\xi P'W = \nabla^*_\xi W$.*

Proof. We suppose that $Rad(TN)$ denotes a completely geodesic foliation on N, i.e., for $\xi_1 \in \Gamma(Rad(TN))$, $\nabla_\xi \xi_1 \in \Gamma(Rad(TN))$. Because $\overline{\nabla}$ is a metric connection, we may write

$$g(\nabla_\xi \xi_1, W) = g'(\overline{\nabla}_\xi \xi_1, W) = g'(\xi_1, \overline{\nabla}_\xi W) = 0,$$

we have, by using (7), (14) and (22),.

$$g'(\nabla^*_\xi P'W, P'\xi) - g'(\nabla^*_\xi W, P'\xi) = 0,$$

$$g'(\nabla^*_\xi P'W - \nabla^*_\xi W, P'\xi) = 0,$$

$$\nabla^*_\xi P'W - \nabla^*_\xi W = 0,$$

$$\nabla^*_\xi P'W = \nabla^*_\xi W.$$

Hence proved.

Theorem 4.2 *Let N be a golden semi-Riemannian manifold N' with a screen semi-invariant lightlike submanifold. Then, for each $W, U \in \Gamma(S(TN))$, $S(TN)$ defines a completely geodesic foliation on N if and only if $\nabla^*_W P'U = \nabla^*_W U$.*

Proof. We assume that $S(TN)$ defines a totally geodesic foliation on N, that is, for any $W, U \in \Gamma(S(TN))$, $\nabla_W U \in \Gamma(S(TN))$.

Since $\overline{\nabla}$ is a metric connection, we can write

$$g'(\overline{\nabla}_W U, N) = g'(\overline{\nabla}_W U, N) = 0, \ \forall N \in \Gamma(ltr(TN)),$$

by using (7), (14) and (22), we have

$$g'(\nabla^*_W P'U, P'N) - g'(\nabla^*_W U, P'N) = 0,$$

$$g'(\nabla^*_W P'U - \nabla^*_W U, P'N) = 0,$$

$$\nabla^*_W P'U - \nabla^*_W U = 0,$$

$$\nabla^*_W P'U = \nabla^*_W U.$$

As a result, it has been established.

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