The Sheaf of The *H*-Cogroups

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Abstract: In this study we show that the sheaf which is constructed in [1] is a sheaf of H -cogroups and it is an H -cospace if we equip the set of sections of this sheaf with the compact-open topology. Finally we give some characterizations.

Key Words: Cospace, Sheaf, Cogroup, Homotopy

H-Kogrupların Demeti

Özet: Bu çalışmada [1] deki metod kullanılarak *H*-kogrupların demeti inşa edilmiş ve bu demetin her bir kesitlerinin kümesi kompakt açık topoloji ile donatılmış ise bir *H*-kouzayı olduğu gösterilmiştir. Sonuç olarak bazı karakterizasyonlar verilmiştir.

Anahtar Kelimeler: Kouzay, Demet, Kogrup, Homotopi

Introduction

Let C be the category of topological spaces X satisfying the property that all pointed topological spaces (X,x), $x \in X$ have the same homotopy type. This category includes for example all topological vector spaces. Let us take $X \in C$ as a base set if Q is any abelian H -cogroup, then there exists a sheaf (H,π) over the topological space X which is formed by a Q H -cogroup. For each $x \in X$, $\pi^{-1}(x) = [Q;(X,x)] = H_x$ is the stalk of the sheaf which has a discrete topology. (where $[Q;(X,x)] = H_x$ is a set of homotopy classes of homotopic maps preserving the base points from (Q,q_0) to (X,x)) [1,7].

2. The Sheaf of the H -Cogroups

Definition 2.1. Let (X,x) be a pointed topological space. The diagonal map

$$\Delta_X : X \to X \times X$$

is defined by $\Delta_X(x) = (x,x)$ and the dual of the diagonal map Δ_X , denoted by ∇_X , with

$$\nabla_X : X \vee X \to X$$

defined by

$$\nabla_X(x,x_0) = \nabla_X(x_0,x) = x.[2].$$

Definition 2.2. Let (H,π) be a sheaf over X and $s \in \Gamma(X,H)$ [1]. The folding map

$$\nabla_{H_x} : H_x \vee H_x \to H_x$$

is defined by

$$\nabla_{H_x}\left([f]_x, (s \circ \pi)[f]_x\right) = [f]_x$$

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and

$$\nabla_{H_x}\left(\left(s\circ\pi\right)\left([f]_x\right),[f]_x\right)=[f]_x .$$

Now let us define a comultiplication on H_x as follows: if $[f]_x,[c]_x\in H_x$ then

$$v_x : H_x \to H_x \vee H_x$$
$$v_x\left([f]_x\right)=\left[\nabla_X\circ\left(f,c\right)\circ v\right]_x ,$$

where v is an operation of a Q H-cogroup and

$$c:Q\to\left(X,x\right)$$

is a constant map.

It follows from this definition that the comultiplication v_x is well-defined, closed, continuous and with this comultiplication H_x is an H-cogroup. In fact, the constant map

$$c_x . H_x \to H_x$$

which satisfies

$$c_x\left(H_x\right)=\left[c\right]_x$$

is a homotopy identity of a pointed topological space $\left(H_x,[c]_x\right)$. That is, each of the composites

$$H_x \xrightarrow{v_x} H_x\vee H_x \underset{\left(c_x\vee 1_{H_x}\right)}{\overset{\left(1_{H_x}\vee c_x\right)}{\longrightarrow}} H_x\vee H_x \xrightarrow{\nabla_x} H_x$$

is a homotopic to 1_{H_x}, that is,

$$\nabla_x\left(1_{H_x}\vee c_x\right)v_x \simeq 1_{H_x}\simeq\nabla_{H_x}\left(c_x\vee 1_{H_x}\right)v_x .$$

The continuous map

$$\phi_x : H_x \to H_x$$

is a homotopy inverse for H_x. That is, each of the composites

$$H_x \xrightarrow{v_x} H_x\vee H_x \underset{\left(\phi_x\vee 1_{H_x}\right)}{\overset{\left(1_{H_x}\vee \phi_x\right)}{\longrightarrow}} H_x\vee H_x \xrightarrow{\nabla_x} H_x$$

is homotopic to c_x, that is

$$\nabla_x\left(1_{H_x}\vee\phi_x\right)v_x \simeq c_x\simeq\nabla_{H_x}\left(\phi_x\vee 1_{H_x}\right)v_x ,$$

where $\phi(x)$ is defined by

$$\phi_x\left([f]_x\right)=\left[f\circ\phi\right]_x$$

and ϕ is homotopy inverse of a Q H-cogroup.

It can be shown also that v_x is a homotopy associative.

Now we have the following results:

Result: 1. H_x is an H-cogroup.

Result: 2. The sheaf which is constructed in [1] is a sheaf of H-cogroups.

Let $\Gamma\left(X,H\right)$ denote the set of global section of H with the compact-open topology. Then the mapping

$$v' : \Gamma\left(X,H\right)\to\Gamma\left(X,H\right)\vee\Gamma\left(X,H\right)$$

which is given by

$$v'(s)(x)=v_x\left(s(x)\right)=v_x\left([f]_x\right)=\left[\nabla_x\circ\left(f,c\right)\circ v\right]_x$$

$\left(s\in\Gamma\left(X,H\right), x\in X\right)$ defines a comultiplication on $\Gamma\left(X,H\right)$ such that v' is well-defined, closed and continuous [3]. In fact, if U is an open neighborhood of $v'(s)$ in $\Gamma\left(X,H\right)\vee\Gamma\left(X,H\right)$

for every $s\in\Gamma(X,H)$, then there exists is finite collection of open sets in the subbasis, $\{M(C_i,O_i)\}_{i\in J}$ (J is finite) such that

$$v'(s)\in\left(\bigcap_{i\in J} M(C_i,O_i)\vee\bigcap_{i\in J} M(C_i,O_i)\right)\subset U.$$

Since the comultiplications v_x are continuous in H_x, we can choose neighborhoods U_i of $s(x)$ such that if $s_i(x')\in U_i$, then

$$v'(s_i)(x')=v_{x'}(s_i(x'))\in(O_i\vee O_i)$$

Thus, $s\in\bigcap M(C_i,U_i)$ and if $s'\in M(C_i,U_i)$ then

$$V'(s)\in(\bigcap M(C_i,O_i)\vee\bigcap M(C_i,O_i))$$

since

$$v'(s')(x)=v_x(s'(x))\leq\bigcap(O_i,O_i)$$

for all $x\in\bigcap C_i$.

Let $I:X\to H$ be the section $(\pi\circ I=1_x)$ satisfying $I(x)\in C(c_x)\subset H_x$. Such an I exists and continuous [4], where $C(c_x)$ is the path component of c_x in H_x. Thus, I is the identity of $\Gamma(X,H)$. Hence, we have the following result:

Result: 3. Since H_x is an H-cogroup, $\Gamma(X,H)$ is an H-cospaces with comultiplication v'.

Let S be family of supports on X and $V\subset X$. Then $\Gamma_{S|V}(V,H)$ is the collection of sections $s\in\Gamma(V,H)$ satisfying $|s|=\{x\in X:s(x)\notin C(c_x)\}$ where $S|V=\{A\subset V:A\in S\}$. The collection $\Gamma_{S|V}(V,H)$ is closed under the comultiplication of $\Gamma(X,H)$ which restricted to $\Gamma_{S|V}(V,H)$ for if $s\in\Gamma_{S|V}(V,H)$ then $|s|\in S|V$. Since

$$\left|v'_{S|V}(s)\right|^c=\left\{x\in X:v_{_{S|V}}(V,H)\in C(c_x)\right\}\supset|s|^c,$$

it follows that $\left|v'_{S|V}(s)\right|\subset|s|$ and so $\left|v'_{S|V}(s)\right|\in S|V$. Also $I\in\Gamma_{S|V}(V,H)$ (because I = $\{x\in X:I(x)\notin C(c_x)\}=\varnothing$) and hence $\Gamma_{S|V}(V,H)$ is an H-cospace.

Thus for any open subset U is of $X,\Gamma(X,H)$ is an H-cospace (Result 3). If V is another open subset of X such that $V\subset U$ we can define a map $\gamma^U_V:\Gamma(U,H)\to\Gamma(V,H)$ which is called restricted map, that is $\gamma^U_V(S)=S|V$.

If U,V and W are any three open subsets of X such that $W\subset V\subset U$ then one can observe that $\gamma^U_W=\gamma^V_W\circ\gamma^U_V$ and so $\left\{\Gamma(U,H),\gamma^U_V,X\right\}$ is called direct system [5].

3. The Characterizations

Let Q be any H-cogroup and X_1, X_2 be two topological spaces in Category C. Let H_1,H_2 be the corresponding sheaves which is given Result 2, respectively. Let us denote these as the pairs (X_1,H_1) and (X_2,H_2).

Definition 3. Let the pairs (X_1,H_1) and (X_2,H_2) be given. We say that there is a homomorphism between these pairs and write

$$F(\beta^*,\beta):(X_1,H_1)\to(X_2,H_2),$$

if there exist a pair $F(\beta^*,\beta)$ such that

1. $\beta:X_1\to X_2$ is a surjective and continuous map.

2. $\beta^*: H_1 \to H_2$ is a continuous map.

3. β^* preserves the stalks with respect to β. That is, the following diagram is commutative

$$\begin{array}{ccc} H_1 & \xrightarrow{\beta^*} & H_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ X_1 & \xrightarrow{\beta} & X_2 \end{array}$$

4. For every $x_1 \in X_1$ the restricted map $\beta^* \big| H_{x_1} : H_{1x_1} \to H_{2\beta(x_1)}$ is a homomorphism.

Theorem 3. Let the pairs (X_1, H_1) and (X_2, H_2) be given. If the map $\beta: X_1 \to X_2$ is surjective and continuous, then there exists a homomorphism between the pairs (X_1, H_1) and (X_2, H_2).

Proof. Let $x_1 \in X$ be arbitrarily fixed point. Then $\beta(x_i) \in X_2$ and

$$[Q:(X, x_1)] = H_{1x_1} \subset H_1, [Q:(X, \beta(x_1))] = H_{2\beta(x_1)} \subset H_2$$

are corresponding H-cogroups or stalks.

Since $(X_1, x_1), (X_2, \beta(x_1))$ are pointed topological spaces and f_1, g_1 are base-points preserving continuous maps from (Q, q_0) to (X_1, x_1) then there exists f_2, g_2 base-points preserving continuous maps from (Q, q_0) to $(X_2, \beta(x_1))$ can be defined as $f_2 = \beta \circ f_1$, $g_2 = \beta \circ g_1$, respectively. Moreover, if $f_1 \sim g_1 \, rel. q_0$, then it can be easily shown that $f_2 \sim g_2 \, rel. q_0$. Thus the correspondence $[f]_{x_1} \to [\beta \circ f]_{\beta(x_1)}$ is well-defined [6] and it maps homotopy classes of base-points preserving continuous maps from (Q, q_0) to (X_1, x_1), to the homotopy classes of base-points preserving continuous maps from (Q, q_0) to $(X_2, \beta(x_1))$. That is, to each element $[f]_{x_1}$ there corresponds a unique element $[\beta \circ f]_{\beta(x_1)}$.

Since the point $x_1 \in X$ is arbitrarily fixed, the above correspondence gives us a map $\beta^*: H_1 \to H_2$ such that $\beta^*([f]) = [\beta \circ f] \in H_2$, for every $[f] \in H_1$.

1) β^* is continuous. Because if $U_2 \subset H_2$ is any open set, then it can be shown that $\beta^{*-1}(U_2) = U_1 \subset H_1$ is an open set. In fact, if $U_2 \subset H_2$ is an open set, then $U_2 = \vee_{i \in I} s_i^2(W_i)$ and $\pi_2(U_2) = \vee_{i \in I} W_i$, where the W_i' s are open neighborhoods and the s_i^2 are sections over W_i. Thus $\bigcup_{i \in I} W_i \subset X_2$ is an open set and $\beta^{-1}\left(\vee_{i \in I} W_i\right) \subset X_1$ is an open set since β is a surjective and continuous map. Furthermore, since $\beta^{-1}(W_i)$, $i \in I$ are open in X_1, there exist sections

$$s_i^1 : \beta^{-1}(W_i) \to H_1$$

such that

$$\vee_{i \in I} s_i^1\left(\beta^{-1}(W_i)\right) \subset H_1$$

is an open set. It can be shown that

$$U_1 = \vee_{i \in I} s_i^1\left(\beta^{-1}(W_i)\right).$$

2. β^* preserves the stalks with respect to β. In fact, for any

$$[f]_{x_1} \in H_{1x_1} \subset H_1,$$

$$(\beta \circ \pi_1)\left([f]_{x_1}\right) = \beta\left(\pi_1\left([f]_{x_1}\right)\right) = \beta(x_1) = x_2$$

$$\left(\pi_2 \circ \beta^*\right)\left([f]_{x_1}\right) = \pi_2\left(\beta^*[f]_{x_1}\right) = \pi_2\left([\beta \circ f]_{x_2}\right) = x_2.$$

3.It can be easily shown that $\beta^*\big|H_{1x_1}$ is a homomorphism.

As a result $F=\left(\beta^*,\beta\right)$is a homomorphism.

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