Obtained Results on the Discrete Asymptotic Stability of Linear Discrete Systems with Periodic Coefficients

Kemal USLU[1]

Selçuk Üniversitesi Fen Edebiyat Fakültesi Matematik Bölümü Kampus-Konya,

Abstract: It is very important problem for periodic systems that they are discrete asymptotically stable or practical discrete asymptotically stable. Main goal of this study is to obtain some results related to discrete asymptotic stability of such systems.

Key words: Discrete asymptotic stability, Linear difference systems with periodic coefficients, Lyapunov methods.

Periyodik Katsayılı Lineer Fark Sistemlerinin Fark Asimtotik Kararlılığı Üzerine Elde Edilmiş Sonuçlar

Özet: Periyodik sistemler için fark asimtotik kararlılık ve pratik fark asimtotik kararlılık önemli bir problemdir. Bu çalışmanın ana amacı böyle sistemlerin fark asimtotik kararlılığı ile ilgili bazı sonuçlar elde etmektir.

Anahtar kelimeler: Fark asimtotik kararlılık, periyodik katsayılı lineer fark sistemleri, Lyapunov metodları.

1. Introduction

In this study, we have obtained some inequalities related to discrete asymptotic stability or practical discrete asymptotic stability of the periodic systems. There are various criteria for the discrete asymptotic stability of these systems in applications. One of these is discrete asymptotic stability parameters. Main goal of this study is to obtain results with connected discrete asymptotic stability parameters of such systems.

2. Preliminaries

In this section, it is given some preliminary definitions and results to be used in later. Now, we consider the linear difference system with periodic coefficients

$$\begin{aligned} x(n+1) &= A(n)x(n),\ n \geq 0 \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

[1] E-mail: kuslu@selcuk.edu.tr

where $A(n+T)=A(n),\ T>0$, is an $(N\times N)$ matrix and x_0 and $x(n)$ are $(N\times 1)$ vectors.

If the fundamental matrix of this system is $X(n)$, then $X(T)$ is called the monodromy matrix and given in the following form

$$X(T)=\prod_{j=0}^{T-1}A(j)=A(T-1)A(T-2)...A(0).$$

The solution of this systems with $x(0)=x_0$ is

$$x(n)=x(kT+m)=X(m)[X(T)]^k x_0\,,$$

where $n=kT+m,\ k\geq 0,\ 0\leq m\leq T-1.$

There are various criteria determining discrete asymptotic stability of the system (2.1). One of them is $\left|\lambda_i[X(T)\right|<1,\ i=1,2,...,N$ [2]. On the other hand it has been defined discrete stability parameters to determine the discrete asymptotic stability in [3].

According to Lyapunov criterion, the zero solution to the system (2.1) is discrete asymptotically stable if and only if the discrete matrix equation

$$X^*(T)FX(T)-F+I=0 \tag{2.2}$$

has a unique matrix solution *F* such that *F* is symmetric and positive definite (i.e. $F = F^* > 0$). In this case a solution to (2.2) is given by the following formula

$$F=\sum_{k=0}^{\infty}[X^*(T)]^k[X(T)]^k \tag{2.3}$$

where $X^*(T)$ is conjugate-transpose of $X(T)$, and I is (N×N) identity matrix. Thus the discrete stability parameter of the system (2.1) has been defined as

$$\{w_1(A,T),\rho(A,T)\} \tag{2.4}$$

where

$$w_1(A,T)=\|F\|=\left\|\sum_{k=0}^{\infty}[X^*(T)]^k[X(T)]^k\right\| \tag{2.5}$$

and

$$\rho(A,T)=\max_{0\leq s\leq T-1}\|X(s)\| \tag{2.6}$$

[3]. It is noted that we henceforth consider the spectral norm of square matrices , i.e.,

$$\|B\|=\max_{\|x\|=1}\|Bx\|$$

where

$$\|x\|=\left(\sum_{i=1}^{N}|x_i|^2\right)^{1/2}$$

denotes the norm of a vector $x=(x_1,x_2,...,x_N)$ in the linear Euclidian space E_N with the inner product

$$\langle x,y\rangle=\sum_{i=1}^{N}x_i\,\bar{y}_i$$

where the bar over a number denotes complex conjugation.

If the inequality $\|X(T)\|<1$ holds then, by norm properties, the inequality holds

$$w_1(A,T)<\frac{1}{1-\|X(T)\|^2}=\tilde{w}_1(A,T)$$

[1].

Definition 2.1. If $w_1(A,T)<\infty$ then system (2.1) is called as discrete asymptotically stable, otherwise (i.e. $w_1(A,T)=\infty$) it is not discrete asymptotically stable [3].

3. Main Results

In this section we mainly used the results of [1] for the following analysis. To find the approximate solution of the matrix series (2.3), the recurrence relation

$$F_k = F_{k-1} + B_{k-1}^* F_{k-1} B_{k-1}\,,\; B_k = \left(B_{k-1}\right)^2,\; k = 1,2,...,M \qquad (3.1)$$

with $F_0 = I$, $B_0 = X(T)$ is required [1]. At the end of *M* steps of the iterations, the relation (3.1) yields the following sum

$$F_M = \sum_{k=0}^{2^M-1} [X^*(T)]^k [X(T)]^k, \qquad (3.2)$$

$$B_M = [X(T)]^{2^M},$$

where the matrix F_M is an approximation to the matrix *F* (see [1]). Then an approximation of discrete stability parameter given by (2.5) can be denoted by

$$\hat{w}_1(A,T) = \|F_M\|. \qquad (3.3)$$

Lemma 3.1. Let the matrix *X*(*T*) is a discrete asymtotically stable matrix. If *F* and F_M are the matrices defined by (2.3) and (3.2) then

$$\|F - F_M\| \le [w_1(A,T)]^2 \left(1 - \frac{1}{w_1(A,T)}\right)^{2^M}. \qquad (3.4)$$

Proof: Substracting (3.2) from (2.4) and using norm given above, we get

$$\|F - F_M\| = \left\| \sum_{k=0}^{\infty} [X^*(T)]^k [X(T)]^k - \sum_{k=0}^{2^M-1} [X^*(T)]^k [X(T)]^k \right\|$$

$$= \left\| \sum_{k=2^M}^{\infty} [X^*(T)]^k [X(T)]^k \right\|$$

$$= \left\| [X^*(T)]^{2^M} (I + X^*(T)X(T) + ...)[X(T)]^{2^M} \right\|$$

$$= \left\| [X^*(T)]^{2^M} F [X(T)]^{2^M} \right\|$$

From the norm properties, we can write the following:

$$\|F - F_M\| \le \left\| [X(T)]^{2^M} \right\| \|F\|.$$

Since the matrix *X*(*T*) is discrete asymptotically stable, the following inequality will be satisfied

$$\left\| X(T)^{2^M} \right\| \le \sqrt{w_1(A,T)} \left(1 - \frac{1}{w_1(A,T)}\right)^{2^M/2} \qquad (3.5)$$

[3]. Thus we write

$$\|F - F_M\| \le [w_1(A,T)]^2 \left(1 - \frac{1}{w_1(A,T)}\right)^{2^M}.$$

If $M = M(w_1(A,T),\delta)$ the smallest integer number satisfying the following

$$[w_1(A,T)]^2 \left(1 - \frac{1}{w_1(A,T)}\right)^{2^M} \le \delta,\; (0 < \delta < \frac{1}{1 + \|X(T)\|^2}),$$

then, from Lemma 3.1, the inequality

$$\|F - F_M\| \le \delta$$

follows. A natural consequence of Lemma 3.1 gives

$$\lim_{M\to\infty} B_M = 0 .$$

In this case, it is also obvious that

$$\lim_{M\to\infty} F_M = F$$

and

$$\lim_{M\to\infty} \Delta_M = \lim_{M\to\infty} (X^*(T)F_M X(T) - F_M + I) = 0 .$$

Lemma 3.2. Let $X(T)$ be a discrete asymptotically stable matrix. In this case, the inequality

$$\|\Delta_M\| \le \frac{\delta}{w_1(A,T)}$$

holds.

Proof: By using equation (3.2) in the matrix equality

$$\Delta_M = (X^*(T)F_M X(T) - F_M + I),$$

it can be written

$$\Delta_M = [X^*(T)]^{2^M} [X(T)]^{2^M} .$$

By using norm properties in the last equality, we have

$$\|\Delta_M\| \le \left\| [X(T)]^{2^M} \right\|^2 .$$

Furthermore, from the inequality (3.5), we write

$$\|\Delta_M\| \le w_1(A,T)\left(1 - \frac{1}{w_1(A,T)}\right)^{2^M} .$$

Reminding $[w_1(A,T)]^2 \left(1 - \frac{1}{w_1(A,T)}\right)^{2^M} \le \delta$, from Lemma 3.1, it is obtained that

$$\|\Delta_M\| \le \frac{\delta}{w_1(A,T)} .$$

Kaynaklar

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