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## A Conformal Fractional Derivative-based Leaky Integrate-and-Fire Neuron Model

İsmail DEVECİOĞLU\*<sup>1</sup>, Reşat MUTLU<sup>1</sup>

### Abstract

Neuron model have been extensively studied and different models have been proposed. Nobel laureate Hodgkin-Huxley model is physiologically relevant and can demonstrate different neural behaviors, but it is mathematically complex. For this reason, simplified neuron models such as integrate-and-fire model and its derivatives are more popular in the literature to study neural populations. Lapique's integrate-and-fire model is proposed in 1907 and its leaky integrate-and-fire version is very popular due to its simplicity. In order to improve this simple model and capture different aspects of neurons, a variety of it have been proposed. Fractional order derivative-based neuron models are one of those varieties, which can show adaptation without necessitating additional differential equations. However, fractional-order derivatives could be computationally costly. Recently, a conformal fractional derivative (CFD) is suggested in literature. It is easy to understand and implement compared to the other methods. In this study, a CFD-based leaky integrate-and-fire neuron model is proposed. The model captures the adaptation in firing rate under sustained current injection. Results suggest that it could be used to easily and efficiently implement network models as well as to model different sensory afferents.

**Keywords:** Fractional order derivatives, conformal fractional derivative, leaky integrate-and-fire, neuron model

### 1. INTRODUCTION

Integrate-and-fire (IF) model of a neuron, which is first suggested by Louis Lapique [1], has influenced many studies in neuroscience [2]. Even today, it is still used in solution of many

different neuroscience problems thanks to its simplicity [2]. Lapique's neuron model paved the way for a more complex neuron model by Hodgkin-Huxley [3]. Although Hodgkin-Huxley neuron model is physiologically consistent and can interpret different behaviors of a real neuron,

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it is mathematically complex. For this reason, simplified neuron models are usually preferred where the behavior of individual neurons is not that critical. A variety of IF-models has been published in the literature. The most popular one is the leaky IF (LIF) model, in which the membrane potential decreases if the input is ceased. The non-linear models such as Fitzhugh-Nagumo model [4-6] and Izhikevich neuron model [7-10] are also popular since they can capture different neural behaviors. However, implementation of a simple LIF is still easier and efficient compared to these models. The drawback of a LIF neuron is the lack of adaptation term (i.e., the decrement of firing rate of the neuron with elongated stimulation). In order to implement adaptation in LIF, an additional equation which increases the threshold to generate an action potential (hereafter called spike) with every spike is used. On the other hand, if the change of membrane potential is defined with a fractional order derivative, a LIF model can capture the adaptive behavior without addition of a second differential equation.

Fractional order derivatives (FD) have emerged as a hot research area especially in the half century [11-15]. Their application areas in engineering and applied science have been rising continuously [12-15]. They are also used to model biological systems [16]. Since the suggestion of their existence in 1695 [17], different types of fractional derivatives are found [11-15, 18]. The conformal fractional derivative (CFD) has been one of the recently found fractional order derivatives [19]. It has been a major interest since it is easier to understand and implement compared to other types of the fractional derivatives [19-20]. The CFD is the multiplication of the ordinary derivative with  $t^{(1-\alpha)}$  (i.e., time ( $t$ ) to the  $(1-\alpha)$ th power). Furthermore, it has a valid physical interpretation that the other fractional derivative types lack [21]. More on the usage of fractional derivatives in electrical circuits can be found in [22]. The CFD has also been used to model supercapacitors [23-29]. The CFD capacitors and analysis of their usage with other circuit elements have also come out as a popular topic [22, 27-31]. Some circuits which have been analyzed with the CFD can be found in [32-34].

Fractional order capacitors have already been used to model neurons in the literature [16, 35-36]. FD-based leaky integrate and fire neuron models can be found in [35-36]. Additionally, a CFD-based FitzHugh–Nagumo neuron model was presented recently [37]. To the best of our knowledge a CFD-based leaky integrate and fire neuron model has not been examined in literature yet. In this study, a CFD-based leaky integrate and fire neuron model is made, and its spiking pattern is inspected.

The remainder of the paper is organized as follows; In the second section, a brief introduction to the IF neuron model is given. In the third section, the conformal fractional derivative and CFD capacitor model are briefly explained. In the fourth section, the CFD capacitor neuron model and its differential equation are given, and its spiking pattern was inspected under a constant current input. The paper is concluded with the last section.

## 2. A SHORT REVIEW OF INTEGRATE-AND-FIRE NEURON MODELS

Lapicque's IF model demonstrates a neuron in its very basic form, and it can be considered as the fundamental component of other models. LIF model is widely known and used in different studies, and it includes another fundamental component for neurons which is "forgetting" (i.e., decrement of membrane potential if the current input is ceased). Fractional order model introduces the adaptation behavior of neurons in the same equation that is used to calculate neuron's membrane potential. Therefore, we briefly explained IF, LIF and fractional-order neuron models in this section.

### 2.1. Lapicque's Integrate-and-Fire Model

After his work on frog nerves, Louis Lapicque introduced the IF model in 1907 [1,2]. Actually, he realized that the time to excite a nerve with a constant current stimulation can be estimated with a capacitor's current-voltage equation:

$$\frac{dV(t)}{dt} = \frac{1}{C}I(t) \quad (1)$$

where  $V$  is the membrane potential of the neuron and  $C$  is the membrane capacitance. If a current,  $I(t)$ , flows through the membrane of the neuron, the membrane potential increases as the integral of the current flowing. When the membrane potential reaches a certain value (e.g., a spike threshold,  $V_{TH}$ ), it generates a spike which is modelled as a Dirac delta function, and the membrane potential returns to its resting state ( $V_r$ ). Then, the neuron continues to integrate and fire periodically until the current stimulation is ceased. The firing rate (i.e., frequency of spike generation) of IF model,  $r$ , is proportional to the input current:

$$r = \frac{I}{CV_{TH}} \quad (2)$$

$r$  linearly increases without any limits as the current amplitude increases. However, physiological neurons do not do so, their firing rate is limited by their refractory period,  $t_{ref}$ . Therefore, a  $t_{ref} > 0$  is used to limit the firing rate of IF neuron:

$$r = \frac{I}{CV_{TH} + t_{ref}I} \quad (3)$$

When the input current is ceased, however, IF neuron keeps its membrane potential indefinitely. If the stimulation is continued at a late time, then the neuron's membrane potential starts to increase from where it remained. This behavior is inconsistent with physiological neurons which return to their resting state with a time constant when the input current is ceased. This shortcoming is solved by introducing a leakage term in the equation as in LIF model.

## 2.2. Leaky Integrate-and-Fire Model

LIF model consists of a capacitor connected in parallel with a resistor. The resistor,  $R$ , introduced in the model represents the leakage resistance for ions diffusing through the cell membrane, and hence, it acts to reduce the membrane potential towards its resting state. The model equation is given as following:

$$\frac{dV(t)}{dt} = \frac{1}{RC} (I(t)R - V(t)) \quad (4)$$

where  $V$  is the membrane potential and  $R$  is the resistance of leakage channels on the membrane. The membrane potential increases until a threshold,  $V_{TH}$ , is reached, then, a spike is generated, and the membrane potential is reset to its resting value.

For a constant current input, the minimum current amplitude to generate a spike is defined as  $I_{TH} = V_{TH}/R$ . Assuming a reset to zero, the firing rate can be estimated as following:

$$r = \begin{cases} 0, & I \leq I_{TH} \\ \left[ t_{ref} - RC \log(RI - V_{TH}) \right]^{-1}, & I > I_{TH} \end{cases} \quad (5)$$

LIF model, as well as IF model, fires at a constant rate for a given constant current input. However, physiological neurons fire with increasing intervals as the current input is sustained. This behavior is called adaptation. Nonlinear generalized IF models show adaptation by including an adaptation term which is also defined with a differential equation [38].

## 2.3. Fractional-Order Leaky Integrate-and-Fire Model

Unlike LIF and other generalized IF models, fractional-order LIF model can demonstrate adaptation in firing rate in a single equation. The model equation is defined as [36]:

$$\frac{d^\alpha V(t)}{dt^\alpha} = \frac{1}{RC} (I(t)R - V(t)) \quad (6)$$

In [36], the change of membrane potential is also rewritten as:

$$\frac{dV(t)}{dt} = \frac{1}{RC} \frac{d^{1-\alpha}}{dt^{1-\alpha}} (I(t)R - V(t)) \quad (7)$$

The Caputo fractional derivative is used to solve the differential equation in [36]. Spike generation progresses as in IF and LIF models explained in previous sections.

### 3. THE CONFORMAL FRACTIONAL DERIVATIVE AND THE CFD CAPACITOR MODEL

The fractional calculus is used to explain and model the fractional order elements [14, 16]. The CFD has been becoming more prominent fractional derivative method although it is developed in the recent decade [19-21]. It is easy to use and to understand compared to the other types of fractional derivatives [19-21]. In [17], The CFD is described as:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{df(t)}{dt} t^{1-\alpha} \quad (8)$$

This FD is easy and efficient to implement compared to other fractional derivatives.

Some capacitors can be modeled using fractional order derivatives [23-29]. The symbol of a CFD capacitor is shown in Figure 1. The constitutional law of a capacitor modeled using the CFD is given as:

$$i_C(t) = C_\alpha \frac{d^\alpha v_C(t)}{dt^\alpha} = C_\alpha \frac{dv_C(t)}{dt} t^{1-\alpha} \quad (9)$$

where  $C_\alpha$  is the CFD capacitor coefficient,  $i_C(t)$  is the CFD capacitor current, and  $v_C(t)$  is the CFD capacitor voltage.

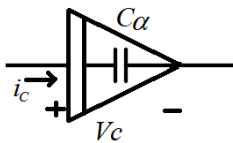


Figure 1 The CFD capacitor symbol

### 4. THE PROPOSED CFD-BASED NEURON MODEL

#### 4.1. Theoretical Analysis of the Model

In this section, we propose a novel CFD-based neuron model. The neuron's electrical circuit is shown in Figure 2. It consists of a switch ( $S$ ), a current source ( $I_S$ ), a leakage resistor ( $R$ ), and the CFD capacitor ( $C_\alpha$ ).  $R$  and  $C_\alpha$  shares the current injected into the model, which prevents  $C_\alpha$  to charge too quickly. The charging time of  $C_\alpha$

depends on the constant  $\tau=RC_\alpha$ . The switch  $S$  is used to discharge  $C_\alpha$  when it reaches a certain threshold voltage,  $V_{TH}$ . This event is registered as a spike. As soon as the discharge occurs, the switch is open again. Then, the integrate and fire operation of the neuron continues until the current injection is ceased.

When an input current ( $I_S$ ) flows through the neuron, the following differential equation can be written using Kirchhoff's laws:

$$I_S = i_C(t) + i_R(t) \quad (10)$$

$$I_S = C_\alpha \frac{d^\alpha v_C(t)}{dt^\alpha} + \frac{v_C(t)}{R} \quad (11)$$

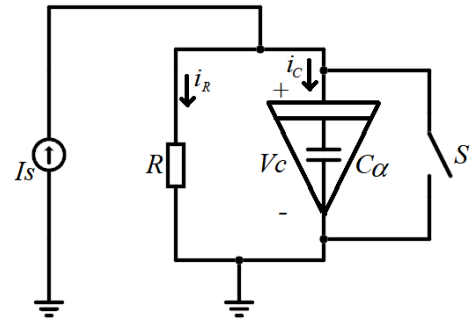


Figure 2 The proposed CFD-based neuron model

$$I_S = C_\alpha \frac{dv_C(t)}{dt} t^{1-\alpha} + \frac{v_C(t)}{R} \quad (12)$$

After multiplying each term with  $R$  and rearranging terms, derivative of the capacitor voltage ( $v_C$ ; called membrane potential hereafter) can be found as following:

$$RI_S = RC_\alpha \frac{dv_C(t)}{dt} t^{1-\alpha} + v_C(t) \quad (13)$$

$$RI_S - v_C(t) = RC_\alpha \frac{dv_C(t)}{dt} t^{1-\alpha} \quad (14)$$

$$\frac{dv_C(t)}{dt} = \frac{1}{RC_\alpha t^{1-\alpha}} (RI_S - v_C(t)) \quad (15)$$

If the terms are rearranged, a solution for  $v_C(t)$  can also be found:

$$\frac{dt}{t^{1-\alpha}} = RC_\alpha \frac{dv_C(t)}{RI_S - v_C(t)} \quad (16)$$

$$\frac{dt}{t^{1-\alpha}} = -RC_\alpha \frac{dv_C(t)}{v_C(t) - RI_S} \quad (17)$$

Taking indefinite integration of each side:

$$\int \frac{dt}{t^{1-\alpha}} = -RC_\alpha \int \frac{dv_C(t)}{v_C(t) - RI_S} \quad (18)$$

$$-\frac{1}{RC_\alpha} \int t^{\alpha-1} dt = \int \frac{dv_C(t)}{v_C(t) - RI_S} \quad (19)$$

$$-\frac{t^\alpha}{\alpha RC_\alpha} + K = \ln(v_C(t) - RI_S) \quad (20)$$

where  $K$  is the integration constant. Rearranging the last equation, the membrane potential is found as:

$$v_C(t) - RI_S = e^{-t^\alpha/\alpha RC_\alpha + K} = e^K e^{-t^\alpha/\alpha RC_\alpha} \quad (21)$$

$$v_C(t) = RI_S + e^K e^{-t^\alpha/\alpha RC_\alpha} \quad (22)$$

Assuming the CFD neuron's membrane potential is zero at  $t=0$  s ( $v_C(0) = 0$  V), the integration constant  $e^K$  can be found as:

$$v_C(0) = RI_S + e^K e^0 = RI_S + e^K = 0 \quad (23)$$

$$e^K = -RI_S \quad (24)$$

By substituting  $e^K$  back into the equation (22), the neuron's membrane potential is found as:

$$v_C(t) = RI_S - RI_S e^{-\frac{t^\alpha}{\alpha RC_\alpha}} \quad (25)$$

At  $t=t_{reset}$ , the neuron voltage becomes equal to the threshold voltage:

$$v_C(t_{reset}) = V_{TH} = RI_S - RI_S e^{-t_{reset}^\alpha/\alpha RC_\alpha} \quad (26)$$

Therefore, the reset time (firing time) is found as:

$$V_{TH} - RI_S = -RI_S e^{-t_{reset}^\alpha/\alpha RC_\alpha} \quad (27)$$

$$(RI_S - V_{TH})/RI_S = e^{-t_{reset}^\alpha/\alpha RC_\alpha} \quad (28)$$

$$\ln((RI_S - V_{TH})/RI_S) = \ln(e^{-t_{reset}^\alpha/\alpha RC_\alpha}) \quad (29)$$

$$t_{reset}^\alpha = -\alpha RC_\alpha \ln((RI_S - V_{TH})/RI_S) \quad (30)$$

$$t_{reset}^\alpha = \alpha RC_\alpha \ln(RI_S/(RI_S - V_{TH})) \quad (31)$$

$$t_{reset} = \sqrt[\alpha]{\alpha RC_\alpha \ln(RI_S/(RI_S - V_{TH}))} \quad (32)$$

If the refractoriness is also assumed, a dead time  $t_{ref}$  after a spike can be used, which would result an inter-spike interval as following:

$$T = t_{ref} + t_{reset} \quad (33)$$

$$T = t_{ref} + \sqrt[\alpha]{\alpha RC_\alpha \ln(RI_S/(RI_S - V_{TH}))} \quad (34)$$

Then, the initial firing rate at the very beginning of current injection can be estimated as:

$$r = \frac{1}{T} = \frac{1}{t_{ref} + \sqrt[\alpha]{\alpha RC_\alpha \ln(RI_S/(RI_S - V_{TH}))}} \quad (35)$$

Beginning from this rate, instantaneous firing rate would decrease along with a sustained current injection. The instantaneous firing rate is dependent on  $t^{1-\alpha}$  and, theoretically, the firing rate never reaches to zero.

## 4.2. Simulation of the Model

In this study, we chose the membrane capacitance as 200 pF and the leakage resistance as 50 M $\Omega$ , which resulted a time constant of 10 ms [39]. The simulations were performed by solving following differential equation with Euler's method in python.

$$\frac{dv_C(t)}{dt} = \begin{cases} 0, & t \leq t_{onset} \\ \frac{1}{RC_\alpha(t-t_{onset})^{1-\alpha}} (RI_S - v_C(t)), & t > t_{onset} \end{cases} \quad (36)$$

where  $t_{onset}$  is the onset time of the current stimulation.

Resting state membrane potential was assumed to be zero ( $V_R=0V$ ), and threshold voltage ( $V_{TH}$ ) to fire a spike was 10 mV. If the membrane potential ( $v_C$ ) reached  $V_{TH}$  at time  $t$ , then  $v_C(t+dt)$  is set to  $V_R$ . While solving for the membrane potential, we recorded its value in a separate variable. In order to emphasize spikes, we recorded the membrane potential at the time of spike as a peak spike voltage of 30 mV in the recording variable (Figure 3).

The Figure 3 shows the model response when stimulated with a current amplitude of 0.21 nA while the simulations are given for four different  $\alpha$  values. The onset of the stimulus was 0.1 s, and its duration was 0.3 s. In order to understand the model's behavior, we ignored refractory period in these simulations (e.g.,  $t_{ref}=0$  s). If  $\alpha$  is chosen 1, then the model equation turns into the basic LIF model, i.e., a model with exponential charging. It fires spikes at a constant rate during the stimulation period. As the  $\alpha$  is decreased, the model starts to generate spikes with increasing interspike intervals as the time proceeds, which is considered as adaptation. Additionally, decreasing  $\alpha$  increased the instantaneous firing rates at the onset of the stimulation period. However, the spike patterns (especially, at the very beginning of stimulation) generated with  $\alpha$  values smaller than 0.5 are not plausible due to the refractoriness of the physiological neurons. Therefore,  $\alpha$  should be chosen  $<0.5$  and/or a non-zero  $t_{ref}$  should be used.

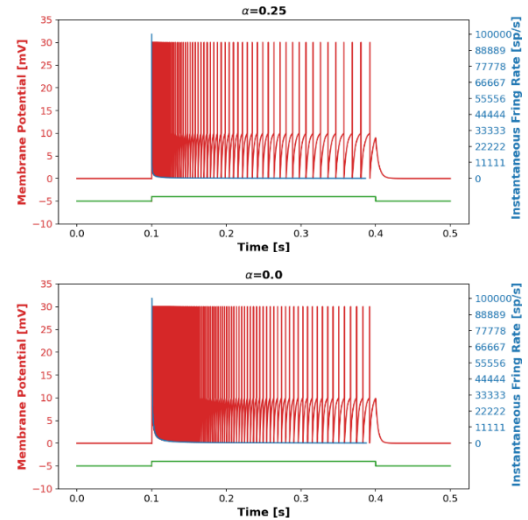
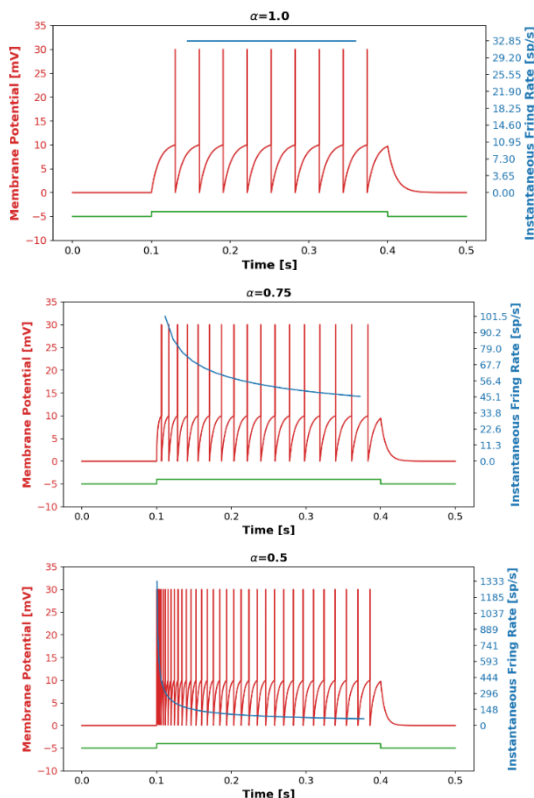
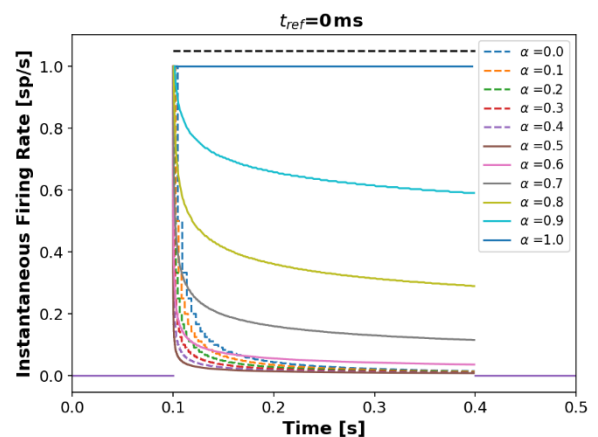


Figure 3 The CFD-based neuron model's membrane potential. Injected current was 0.21 nA, and  $\alpha$  is varied between 0 and 1. Spikes are virtually demonstrated with a peak voltage of 30 mV. Stimulation period is shown with the green line

Although the adaptation rate seems to be increased with decreasing  $\alpha$ , it actually didn't do so along the interval of  $\alpha$ . Figure 4 shows the change of instantaneous firing rate normalized to its initial rate (i.e.,  $r/r_{onset}$ ). The adaptation rate increases with decreasing  $\alpha$  until  $\alpha=0.5$ , and then, it decreases with decreasing  $\alpha$ . The same trend is observed with  $t_{ref}=0$  ms and  $t_{ref}=5$  ms. However, the adaptation rates were decreased for  $t_{ref}=5$  ms.



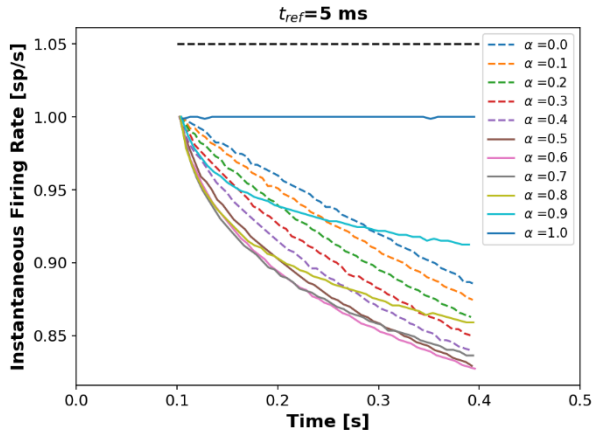


Figure 4 Adaptation profile of the model neuron. If refractoriness is ignored (upper panel;  $t_{ref}=0s$ ), adaptation rate (rate of decrement of normalized instantaneous firing rate) increased with decreasing  $\alpha$  until  $\alpha=0.5$ , and then, it decreased as  $\alpha$  was decreased further. Although a similar trend still exists with a refractory period of 5 ms (lower panel), the rate of adaptation significantly changed

In Figure 5, we inspected how the average firing rate of the model neuron changes with increasing stimulation current for different  $\alpha$  values. As expected, the model doesn't fire until a certain threshold is reached ( $I_{TH}=0.2$  nA with the chosen  $R$  and  $V_{TH}$ ). Then, the firing rate increases with a decreasing rate. However, if the refractoriness is ignored, then the firing rate starts to exponentially increase after a certain input current. This is inconvenient with the behavior of the physiological neurons. On the other hand, if refractoriness is accounted for, then the firing rate plateaus at  $1/t_{ref}$ .

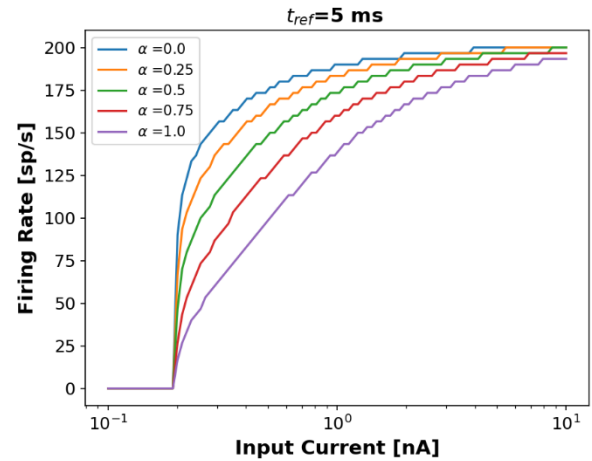
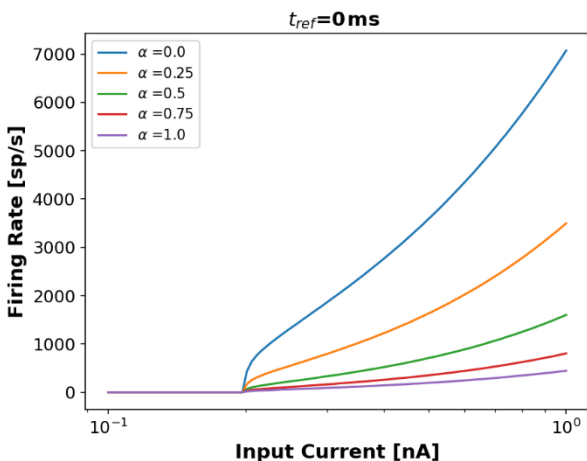


Figure 5 Change of average firing rate with increasing stimulation current. If the refractoriness is ignored (upper panel), then the firing rate doesn't reach a plateau but increases with increasing stimulation current. If a refractory period is chosen (lower panel;  $t_{ref}=5$  ms), then the firing rate plateaus at  $1/t_{ref}$

## 5. CONCLUSION

In this study, we demonstrated a conformal fractional derivative-based leaky integrate-and-fire neuron model. The model equations are easier and more efficient to implement compared to other fractional derivative-based models in literature. Additionally, it shows the adaptation behavior that is observed in physiological neurons and other fractional-order neuron models [35-37]. However, the model parameters, especially  $t_{ref}$  and  $\alpha$ , should be chosen with care. For example,  $\alpha$  values in the range of  $[0.5, 1.0]$  would probably result in more physiologically consistent spike patterns. Because the  $t^{1-\alpha}$  term in the model equation radically increases the charge rate of the capacitor, if  $\alpha < 0.5$ . For example, if  $\alpha$  is zero, then the initial rate of change in membrane potential would approximately be  $I_S/(C_\alpha \Delta t^{1-\alpha})$  where  $\Delta t$  is the step size of the simulation. On the other hand, using a refractory period also limits the initial firing rate, therefore, resulting in a more physiologically consistent spiking pattern. Additionally, the difference in adaptation profile for  $\alpha < 0.5$  and  $\alpha > 0.5$  can be used in favor of modeling purposes (with a  $t_{ref} > 0$ ). For example, slowly adapting mechanoreceptive fibers innervating Merkel cells (SA-I) and Ruffini endings (SA-II) in the skin shows adaptation



under sustained indentation of the skin [40,41]. However, the fibers innervating these two types of mechanoreceptors differ in means of the rate and pattern of adaptation; SA-II fibers adapt to the sustained stimulus at a constant rate, while SA-I fibers' adaptation consists of two phases (first a steeper decrement and then relatively slower decrement) [41]. Therefore, SA-I afferents can be modelled with  $\alpha$  values greater than 0.5, and SA-II fiber can be modelled with  $\alpha$  values smaller than 0.5. Finally, due to its mathematical simplicity, the proposed model can also be used in microcontroller- or microprocessor-based emulators to demonstrate spiking of a neuron silica conditions [42,43].

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No conflict of interest or common interest has been declared by the authors.

### ***Authors' Contribution***

The authors contributed equally to the study.

### ***The Declaration of Ethics Committee Approval***

This study does not require ethics committee permission or any special permission.

### ***The Declaration of Research and Publication Ethics***

The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they

do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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