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Araştırma Makalesi–Research Article

Gaussian Bronze Lucas Numbers

Gauss Bronz Lucas Sayıları

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ABSTRACT

The present work aims to introduce and study the Gaussian Bronze Lucas number sequence. Firstly, we define Gaussian Bronze Lucas numbers by extending the Bronze Lucas numbers. Then, we find the Binet formula and generating function for this number sequence. We also investigate some sum formulas and matrices related to the Gaussian Bronze Lucas numbers. Finally, we obtain some known equalities like Catalan, Cassini and d'Ocagne identities by considering the Binet formula of this sequence.

Keywords- Bronze Lucas Numbers, Gaussian Bronze Lucas Numbers, Generating Function, Binet Formula

ÖZ

Bu çalışmanın amacı Gauss Bronz Lucas sayı dizisini tanıtmak ve incelemektir. İlk olarak Bronz Lucas sayılarını genişleterek Gauss Bronz Lucas sayılarını tanımladık. Daha sonra bu sayı dizisi için Binet formülü ve üreteç fonksiyonunu bulduk. Ayrıca Gauss Bronz Lucas sayıları ile ilgili bazı toplam formülleri ve matrisleri araştırdık. Son olarak, bu dizinin Binet formülünü dikkate alarak Catalan, Cassini ve d'Ocagne özdeşlikleri gibi bilinen eşitlikleri elde ettik.

Anahtar Kelimeler- Bronz Lucas Sayıları, Gauss Bronz Lucas Sayıları, Üreteç Fonksiyonu, Binet Formülü

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I. INTRODUCTION

Recently, we have seen so many studies on the different number sequences. The most famous examples for number sequences are the Fibonacci, Lucas, Pell, Jacobsthal sequences etc, see [1-5] for details. In [1], the Fibonacci and Lucas number sequences are given recurrently by

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \tag{1}$$

$$L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2} \tag{2}$$

respectively.

The Gaussian shapes of these number sequences have been a center of attention lately. Horadam defined the complex Fibonacci numbers in 1963. Since then, many authors investigated the Gaussian shapes of these sequences and their properties; see for example [6-16]. In [8], the Gaussian forms of the Fibonacci and Lucas number sequences are given recurrently by

$$GF_0 = i, GF_1 = 1, GF_n = GF_{n-1} + GF_{n-2} \tag{3}$$

$$GL_0 = 2 - i, GL_1 = 1 + 2i, GL_n = GL_{n-1} + GL_{n-2} \tag{4}$$

respectively.

On the other hand, the Bronze Fibonacci (sequence A006190 in [17]) and Bronze Lucas (sequence A006497 in [17]) number sequences are given by the following recursion formulas

$$B_0 = 0, B_1 = 1, B_n = 3B_{n-1} + B_{n-2}, n \geq 2 \tag{5}$$

$$BL_0 = 2, BL_1 = 3, BL_n = 3BL_{n-1} + BL_{n-2}, n \geq 2 \tag{6}$$

respectively.

Furthermore, Akbıyık and Alo [18] studied on the third-order Bronze Fibonacci numbers.

Also, Kartal [19] introduced the Gaussian Bronze Fibonacci sequence as follows:

$$GB_0 = i, GB_1 = 1; GB_n = 3GB_{n-1} + GB_{n-2}. \tag{7}$$

Additionally, the generating function of the Gaussian Bronze Fibonacci sequence is

$$g(x) = \frac{x + i(1 - 3x)}{1 - 3x - x^2}. \tag{8}$$

Also, the Binet's formula of this sequence is

$$GB_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta} \tag{9}$$

where α, β are the roots of the characteristic equation.

Moreover, many properties of this sequence are investigated in the same study.

In this study, we extend the Bronze Lucas sequence to the Gaussian Bronze Lucas number sequence. Then, we derive the Binet formula and generating function for this sequence. We also find sums and various identities for the Gaussian Bronze Lucas sequence.

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II. MAIN RESULTS

In this section, firstly, we define the recurrence relation of the Gaussian Bronze Lucas sequence. Then we find the Binet formula, generating function, some sum formulas and identities of the Gaussian Bronze Lucas sequence.

Definition 1. The Gaussian Bronze Lucas sequence $\{GBL_n\}_{n=0}^{\infty}$ is given recursively by

$$GBL_n = 3GBL_{n-1} + GBL_{n-2} \tag{10}$$

with initial values $GBL_0 = 2 - 3i$ and $GBL_1 = 3 + 2i$ and $n \geq 2$.

Also, note that for $n \geq 0$,

$$GBL_n = BL_n + iBL_{n-1}. \tag{11}$$

The first few values of GBL_n are: $2 - 3i, 3 + 2i, 11 + 3i, 36 + 11i, 119 + 36i, 393 + 119i$ and so on.

Theorem 1. The sequence $\{GBL_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GBL_{-n} = -3GBL_{-(n-1)} + GBL_{-(n-2)} \tag{12}$$

for $n \geq 1$.

Proof. From the recurrence relation of Gaussian Bronze Lucas sequence, we have

$$GBL_{n-2} = GBL_n - 3GBL_{n-1}. \tag{13}$$

Then, for $n = -n + 2$, we obtain

$$GBL_{-n} = GBL_{-n+2} - GBL_{-n+1} \tag{14}$$

$$= GBL_{-(n-2)} - 3GBL_{-(n-1)} \tag{15}$$

$$= -3GBL_{-(n-1)} + GBL_{-(n-2)} \tag{16}$$

as required.

The next theorem gives the generating function for the Gaussian Bronze Lucas sequence.

Theorem 2. The generating function of the Gaussian Bronze Lucas number sequence $\{\mathrm{GBL_n}\}_{\mathrm{n=0}}^{\infty}$ is given as

$$f(x) = \frac{(2 - 3x) + (11x - 3)i}{1 - 3x - x^2}. \tag{17}$$

Proof. Suppose that $f(x)$ is the generating function of $\{GBL_n\}_{n=0}^{\infty}$. Hence, $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} GBL_n x^n = GBL_0 + GBL_1 x + GBL_2 x^2 + GBL_3 x^3 + GBL_4 x^4 + \cdots. \tag{18}$$

Now, multiplying both sides of this equality with the term $3x$ and x^2, respectively, we write

$$3xf(x) = 3GBL_0 x + 3GBL_1 x^2 + 3GBL_2 x^3 + 3GBL_3 x^4 + \cdots \tag{19}$$

and

$$x^2 f(x) = GBL_0 x^2 + GBL_1 x^3 + GBL_2 x^4 + \cdots \tag{20}$$

Thus, we obtain

$$f(x)(1 - 3x - x^2) = 2 - 3i + 3x + 2xi - 6x + 9xi. \tag{21}$$

Hence, we have

$$f(x) = \frac{(2 - 3x) + (11x - 3)i}{1 - 3x - x^2} \tag{22}$$

which is desired.

Now, the Binet formula of $\{GBL_n\}_{n=0}^{\infty}$ will be expressed and proved.

Theorem 3. Let $\alpha = \frac{3+\sqrt{13}}{2}$ and $\beta = \frac{3-\sqrt{13}}{2}$ be the roots of the equation $x^2 - 3x - 1 = 0$. The Binet formula of the Gaussian Bronze Lucas sequence is

$$GBL_n = (\alpha^n + \beta^n) + (\alpha^{n-1} + \beta^{n-1})i\,. \tag{23}$$

Proof. The general term of the Gaussian Bronze Lucas sequence can be written in the following form:

$$GBL_n = c_0 \alpha^n + c_1 \beta^n \tag{24}$$

where c_0 and c_1 are coefficients.

Then, using the initial values imply that

$$c_0 + c_1 = 2 - 3i \tag{25}$$

$$c_0 \left(\frac{3 + \sqrt{13}}{2}\right) + c_1 \left(\frac{3 - \sqrt{13}}{2}\right) = 3 + 2i. \tag{26}$$

Solving the above system, we have

$$c_0 = \frac{(3 - 2\beta) + (2 + 3\beta)i}{\alpha - \beta} \tag{27}$$

$$c_1 = \frac{(2\alpha - 3) - (3\alpha + 2)i}{\alpha - \beta} \tag{28}$$

Hence, the Binet formula for this sequence is obtained

$$GBL_n = (\alpha^n + \beta^n) + (\alpha^{n-1} + \beta^{n-1})i \tag{29}$$

as required.

Theorem 4. For the Gaussian Bronze Lucas sequence, we have

$$\sum_{k=1}^{n} GBL_k = \frac{1}{3}(GBL_{n+1} + GBL_n - 5 + i), \quad n \geq 1. \tag{30}$$

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Proof. From the recurrence relation of the Gaussian Bronze Lucas sequence, we have

$$GBL_{n-1} = \frac{1}{3}(GBL_n - GBL_{n-2}). \tag{31}$$

From this equation, we get

$$GBL_1 = \frac{1}{3}(GBL_2 - GBL_0) \tag{32}$$

$$GBL_2 = \frac{1}{3}(GBL_3 - GBL_1) \tag{33}$$

$$GBL_3 = \frac{1}{3}(GBL_4 - GBL_2) \tag{34}$$

$$\vdots$$

$$GBL_n = \frac{1}{3}(GBL_{n+1} - GBL_{n-1}) \tag{35}$$

After performing necessary calculations we get

$$\sum_{k=1}^{n} GBL_k = \frac{1}{3}(GBL_{n+1} + GBL_n - GBL_1 - GBL_0) \tag{36}$$

$$= \frac{1}{3}(GBL_{n+1} + GBL_n - 5 + i) \tag{37}$$

which is desired.

The undermentioned result follows from the supra theorem.

Theorem 5. The following sum formulas hold for n $\in \mathbb{Z}^+$.

$$(i) \sum_{k=1}^{n} GBL_{2k} = \frac{1}{3}(GBL_{2n+1} - 3 - 2i) \tag{38}$$

$$(ii) \sum_{k=1}^{n} GBL_{2k-1} = \frac{1}{3}(GBL_{2n} - 2 + 3i) \tag{39}$$

Presently, we introduce the matrices $\boldsymbol{Q}$ and $\boldsymbol{B}$. Let $\boldsymbol{Q}$ and $\boldsymbol{B}$ denote the $2x2$ matrices defined as $\boldsymbol{Q} = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 11+3i & 3+2i \\ 3+2i & 2-3i \end{pmatrix}$.

Theorem 6. For n ≥ 1, we have

$$\boldsymbol{Q}^n \boldsymbol{B} = \begin{pmatrix} GBL_{n+2} & GBL_{n+1} \\ GBL_{n+1} & GBL_n \end{pmatrix} \tag{40}$$

Proof. We can prove the theorem by induction method on n. For $n = 0$, we get

$$B = \begin{pmatrix} GBL_2 & GBL_1 \\ GBL_1 & GBL_0 \end{pmatrix} = \begin{pmatrix} 11+3i & 3+2i \\ 3+2i & 2-3i \end{pmatrix} \tag{41}$$

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Assume that the equality holds for $n = k$, namely

$$Q^k B = \begin{pmatrix} GBL_{k+2} & GBL_{k+1} \\ GBL_{k+1} & GBL_k \end{pmatrix} \tag{42}$$

Now, we need to demonstrate it is true for $n = k + 1$. Hence, we get

$$Q^{k+1} B = Q(Q^k B) \tag{43}$$

$$= \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} GBL_{k+2} & GBL_{k+1} \\ GBL_{k+1} & GBL_k \end{pmatrix} \tag{44}$$

$$= \begin{pmatrix} 3GBL_{k+2} + GBL_{k+1} & 3GBL_{k+1} + GBL_k \\ GBL_{k+2} & GBL_{k+1} \end{pmatrix} \tag{45}$$

$$= \begin{pmatrix} GBL_{k+3} & GBL_{k+2} \\ GBL_{k+2} & GBL_{k+1} \end{pmatrix} \tag{46}$$

So, we obtain the desired result.

Theorem 7. (Cassini Identity)**.** For $n \in \mathbb{Z}^+$, the Gaussian Bronze Lucas sequence, we have the following equation

$$GBL_{n-1} GBL_{n+1} - GBL_n{}^2 = 13(-1)^{n-1}(2 - 3i). \tag{47}$$

Proof. It is clear that $\det \mathbf{Q^{n-1}} = (-1)^{n-1}$ and $\det \mathbf{B} = 13(2 - 3i)$. If the determinant of both sides of the equation below is taken

$$\boldsymbol{Q^{n-1} B} = \begin{pmatrix} GBL_{n+1} & GBL_n \\ GBL_n & GBL_{n-1} \end{pmatrix} \tag{48}$$

we get

$$GBL_{n-1} GBL_{n+1} - GBL_n{}^2 = 13(-1)^{n-1}(2 - 3i) \tag{49}$$

which is completes the proof.

Theorem 8. The Catalan and d'Ocagne identities for Gaussian Bronze Lucas sequence

$$(i) GBL_{n-k} GBL_{n+k} - GBL_n{}^2 = (-1)^n (2 - 3i)[(-1)^k (\alpha^k + \beta^k)^2 - 4] \quad n, k \in \mathbb{Z}^+ \tag{50}$$

$$(ii) GBL_m GBL_{n+1} - GBL_n GBL_{m+1} = \sqrt{13}(2 - 3i)(-1)^{n+1}(\alpha^{m-n} - \beta^{m-n}) \quad m, n \in \mathbb{Z}^+ \tag{51}$$

respectively.

Proof. By using the Binet formula, the proof can be easily seen.

In addition, if $k = 1$istaken in Theorem $8(i)$, Cassini formula of the Gaussian Bronze Lucas sequence is obtained again.

III. CONCLUSION

Firstly, we have defined the Gaussian Bronze Lucas number sequence. Then, we were examined some properties like Binet formula, generating function, several summation formula and matrix representation. Additionally, by using the Binet formula, various identities like Catalan, Cassini and d'Ocagne related to the Gaussian Bronze Lucas sequence were also derived.

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