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On the Exponential Diophantine Equation $(6m^2+1)^x+(3m^2-1)^y=(3m)^z$

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Abstract

Let m be a positive integer. In this paper, we consider the exponential Diophantine equation $(6m^2+1)^x+(3m^2-1)^y=(3m)^z$ and we show that it has only unique positive integer solution $(x,y,z)=(1,1,2)$ for all $m>1$. The proof depends on some results on Diophantine equations and the famous primitive divisor theorem.

1. Introduction

Let u, v, w be relatively prime positive integers greater than one. Consider the exponential Diophantine equation

$$u^x+v^y=w^z, \quad x,y,z\in\mathbb{N}. \tag{1.1}$$

In 1956, Jeśmanowicz conjectured that if (u,v,w) is a Pythagorean triple then the above equation has only the unique positive integer solution $(x,y,z)=(2,2,2)$ [1]. In [2], Terai proposed that if $u^p+v^q=w^r$ with $p,q,r\in\mathbb{N}$, $r\geq 2$ then (1.1) has only the positive integer solution $(x,y,z)=(p,q,r)$ except for a few triples (u,v,w). The following combined version of these two conjectures are called the Terai-Jeśmanowicz conjecture [3].

Conjecture 1. *[3, Conjecture 3.2] If $(x,y,z)=(p,q,r)$ is a solution of (1.1) with $\min\{p,q,r\}>1$ then the only solution to (1.1) with $\min\{x,y,z\}>1$ is $(x,y,z)=(p,q,r)$.*

Many research confirmed that these conjectures are true in many special cases [4]-[11]. Especially, the positive integer solutions of the exponential Diophantine equation

$$(am^2+1)^x+(bm^2-1)^y=(cm)^z \tag{1.2}$$

which is a special case of (1.1) with a,b,c,m are positive integers such that $a+b=c^2$ has already been investigated by a number of authors and all of them justify Terai's conjecture in their special cases. In [12], Terai consider the equation (1.2) with $(a,b,c)=(4,5,3)$ and he proved that $(x,y,z)=(1,1,2)$ is the only positive integer solution of $(4m^2+1)^x+(5m^2-1)^y=(3m)^z$ under some conditions. Remaining cases of this equation are completed in [13]-[15]. As a recent study, in [16], the complete solution of (1.2) with $(a,b,c)=(4,21,5)$ is also given. For some similar problems, see for example [8], [17]-[24]. In this paper, we consider the exponential Diophantine equation

$$(6m^2+1)^x+(3m^2-1)^y=(3m)^z \tag{1.3}$$

and we give the complete solution of this equation by proving the following theorem.

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Theorem 1.1. *Let m be a positive integer. Then the equation (1.3) has only the unique positive integer solution* $(x,y,z)=(1,1,2)$ *for all* $m>1$.

We state the above theorem for $m>1$, since for $m=1$ the equation (1.3) turns intu the equation $7^x+2^y=3^z$ which is already known that it has exactly two solutions as $(x,y,z)=(1,1,2),(2,5,4)$ [25]. It is also worth to note that this equation $7^x+2^y=3^z$ is one of the a few known exceptional cases of Terai's conjecture [10]. So from now on we take $m>1$. We refer to section 3 of [3] for various version of above conjecture and for a complete list of all known examples of (1.1) which has at least two distinct solutions. The proof of the Theorem 1.1 mainly depends on the combinations of two methods. One of them is due to [26, 27] which enable us to find the other possible solutions of the Diophantine equations $X^2+DY^2=k^Z$ and $aX^2+bY^2=k^Z$ from the known solutions under some conditions and the other one is the famous primitive divisor theorem [28, 29]. The details of these methods are given in the next chapter

2. Preliminaries

Let D be any positive integer. By $h(-4D)$, we denote the class number of positive binary quadratic forms of discriminant $-4D$.

Lemma 2.1. *[30, Theorems 11.4.3, 12.10.1 and 12.14.3]*

$$h(-4D)<\frac{4}{\pi}\sqrt{D}\log(2e\sqrt{D}).$$

Lemma 2.2. *[27, Theorems 1 and 2] Let D and k be relatively prime positive integers such that* $D>1$ *and k is an odd integer. If the equation*

$$U^2+DV^2=k^W,\quad U,V,W\in\mathbb{Z},\quad \gcd(U,V)=1,\quad W>0$$

has solutions (U,V,W), *then any solution of the above equation can be expressed as*

$$U+V\sqrt{-D}=\lambda_1(U_1+\lambda_2V_1\sqrt{-D})^t,$$

$$W=W_1t,\quad t\in\mathbb{N},$$

where $\lambda_{1,2}\in\{\pm1\}$, U_1,V_1,W_1 *are positive integers satisfying* $U_1^2+DV_1^2=k^{W_1}$, $\gcd(U_1,V_1)=1$ *and* $W_1\mid h(-4D)$.

Let D_1,D_2 be relatively prime positive integers greater that 1 and let (X,Y,Z) be a fixed solution of the equation

$$D_1X^2+D_2Y^2=k^z,\ \gcd(X,Y)=1,\ 2\nmid k,\ Z>0\quad\text{and}\quad X,Y,Z\in\mathbb{Z}. \tag{2.1}$$

Then there exists a unique positive integer l such that $l=D_1\alpha X+D_2\beta Y$, $0<t<k$, where α,β are integers with $\beta X-\alpha Y=1$ [27, Lemma 1]. The positive integer l is called the characteristic number of this particular solution (X,Y,Z) and it is denoted by $<X,Y,Z>$. if $<X,Y,Z>=l$ then it is known that $D_1X\equiv -lY \pmod k$ [27, Lemma 6]. Let (X_0,Y_0,Z_0) be a solution of (2.1) and let $<X_0,Y_0,Z_0>=l_0$. Then the set of all solutions (X,Y,Z) with $<X,Y,Z>\equiv \pm l_0 \pmod k$ is called a solution class of (2.1) and it is denoted by $S(l_0)$.

Lemma 2.3. *[27, Theorems 1 and 2] For any fixed solution class* $S(l_0)$ *of* (2.1), *there exists a unique solution* $(X_1,Y_1,Z_1)\in S(l_0)$ *such that* $X_1>0$, $Y_1>0$ *and* $Z_1\geq Z$, *where Z runs through all solutions* $(X,Y,Z)\in S(l_0)$. *The solution* (X_1,Y_1,Z_1) *is called the least solution of* $S(l_0)$. *If* (X,Y,Z) *is a solution belongs to* $S(l_0)$ *then*

$$Z=Z_1t,\ 2\nmid t,\ t\in\mathbb{N},$$

$$X\sqrt{D_1}+Y\sqrt{-D_2}=s_1\left(X_1\sqrt{D_1}+s_2Y_1\sqrt{-D_2}\right)^t,\ s_1,s_2\in\{-1,1\}.$$

Lemma 2.4. *[26, Theorem 2] Let* (X_1,Y_1,Z_1) *be the least solution of* $S(l_0)$. *If* (2.1) *has a solution* $(X,Y,Z)\in S(l_0)$ *satisfying* $X>0$ *and* $Y=1$, *then* $Y_1=1$. *Further, if* $(X,Z)\neq(X_1,Z_1)$, *then one of the following conditions is satisfied:*

(i) $D_1X_1^2=\frac{1}{4}(k^{Z_1}\pm1), D_1=\frac{1}{4}(3k^{Z_1}\pm1), (X,Z)=(X_1|D_1X_1^2-3D_2|,3Z_1)$.

(ii) $D_1X_1^2=\frac{1}{4}F_{3r+3\varepsilon}, D_2=\frac{1}{4}L_{3r}, k^{Z_1}=F_{3r+\varepsilon}$,
$(X,Z)=(X_1|D_1^2X_1^4-10D_1D_2X_1^2+5D_2^2|,5Z_1)$, *where* $\varepsilon\in\{-1,1\}$, *r is a positive integer, and* F_n *is nth Fibonacci number.*

The primitive divisor theorem is another powerful tool for solving some Diophantine equations. Let α,β be algebraic integers. A Lucas pair is a pair (α,β) such that $\alpha+\beta$ and $\alpha\beta$ are non-zero relatively prime integers and $\dfrac{\alpha}{\beta}$ is not a root of unity. If (α,β) is any Lucas pair then the corresponding sequences of Lucas numbers are defined by

$$L_n(\alpha,\beta)=\frac{\alpha^n-\beta^n}{\alpha-\beta},\ n=0,1,2,\ldots.$$

Recall that primitive divisors of $L_n(\alpha,\beta)$ are the prime numbers p such that $p\mid L_n(\alpha,\beta)$ and $p\nmid(\alpha-\beta)^2L_1(\alpha,\beta)\ldots L_{n-1}(\alpha,\beta)$ $(n>1)$. Any two Lucas pairs (α_1,β_1) and (α_2,β_2) are called equivalent if $\dfrac{\alpha_1}{\alpha_2}=\dfrac{\beta_1}{\beta_2}=\pm1$.

Lemma 2.5. *[28] If $n > 30$ then $L_n(\alpha, \beta)$ has a primitive divisor.*

Lemma 2.6. *[29] If $4 < n \leq 30$ and $n \neq 6$ then, up to equivalence, $L_n(\alpha, \beta)$ has a primitive divisor except for the following parameters (e, f)*

- $(1,5), (1,-7), (2,-40), (1,-11), (1,-15), (12,-76)$ *or* $(12,-1364)$ *if* $n = 5$,
- $((1,-7)$ *or* $(1,-19)$ *if* $n = 7$,
- $(1,-7)$ *or* $(2,-24)$ *if* $n = 8$,
- $(2,-8), (5,-3)$ *or* $(5,-47)$ *if* $n = 10$,
- $(1,-5), (1,-7), (1,-11), (2,-56), (1,-15)$ *or* $(1,-19)$ *if* $n = 12$,
- $(1,-7)$ *if* $n = 13, 18$ *or* 30.

where $(\alpha, \beta) = \left(\dfrac{e + \sqrt{f}}{2}, \dfrac{e - \sqrt{f}}{2} \right)$.

3. Proof of Theorem 1.1

We treat the (1.3) according to the parity of m. For the case m is even, the proof of Theorem 1.1 easily follows from the next lemma.

3.1. The case $2 \mid m$

Lemma 3.1. *If m is even then $(x, y, z) = (1, 1, 2)$ is the unique positive integer solution of equation (1.3).*

Proof. If $z \leq 2$, then $(x, y, z) = (1, 1, 2)$ is clearly the unique solution of the equation (1.3). So assume that $z \geq 3$. Taking equation (1.3) modulo m^2 we get that $1 + (-1)^y \equiv 0 \pmod{m^2}$ and hence we see that y is odd since $m^2 > 2$. Taking equation (1.3) modulo $3m^3$ we find that

$$1 + 6m^2x + (-1) + 3m^2y \equiv 0 \pmod{3m^3}$$
$$2x + y \equiv 0 \pmod{m},$$

which is false because y is odd and m is even. So we conclude that the equation (1.3) has no positive integer solution when $z \geq 3$. Therefore (1.3) has only the unique positive integer solution $(1, 1, 2)$ when m is even. □

From now on we deal with the case m is odd.

3.2. The case $2 \nmid$ m

Let (x, y, z) be any solution of (1.3). Clearly $(x, y, z) = (1, 1, 2)$ is a solution of (1.3). Since $m > 1$, taking (1.3) modulo m^2 we see that, as in the previous case, y is odd.
From now on we separate two cases according to the parity of x. First suppose that x is also odd. Now consider the Diophantine equation

$$(6m^2 + 1)X^2 + (3m^2 - 1)Y^2 = (3m)^Z, \ Z > 0 \quad \text{and} \quad X, Y, Z \in \mathbb{Z}. \tag{3.1}$$

Since (x, y, z) is any solution of (1.3), we see that

$$(X, Y, Z) = \left((6m^2 + 1)^{\frac{x-1}{2}}, (3m^2 - 1)^{\frac{y-1}{2}}, z \right) \tag{3.2}$$

is a solution of (3.1). Let $l =< (6m^2 + 1)^{\frac{x-1}{2}}, (3m^2 - 1)^{\frac{y-1}{2}}, z >$ be a characteristic number of the solution given in (3.2). Then, from the congruence

$$(6m^2 + 1)^{\frac{x+1}{2}} \quad \equiv -l(3m^2 - 1)^{\frac{y-1}{2}} \pmod{3m},$$

we see that $l \equiv \pm 1 \pmod{3m}$.
Note that $(X_1, Y_1, Z_1) = (1, 1, 2)$ is also a solution of the equation (3.1) and let $l_0 =< 1, 1, 2 >$ be the characteristic number of this solution. So, we have that

$$\begin{gathered} (6m^2 + 1) \cdot 1 \equiv -l_0 \cdot 1 \pmod{3m}, \\ l_0 \equiv -1 \pmod{3m}. \end{gathered} \tag{3.3}$$

So we see that $l \equiv \pm l_0 \pmod{3m}$, which implies that the solutions $(X_1, Y_1, Z_1) = (1, 1, 2)$ and one in (3.2) are in the same solution class $S(l_0)$ of (3.1). Further $(X, Y, Z) = (1, 1, 2)$ is clearly the least solution of $S(l_0)$. So by Lemma (2.3), we get that

$$(6m^2 + 1)^{\frac{x-1}{2}} \sqrt{6m^2 + 1} + (3m^2 - 1)^{\frac{y-1}{2}} \sqrt{1 - 3m^2} = \lambda_1 (\sqrt{6m^2 + 1} + \lambda_2 \sqrt{1 - 3m^2}\ )^t, \tag{3.4}$$

with

$$z = 2t,\ 2 \nmid t,\ t \in \mathbb{N} \quad \text{and} \quad \lambda_{1,2} \in \{-1, 1\}.$$

Expanding the right hand side of (3.4) and equating the coefficients of $\sqrt{1-3m^2}$, we find that

$$(3m^2-1)^{\frac{y-1}{2}} \quad = \lambda_1\lambda_2 \sum_{i=0}^{\frac{t-1}{2}} \binom{t}{2i+1}(6m^2+1)^{\frac{t-1}{2}-i}(1-3m^2)^i \tag{3.5}$$

At this point we claim that $y = 1$. For this purpose, assume that $y > 1$. Then from (3.5), we find that

$$0 \equiv \lambda_1\lambda_2 t\ (6m^2+1)^{\frac{t-1}{2}} \pmod{(3m^2-1)}$$

$$0 \equiv \pm 3^{\frac{t-1}{2}} t \pmod{(3m^2-1)},$$

which is a contradiction, since it implies that $2 \mid 3^{\frac{t-1}{2}} t$ since m is odd. So we have that $y = 1$ and hence $Y = (3m^2-1)^{\frac{y-1}{2}} = 1$. Now we check two conditions in Lemma 2.4. Since $(X_1, Y_1, Z_1) = (1, 1, 2)$ is the least solution of $S(l_0)$, by Lemma 2.4, we have that either

$$6m^2 + 1 = \frac{1}{4}(3^2m^2 \mp 1)$$

or

$$F_{3r+\varepsilon} = (3m)^2$$

where $\varepsilon = \pm 1$. The first one implies that $4(6m^2+1) = (3^2m^2 \mp 1)$. But this means that $4 \equiv \pm 1 \pmod{m^2}$, which is impossible. On the other hand since only square Fibonacci number greater than 1 is $F_{12} = 12^2$ [31], the second one implies that $3m = 12$ which is also false because of parity of m. Thus, by Lemma 2.4, we conclude that $(X, Z) = ((6m^2+1)^{\frac{x-1}{2}}, z) = (X_1, Z_1) = (1, 2)$. Hence the equation (1.3) has no positive integer solution other than $(x, y, z) = (1, 1, 2)$ when x is odd.
Now we treat the case x is even. Then from (1.3), the equation

$$U^2 + (3m^2-1)V^2 = (3m)^W,\ \gcd(U, V) = 1,\ W > 0$$

has a solution

$$(U, V, W) = \left((6m^2-1)^{\frac{x}{2}}, (3m^2-1)^{\frac{y-1}{2}}, z\right)$$

Thus from Lemma 2.2, we have that

$$\begin{gathered} z = W_1 t,\ t \in \mathbb{N}, \\ (6m^2+1)^{\frac{x}{2}} + (3m^2-1)^{\frac{y-1}{2}}\sqrt{1-3m^2} = \lambda_1\left(U_1 + \lambda_2 V_1\sqrt{1-3m^2}\right)^t, \end{gathered} \tag{3.6}$$

where $\lambda_{1,2} \in \{-1, 1\}$ and U_1, V_1, W_1 are positive integers satisfying

$$U_1^2 + (3m^2-1)V_1^2 = (3m)^{W_1},\ \gcd(U_1, V_1) = 1 \tag{3.7}$$

$$h(-4(3m^2-1)) \equiv 0 \pmod{W_1}. \tag{3.8}$$

Suppose that $2|t$ and let

$$U_2 + V_2\sqrt{1-3m^2} = \left(U_1 + \lambda_2 V_1\sqrt{1-3m^2}\right)^{\frac{t}{2}}. \tag{3.9}$$

Taking the norm of both sides of (3.9) in $\mathbb{Q}(\sqrt{1-3m^2})$ and taking into account (3.7), we get that

$$U_2^2 + (3m^2-1)V_2^2 = (3m)^{\frac{W_1 t}{2}} = (3m)^{\frac{z}{2}}. \tag{3.10}$$

Substituting (3.9) into (3.6), we have that

$$(6m^2+1)^{\frac{x}{2}} + (3m^2-1)^{\frac{y-1}{2}}\sqrt{1-3m^2} = \lambda_1\left(U_2 + V_2\sqrt{1-3m^2}\right)^2$$

and therefore it follows that

$$(6m^2+1)^{\frac{x}{2}} = \lambda_1(U_2^2 - V_2^2(3m^2-1)). \tag{3.11}$$

$$(3m^2-1)^{\frac{y-1}{2}} = 2\lambda_1 U_2 V_2. \tag{3.12}$$

Since $\gcd(6m^2+1, 3m^2-1)=1$, from (3.11) and (3.12) we deduce that $|U_2|=1$. So $|V_2|=\frac{1}{2}(3m^2-1)^{\frac{y-1}{2}}$. Substituting $|U_2|$ and $|V_2|$ into (3.10), we get that

$$1+\frac{1}{4}(3m^2-1)^y=(3m)^{\frac{z}{2}},$$

which implies that $3\equiv 0 \pmod{3m}$, a contradiction since $3m>3$. So we conclude that $2\nmid t$. Let

$$\alpha=U_1+V_1\sqrt{1-3m^2},\ \beta=U_1-V_1\sqrt{1-3m^2}$$

Then, from (3.6), taking its complex conjugate, we get that

$$(6m^2+1)^{\frac{x}{2}}-(3m^2-1)^{\frac{y-1}{2}}\sqrt{1-3m^2}=\lambda_1\left(U_1-\lambda_2V_1\sqrt{1-3m^2}\right)^t. \tag{3.13}$$

By subtracting (3.13) from (3.6) we get that

$$(3m^2-1)^{\frac{y-1}{2}}=V_1\left|\frac{\alpha^t-\beta^t}{\alpha-\beta}\right|=V_1|L_t(\alpha,\beta)|. \tag{3.14}$$

By (3.7), we have $\alpha+\beta=2U_1$, $\alpha-\beta=2V_1\sqrt{1-3m^2}$, $\alpha\beta=(3m)^{W_1}$. Since $\gcd(U_1,V_1)=1$, the integers $\alpha+\beta=2U_1$ and $\alpha\beta=(3m)^{W_1}$ are also relatively prime by (3.7) and $\frac{\alpha}{\beta}\neq\pm1$, units of ring of algebraic integers of $\mathbb{Q}(\sqrt{1-3m^2})$. So $L_t(\alpha,\beta)$ is a Lucas sequence. From (3.14), we see that the Lucas numbers $L_t(\alpha,\beta)$ have no primitive divisors. So, from Lemma 2.5 and Lemma 2.6, we get that $t\leq 30$ and if $4<t\leq 30$ and $t\neq 6$ then the parameters

$$(e,f):=(2U_1,4V_1^2(1-3m^2))$$

must be one of the parameters given in Lemma 2.6. But none of them match with any one of these parameters. So, it follows that

$$t\leq 3.$$

Now we will show that the case $t=3$ is also not possible. To see this, assume that $t=3$. So, expanding the right hand side of (3.6) for $t=3$ as

$$\left(U_1+\lambda_2V_1\sqrt{1-3m^2}\right)^t=U_1^3+3U_1^2\lambda_2V_1\sqrt{1-3m^2}+3U_1V_1^2(1-3m^2)+\lambda_2V_1^3(1-3m^2)\sqrt{1-3m^2}$$

and equating the coefficients of both sides of it, we get that

$$(6m^2+1)^{\frac{x}{2}}=\lambda_1U_1\left(U_1^2-3(3m^2-1)V_1^2\right) \tag{3.15}$$

and

$$(3m^2-1)^{\frac{y-1}{2}}=\lambda_1\lambda_2V_1\left(3U_1^2-(3m^2-1)V_1^2\right). \tag{3.16}$$

Note that from (3.7) one can see that $\gcd(3U_1, 3m^2-1)=1$, so from (3.16) we have that $3U_1^2-(3m^2-1)V_1^2=\pm1$. In fact taking modulo 3 we see that only the positive sign can occur and

$$3U_1^2-(3m^2-1)V_1^2=1. \tag{3.17}$$

Thus, it follows that

$$|V_1|=(3m^2-1)^{\frac{y-1}{2}}. \tag{3.18}$$

Substituting (3.18) into (3.15) we get that

$$(6m^2+1)^{\frac{x}{2}}=\lambda_1U_1\left(U_1^2-3(3m^2-1)^y\right). \tag{3.19}$$

By reducing (3.17) and (3.18) modulo $3m$, we find that $3X_1^2-(-1)1\equiv\pm1 \pmod{3m}$, which means that $U_1\equiv 0 \pmod{m}$. Then from (3.19) we find that $1^{\frac{x}{2}}\equiv 0 \pmod{m}$, which is clearly false. Thus, we may have only $t=1$. Thus $z=W_1t=W_1$ and by (3.8) we know that $W_1\leq h(-4(3m^2-1))$. Using the upper bound in Lemma 2.1, we get that

$$z<\frac{4}{\pi}\sqrt{3m^2-1}\log\left(2e\sqrt{3m^2-1}\right). \tag{3.20}$$

Assume that $z=3$. Then at least one of x or y must be greater than 1. $x\geq 2$ gives $(3m)^3>(6m^2+1)^x\geq(6m^2+1)^2>6^2m^4$, and hence $3^3>6^2m>36$, a contradiction. Similarly if $y\geq 2$ then the inequality $(3m)^3\geq(3m^2-1)^2+(6m^2+1)$ also leads us a contradiction. So $z\geq 4$. Taking equation (1.3) modulo $(9m^4)$, it implies that

$$6m^2x+3m^2y\equiv 0 \pmod{9m^4}$$

and therefore

$$2x+y\equiv 0 \pmod{3m^2}.$$

So

$$3m^2\leq 2x+y. \tag{3.21}$$

Since $(6m^2+1)^x<(3m)^z$ and $(3m^2-1)^y<(3m)^z$, we see that $x<z$ and $y<z$. So from (3.21) we find $m^2<z$. Thus from the inequality

$$m^2<z<\frac{4}{\pi}\sqrt{3m^2-1}\log\left(2e\sqrt{3m^2-1}\right)$$

we find that $m\leq 11$. Then z and hence x and y are also bounded. Taking into account (3.20) together with $x,y<z$ we wrote a short computer program with Maple to check all possible solutions of (1.3) in the range $3\leq m\leq 11$ and we found no positive integer solutions (m,x,y,z) of (1.3) when $z\geq 3$. This completes the proof.

4. Discussion

In this paper, we take into account the equation (1.2) in the special case with the parameters $(a,b,c)=(6,3,3)$ and we show that the corresponding equation $(6m^2+1)^x+(3m^2-1)^y=(3m)^z$ has only the unique solution $(x,y,z)=(1,1,2)$ when $m>1$. By the results of this paper we get that another support of the Terai's Conjecture. As a generalization of the results of this paper one can consider to solve the equation (1.2) in more general case where $2\mid a$, $2\nmid b$, $a+b=c^2$.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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