On the Anti-Automorphism Of Mod-p Steenrod Algebra in the Language of the Structure of Combs

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Abstract: The method, named bundled and partitioned comb, intoroduced by Judith H.Silverman in [1] for the milnor basis elements in mod-2 Steenrod algebra. This method gives whether a given Milnor element $P(T)$ is a summand in product of anti-automorphisms of the Milnor elements $P(r)$ and $P(s)$, $[\chi(P(r))]\cdot[\chi(P(s))]$, without using Milnor product formula which will mention in section 4. We adopt some results about anti-automorphism into the mod-p Steenrod algebra.

Keywords : Steenrod Algebra, Milnor Basis, Anti-automorphism, Comb, Bundle.

Tarak Yapısı yardımıyla Mod-p Steenrod Cebirinin Anti-Otomorfizması Üzerine

Özet: Parçalanmış ve demetlenmiş taraklar metodu mod-2 Steenrod cebiri için J.H.Silverman tarafından verilmiştir [1]. Bu metod, verilen bir $P(T)$ Milnor elemanının diğer $P(r)$ ve $P(s)$ gibi iki Milnor elemanının anti-otomorfizmalarının $[\chi(P(r))]\cdot[\chi(P(s))]$ şeklindeki çarpımında bir bileşen olup olmadığını, bölüm 4 de yapısını verdiğimiz Milnor çarpım formülünü kullanmadan belirleyebilmektedir. Biz bu çalışmada J.H.Silverman'ın, anti-otomorfizma hakkında elde ettiği bazı sonuçları mod-p Steenrod cebirine genelleştirdik.

Anahtar Kelimeler : Steenrod cebiri, Milnor bazı, Anti-otomorfizma, Tarak, Demet.

Intoroduction

Let p an odd prime number. The Mod-p Steenrod algebra is formed by certain cohomology operations, called Steenrod Squares,

$$P^i : H^k(X;Z_p) \to H^{k+2i(p-1)}(X;Z_p)$$

where H^k is the *k*th cohomology group of *X* with coefficient Z_p and by Bockstein operations

$$\beta : H^q(X;Z_p) \to H^{q+1}(X;Z_p),$$

associated with the exact sequence $0 \to \mathrm{Z_p} \to \mathrm{Z_{p^2}} \to \mathrm{Z_p} \to 0$ with the property

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$$\beta^2 = 0 \text{ and } \beta(\mathrm{x.y}) = \beta(\mathrm{x})\mathrm{y} + (\text{-}1)^{\mathrm{q}} x\beta(y),\ \dim(\mathrm{x}) = \mathrm{q}\,.$$

Adem [2], Cartan [3] and Serre [4] gave the structure of this algebra. The bockstain doesn't have any role in this work, so we will study on the sub-algebra generated only by the elements P^T. But we will continue to use the name Steenrod algebra for this sub-algebra.

There are several bases which are called Admissible basis, Milnor basis, Arnon bases (two bases), Wall basis e.t.c. in the mod-p Steenrod algebra. The structure of Milnor basis was given by John Milnor [5]. The mod-p Steenrod algebra structure in Milnor basis is given by the Milnor product formula. The bundled and partitioned comb method was constructed J.H.Silverman [1] by using the properties dimension, excess of the Milnor elements in the mod-2 Steenrod algebra. One of the consequences of this method is that whether a given Milnor element $P(T)$ is a summand in product of anti-automorphisms of the milnor elements $P(r)$ and $P(s)$, $[\chi(P(r))]\cdot[\chi(P(s))]$, without using the Milnor product formula. We generalize her some results into mod-p Steenrod algebra.

Preliminaries

Let *R* be a field and $M = (M_i)$ where $i \geq 0$ be a sequence of *R*-vector spaces. Then *M* is called ***graded vector space over R*** and M_i has ***degree*** *i* for all *i*. By a ***graded algebra*** (M_*, φ_*) is meant a graded vector space M_* together with a homomorphism $\varphi_* : M_* \otimes M_* \to M_*$ and it is usually required that φ_* be associative and have a unit element $1 \in M_0$. The graded algebra is ***connected*** if the vector space A_0 is generated by 1. By a ***connected Hopf algebra*** $(M_*, \varphi_*, \gamma_*)$ is meant a connected graded algebra with a homomorphism $\gamma_* : M_* \to M_* \otimes M_*$ satisfying the following two conditions; γ_* is a homomorphism of algebras with unit and for $\dim a > 0$, the element $\gamma_*(a)$ has the form $a \otimes 1 + 1 \otimes a + \sum b_i \otimes c_i$ with $\dim b_i, \dim c_i > 0$. If $(M_*, \varphi_*, \gamma_*)$ is a Hopf algebra there is a homomorphism $\chi : M \to M$ defined by the properties:

(1) $\chi(1) = 1$

(2) If $\gamma_*(a) = \sum a_i' \otimes a_i''$ where $\dim a > 0$, then $\sum a_i' \otimes \chi(a_i'') = 0$ [7].

Let $T = (t_1, t_2, \ldots)$ be a sequence of non negative integers almost all of which are zero and the Milnor element associated with this sequence is *P(T)*. If *T* is a sequence for which $t_l = 0$, for $l > m$, we denote the corresponding basis element by $P(T) = P(t_1, t_2, \ldots, t_m)$. The ***dimension*** $|P(T)|$ of the Milnor element *P(T)* is $2|T|$ where $|T| = \sum_{k=1}^{m} (p^k - 1)t_k$. In [6] Kraines showed that the ***excess*** $ex(P(T))$ of Milnor element associated with sequence $T = (t_1, t_2, \ldots, t_m)$ is $2ex(T)$ where $ex(T) = \sum_{k=1}^{m} t_k$ and the excess of a sum of Milnor basis elements is the minimum excesses of the summands.

It is showed in [7] that the mod-p Steenrod algebra is a connected Hopf algebra so there is a unique homomorphism χ satisfying (1) and (2) conditions given above. χ is an antiautomorphism in the sense that $\chi(P(R)P(S)) = \chi(P(R))\chi(P(S))$ and χ is one-to-one and onto. So it carries basis elements of the Steenrod algebra into new basis elements of the algebra. For a non-negative integer *n* let $S(n)$ denote the sum of all Milnor basis elements of the form $P(R)$ in dimension *n* and in [5], Milnor proves that $\chi(P(n)) = (-1)^n S(2n(p-1))$.

For Milnor basis, it is known how to express of two generators as a sum of other generators by Milnor product formula. This product formula involve binomial or multinomial coefficients taken mod-*p*. There are several criterion to compute this coefficient; let $p^{n_\sigma} \cdots p^{n_2} \cdot p^{n_1}$ and $p^{r_\sigma} \cdots p^{r_2} \cdot p^{r_1}$ be the *p*-adic represantation of the integers *n* and *r* respectively. We write $n >_i r$ to mean $n_i \geq r_i$ and we say *n* dominates *r* $(n > r)$ if $n >_i r$ for all *i*. It is known that the coefficient

$\binom{n}{r} \neq o \,(\mathrm{mod}\, p) \Leftrightarrow n > r (n > n - r)$. In other words, each power of *p* appearing in the *p*-adic representation of *n* appears in exactly one of the p-adic representations of *r* and *n-r*. More genarally, if $m \geq 3$ and $\sum_{i=1}^{m} r_i = n$ the multinomial coefficient giving the number of ways to divide a set of *n* elements into *m* subsets of orders $r_1, \ldots, r_m$ is written

$$\binom{n}{r_1 | r_2 | \ldots | r_m} = \binom{s_1}{r_1} \cdot \binom{s_2}{r_2} \cdot \binom{s_3}{r_3} \cdots \binom{s_m}{r_m}$$

where $s_l = r_l + r_{l+1} + \cdots + r_m$ and $1 \leq l \leq m$.

The criterion mentioned above and inductive argument imply that $\binom{n}{r_1 | r_2 | \ldots | r_m} \neq 0 \; (\mathrm{mod}\, p) \Leftrightarrow$ each power of *p* in the *p*-adic representation of *n* occurs in exactly one of the *p*-adic representation of $r_1, \ldots, r_m$.

For more details about the mod p Steenrod Algebra see [8]. Now, to proove main results of this paper let define the structures on combs as given in [9].

Structures on Combs

Combs

We interpret $|T| = \sum_{k=1}^{m} (p^k - 1) t_k$ as the value of the sequence $T = (t_1, t_2, \ldots, t_m)$ in the system in which the *l*-th term counts for $p^l - 1$ times its face value. Since $p^l - 1 = p^* (p^{l-1} + p^{l-2} + \cdots + p^1 + p^0)$ with $p^* = p - 1$ we can represent $|T|$ in a different way ;

$$\begin{aligned}
|T| &= (p^1 (p^1 - 1) t_1 - 1) t_1 + (p^2 - 1) t_2 + \cdots + (p^m - 1) t_m \\
&= p^* t_1 + p^* (p^1 + p^0) t_2 + \cdots + p^* (p^{m-1} + p^{m-2} + \cdots + p^1 + p^0) t_m \\
&= p^* p^0 t_1 + p^* p^0 t_2 + p^* p^0 t_3 + \cdots + p^* p^0 t_m \\
&\qquad + p^* p^1 t_2 + p^* p^1 t_3 + \cdots + p^* p^1 t_m \\
&\qquad\qquad + p^* p^2 t_3 + \cdots + p^* p^2 t_m \\
&\qquad\qquad\qquad \vdots \\
&\qquad\qquad\qquad + p^* p^{m-1} t_m
\end{aligned}$$

Therefore we can represent $|T|$ with the picture below where

i-th row is associated with p^i for all $i = 0, 1, 2, \ldots, m-1$

$$\begin{array}{cccccc}
\overbrace{p^*p^*\ldots p^*}^{l_1\text{ times}} & \overbrace{p^*p^*\ldots p^*}^{l_2\text{ times}} & \cdots & \overbrace{p^*p^*\ldots p^*}^{l_m\text{ times}} & \to & p^0 \\
 & p^*p^*\ldots p^* & \cdots & p^*p^*\ldots p^* & \to & p^1 \\
 & & \ddots & \vdots & & \\
 & & & p^*p^*\ldots p^* & \to & p^{m-1}
\end{array}.$$

This picture, or any obtained from it by a permutation of colums, will be called the ***comb of T*** and denoted **C(T).** A column of l p^*'s is called ***a tooth of length l,*** denoted τ^l**,** and its ***weight*** is $W(T)=p^*\sum_{k=1}^{l-1}p^k=p^l-1$. The ***excess of C(T)*** is the number of teeth which is equal to $ex(T)$ and the ***weight of C(T)***, W(C(T)), is the sum of the weights of the teeth which is equal $|T|$.

Example : Let $p=5$ and find the comb *C(T)* for the sequence $T=(4,3,2)$;

$$|T|=(p^1-1)\cdot 4+(p^2-1)\cdot 3+(p^3-1)\cdot 2$$
$$=p^*\cdot p^0\cdot 4+p^*\cdot p^0\cdot 3+p^*\cdot p^1\cdot 3+p^*\cdot p^2\cdot 2+p^*\cdot p^1\cdot 2+p^*\cdot p^0\cdot 2$$

so the picture of the comb *C(T)* is

$$\begin{array}{ccccccccc}
p^* & p^* & p^* & p^* & p^* & p^* & p^* & p^* & p^* \\
 & & & & p^* & p^* & p^* & p^* & p^* \\
 & & & & & & & p^* & p^*
\end{array}.$$

Bundles

A ***bundle of size*** P^σ is a collection of P^σ teeth of the same length and represented in column form as a sort of generalized tooth having same number of p^*'s as the teeth including it but preceded by σ times zeros as the teeth.

Example : Let $p=5$. The bundle of 5^2 teeth of length 3 (according to the definition above) is

$$\begin{array}{c} 0 \\ 0 \\ p^* \\ p^* \\ p^* \end{array}$$

The sum of the weights of 25 teeth of length 3 is $25(5^3-1)$. The 5-adic representation of this number is ;

$$25(5^3-1)=3100=0\cdot 5^0+0\cdot 5^1+4\cdot 5^2+4\cdot 5^3+4\cdot 5^4 \qquad , \ p^*=5-1$$

If we represent the coefficients of this representation vertically

$$\begin{array}{c} 0 \\ 0 \\ p^* \\ p^* \\ p^* \end{array}.$$

we'll have the bundle of 5^2 teeth of length 3 again. From this point, the number of the form $p^{\sigma}(p^{l}-1)$ identified as a bundle of p^{σ} teeth of length *l*.

Let $T=(t_1,t_2,\ldots,t_m)$ be a sequence. We can write $t_l\ ,1\leq l\leq m,$ as sums of powers of *p* and this writing gives rise a bundle structure on *C(T)* : if

$$t_l=\alpha_{n_1}(t_l)\cdot p^{n_1,l}+\alpha_{n_2}(t_l)\cdot p^{n_2,l}+\cdots+\alpha_{n_s}(t_l)\cdot p^{n_s,l}$$

then the teeth of length *l* are arranged in bundles of sizes $p^{n_1,l},p^{n_2,l},\ldots,p^{n_s,l}$. The orders of the bundles is not important. The comb having the bundles with cofficient $\alpha_{n_1}(t_l),\alpha_{n_2}(t_l),\ldots,\alpha_{n_s}(t_l)$ as columns is called ***canonically bundled comb of T*** and denoted $C_b(T)$.

Example : Let $\mathrm{p}=5$ and find canonically bundled comb $C_b(T)$ of $T=(13,25,10,4)$;

$$\begin{aligned}|T|&=(5^1-1)\cdot 13+(5^2-1)\cdot 25+(5^3-1)\cdot 10+(5^4-1)\cdot 4\\&=(5^1-1)\cdot(3\cdot 5^0+2\cdot 5^1)+(5^2-1)\cdot(1\cdot 5^2)+(5^3-1)\cdot(2\cdot 5^1)+(5^4-1)\cdot(4\cdot 5^0)\\&=3\cdot 5^0\cdot(5^1-1)+2\cdot 5^1\cdot(5^1-1)+1\cdot 5^2\cdot(5^2-1)+2\cdot 5^1\cdot(5^3-1)+4\cdot 5^0\cdot(5^4-1)\end{aligned}$$

therefore the picture of the canonically bundled comb of *T* is below ;

$$\begin{array}{ccccc} 0 & 3\cdot p^* & 0 & 0 & 4\cdot p^* \\ 2\cdot p^* & & 0 & 2\cdot p^* & 4\cdot p^* \\ & & 1\cdot p^* & 2\cdot p^* & 4\cdot p^* \\ & & 1\cdot p^* & 2\cdot p^* & 4\cdot p^* \end{array}.$$

We can find same information about *P(T)* in $C_b(T)$ as does the comb $C(T)$: A bundle of teeth p^{σ} of length *l* has its topmost p^* in the σth row and the number $\alpha_{\sigma}(t_l)\cdot p^{\sigma,l}\cdot(p^{l}-1)$ is called ***weight of this bundle***. The sum of the numbers $\alpha_{\sigma}(t_l)\cdot p^{\sigma,l}$ is the ***excess of*** $C_b(T)$ which equals to $ex(T)$ and the sum of the weights of bundles in the $C_b(T)$ is ***the weight of*** $C_b(T)$ which equals to $|T|$.

Example: Let $p=5$**,** $T=(13,25,10,4)$ and find $ex(T)$ and $|T|$; The topmost rows ,in which the p^*'s are seen first, and the coefficient, which arise in the *p*-adic reprsentation of t_l for all $1\leq l\leq m$, are found easily from the picture of $C_b(T)$ above. So,

$$ex(T)=2\cdot p^1+3\cdot p^0+1\cdot p^2+2\cdot p^1+4\cdot p^0=52$$

$$|T|=1\cdot p^1\cdot(5^1-1)+3\cdot p^0\cdot(5^1-1)+1\cdot p^2\cdot(5^2-1)+2\cdot p^1\cdot(5^3-1)+4\cdot p^0(5^4-1)=4388$$

Partitions

We'll define the ***partitioned comb***, $PC(T)$, as a comb whose each tooth τ is split in two horizantally by choosing a partition number $0\leq\pi(\tau)\leq l$ indicating that the tooth is to be split above the $\pi(\tau)$-th row. Grafically we represent the partition number of each tooth as a horizantal partition line across the tooth.

Example : A partition can be given as below on $C(T)$ of the sequence $T=(4,3,2)$;

$$\begin{array}{ccccccccc} \overline{p^*} & \overline{p^*} & \underline{p^*} & \overline{p^*} & \underline{p^*} & p^* & \underline{p^*} & \overline{p^*} & p^* \\ & & & & p^* & \underline{p^*} & p^* & p^* & \underline{p^*} \\ & & & & & & & p^* & p^* \end{array}.$$

We can also costruct bundle structure on the partitioned comb too as before. The generalized teeth of a ***bundled partitioned comb***, denoted $PC_b(T)$**,** represent p^{σ} teeth with the same length and same partition number . However, we can give a partition to a bundled comb such a way that each generalized tooth τ of size p^{σ} and length *l* is assigned a partition number $0 \le \pi(\tau) \le l$ indicating that the generalized tooth is to be split obove the $(\sigma + \pi(\tau))$-th row.

Compatability

Let *T* be a sequence. A partition of the comb C(T) is ***compatible*** with a bundle structure on $C_b(T)$, if one can indicate the partition on a picture of the bundled comb.That is, given integers $0 \le \pi \le l$, let N_{π}^{l} be the number of teeth of length *l* by the partition structure to have partition number π. The partition is compatible with the bundle structure if the generalized teeth of each length *l* in the bundled comb can be arranged in *l* groups in such away that the number of ordinary teeth represented in the π-th group is N_{π}^{l}. Each generalized tooth in the π-th group is then assigned the partition number π.

Example : Let $p = 5$ and $T = (0,7)$ then the partition of *C(T)* given by

$$\begin{array}{ccccccc} \overline{p^*} & \underline{p^*} & \underline{p^*} & \underline{p^*} & \underline{p^*} & p^* & p^* \\ p^* & p^* & p^* & p^* & p^* & \underline{p^*} & \underline{p^*} \end{array}$$

is compatible with the bundle structure below, with the assigment of partition number to teeth as indicated,

$$\begin{array}{ccc} \overline{p^*} & \underline{4p^*} & 2p^* \\ p^* & 4p^* & \underline{2p^*} \end{array}$$

but is not compatible with the bundle structure below

$$\begin{array}{cc} 0 & 2p^* \\ p^* & 2p^* \\ p^* & \end{array}.$$

If we change the partition of *C(T)* as indicated below

$$\begin{array}{ccccccc} \underline{p^*} & \underline{p^*} & \underline{p^*} & \underline{p^*} & \underline{p^*} & p^* & p^* \\ p^* & p^* & p^* & p^* & p^* & \underline{p^*} & \underline{p^*} \end{array}$$

then this partition will be compatible with the bundle structure below

$$\begin{array}{cc} 0 & 2p^* \\ \underline{p^*} & \underline{2p^*} \\ p^* & \end{array}$$

The ***first part*** (resp. ***second part***) of a partitioned bundled comb of a sequence *T*, is the bundled comb obtained by replacing all the p^*'s of *C(T)* below (resp. above) the partition lines with blanks (resp. 0's).

Milnor Product Formula

Let $R = (r_1, r_2, \mathrm{K}, r_m)$, $S = (s_1, s_2, \mathrm{K}, s_n)$ and $T = (t_1, t_2, \mathrm{K}, t_v)$ be sequences of non negative integers with $|R| + |S| = |T|$. Now we'll find the condition if P(T) is a summand in the product $\mathrm{P(R)} \cdot \mathrm{P(S)}$. In [5] Milnor describes the product in terms of certain matrices :

Let $M_{R,S}$ be the set of infinite matrices

$$X = \begin{pmatrix} * & x_{01} & x_{02} & \Lambda \\ x_{10} & x_{11} & x_{12} & \Lambda \\ x_{20} & x_{21} & x_{13} & \Lambda \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{O} \end{pmatrix}$$

of non negative integers with $x_{00} = *$ such that

$$\sum_{j=0}^{\infty} p^j x_{ij} = r_i \quad \text{for all } i \geq 1 \qquad (1)$$

$$\sum_{i=0}^{\infty} x_{ij} = s_j \quad \text{for all } j \geq 1. \qquad (2)$$

For each matrix $X \in M_{R,S}$, define the sequence $T(X) = (t_1, t_2, \mathrm{K})$ by $t_l = \sum_{i=0}^{l} x_{i,l-i}$ and let $b_l(X)$ be multinomial coefficient

$$\begin{pmatrix} n \\ x_{0,l} \big| x_{1,l-1} \big| x_{2,l-2} \big| \Lambda \ \big| x_{l,0} \end{pmatrix}.$$

Then the product $\mathrm{P(R)} \cdot \mathrm{P(S)}$ is given by

$$\mathrm{P(R)} \cdot \mathrm{P(S)} = \sum_{X \in M_{R,S}} [b_1(X) b_1(X) \mathrm{L}\ ] \quad P(T(X))$$

Thus ;

P(T) is a summand of $\mathrm{P(R)} \cdot \mathrm{P(S)} \Leftrightarrow \mathrm{a(T)} \neq 0 \pmod{p}$ and $b_l(X) \neq 0 \ (\mathrm{mod}\, p)$ for al *l*

where $a(T)$ is the number of matrices $X \in M_{R,S}$ with $T(X) = T$.

Rather than trying to construct such matrices mentioned in the product formula one by one, it is often advantageous to translate the question into the language of combs.

Now our goal is to find the condition if *P(T)* is a summand in the product $\mathrm{P(R)} \cdot \mathrm{P(S)}$ by using the bundled and partitioned comb structures. To be able to do this we must translate the Milnor formula into the language of bundle and partition structures.

Fix R and S and suppose $X \in M_{R,S}$ with $T(X) = T$. The $P_X C(T)$ is the partitioned comb induced with the matrix *X* which for all i,j has x_{ij} teeth of length *i+j* and partition number *j*, The matrix *X* is associated not only to the comb *C(T)* but also to the partitioned comb $P_X C(T)$.

Example : Let $p = 5$, $R = (12,32),\ S = (5,6)$ and the matrix

$$X = \begin{pmatrix} * & 3 & 5 \\ 7 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

With $T(X) = (10,8,1,1)$.

Is the Milnor element *P(T)* a summand of the Milnor product $\mathrm{P(R)} \cdot \mathrm{P(S)}$?

Firstly, let's answer this question by using formula described above : Since the first multinomial coefficient in the product $b_1 \equiv 0 \ (\mathrm{mod}\, 5)$ the answer of the question is no.

Now let's answer the question by the bundle and partition structures. The comb $P_X C(T)$ is

$$\begin{array}{cccccccccccccccccccc}
\overline{p^*} & \overline{p^*} & \overline{p^*} & \overline{p^*} & \overline{p^*} & \overline{p^*} & \overline{p^*} & \underline{p^*} & \underline{p^*} & \underline{p^*} & \overline{p^*} & \overline{p^*} & \underline{p^*} & p^* & p^* & p^* & p^* & p^* & \underline{p^*} & p^* \\
 & & & & & & & & & & p^* & p^* & p^* & \underline{p^*} & \underline{p^*} & \underline{p^*} & \underline{p^*} & \underline{p^*} & p^* & \underline{p^*} \\
 & & & & & & & & & & & & & & & & & & p^* & p^* \\
 & & & & & & & & & & & & & & & & & & & p^*
\end{array}$$

and bundled partitioned comb $P_X C_b(T)$ is

$$\begin{array}{cccccccc}
\underline{0} & 2\overline{p^*} & 3\underline{p^*} & 2\overline{p^*} & \underline{p^*} & 0 & \underline{p^*} & p^* \\
p^* & & & 2p^* & p^* & p^* & p^* & \underline{p^*} \\
 & & & & & \underline{p^*} & p^* & p^* \\
 & & & & & & & p^*
\end{array}$$

But the partition on $P_X C_b(T)$ is not compatible with the bundle structure on $C_b(T)$ below

$$\begin{array}{ccccc}
0 & 0 & 3p^* & p^* & p^* \\
2p^* & p^* & 3p^* & p^* & p^* \\
 & p^* & & p^* & p^* \\
 & & & & p^*
\end{array}.$$

Because, the 1st tooth of length 1 in $C_b(T)$ can not be arranged in three groups such that each group will be the one of the teeth of length 1 in $P_X C_b(T)$

If we examine the comb $P_X C(T)$ carefully we can se that the first part of it is the $C(S)$, after arranging the teeth up to their lengths,

$$\begin{array}{ccccccccccc}
p^* & p^* & p^* & p^* & p^* & p^* & p^* & p^* & p^* & p^* & p^* \\
 & & & & p^* & p^* & p^* & p^* & p^* & & p^*
\end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccccccc}
p^* & p^* & p^* & p^* & p^* & p^* & p^* & p^* & p^* & p^* & p^* \\
 & & & & & p^* & p^* & p^* & p^* & p^* & p^*
\end{array}$$

and the second part of it is the $C_b(R)$, after arranging bundles up to their length and mod p,

$$\begin{array}{cccccccccccc}
p^* & p^* & p^* & p^* & p^* & p^* & p^* & 0 & p^* & p^* & 0 & 0 \\
 & & & & & & & p^* & p^* & p^* & p^* & 0 \\
 & & & & & & & & & & p^* & p^* \\
 & & & & & & & & & & & p^*
\end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccc}
0 & 2p^* & 2p^* & 0 & 0 \\
p^* & & 2p^* & p^* & 0 \\
 & & & p^* & p^* \\
 & & & & p^*
\end{array}$$

It is the fact that, we view $P_X C(T)$ as the result of suspending bundles of teeth of $C(R)$ below the teeth of $C(S)$ of appropriate of length. That is, a tooth of length $i+j$ and partition number *j* is obtained from a tooth of $C(S)$ of length *j* by appending a bundle of p^j teeth of length *i* of $C(R)$. Equations (1) and (2) imply that the first part of $P_X C(T)$ is exactly the comb $C(S)$ and second part of $P_X C(T)$ is a bundled comb for *R*. Conversely, any partition of $C(T)$ whose first and second parts are combs for *S* and *R* respectively is readily seen to be $P_Y C(T)$ for some $Y \in M_{R,S}$ with $T(Y) = T$.

Multiplication and the Anti-Automorphism

Let *n(T)* denote the number of partitions of canonical bundled comb $C_b(T)$ whose first parts are combs for *S* and whose second parts are bundled combs for *R.* A theorem given in [9] is as below :

Theorem 1 : The Milnor element *P(T)* is a summand in the product $\mathrm{P(R)} \cdot \mathrm{P(S)} \Leftrightarrow \mathrm{n(T)} \neq 0$ (mod p).

Proof : From the properties of Milnor product we know that *P(T)* is a summand of $\mathrm{P(R)} \cdot \mathrm{P(S)} \Leftrightarrow \mathrm{a(T)} \neq 0 \pmod p$ and $b_l(X) \neq 0 \pmod p$ for all *l* where $a(T)$ is the number of matrices $X \in M_{R,S}$ with $T(X) = T$. As discussed in section 2 $b_l(X) \neq 0 \pmod p$ when each power of *p* in the *p*-adic representation of the $t_l = \sum_{i=0}^{l} x_{i,l-i}$ occurs in the binary representation of exactly one of the $x_{i,l-i}$. But these powers of *p* are exactly the sizes of the generalized teeth of length *l* in canonical bundle comb $C_b(T)$. Therefore the above condition may be rephrased as the requirement that the generalized teeth of length *l* of $C_b(T)$ can be divided into *l* groups in such a way that the sizes of the teeth in the *i*-th group add up to $x_{i,l-i}$. This is the case for for all *l* $\Leftrightarrow$ the partition structure of $P_X C(T)$ is compatible with the canonical bundle structure on $C(T)$. Accordingly, the number *a(T)* is exactly the number of partition of canonical bundle comb $C_b(T)$ whose first parts are combs for *S* and whose second parts are bundled combs for *R.* That is, $a(T) = n(T)$.

Recall that χ denote the automorphism of mod-p Steenrod algebra and let *r* and *s* be integers. Recall that it is known how to express the product of two generators, especially $\chi(P(r))$ and $\chi(P(s))$, as a sum of other generators. However, the comb method lends itself particularly well to identifying terms of the product $[\chi(P(r))] \cdot [\chi(P(s))]$.

Now we can proove the main theorem of this paper :

Theorem 2 : The Milnor element $P(T)$ is a summand of the product $[\chi P(r)] \cdot [\chi P(s)] \Leftrightarrow$ there is a number, which is not equal to zero in mod-p, of partitions of the canonical comb $C_b(T)$ whose first parts have weight $2s(p-1)$ and second parts have weight $2r(p-1)$.

Proof : We remember from section 2 that for any non-negative integer *r*,

$$\chi(P(r)) = (-1)^r \sum P(R)$$

where the sum extends over all Milnor basis elements $P(R)$ having the dimension $|P(R)| = 2r(p-1)$ and similarly, for a non-negative integer *s*, we have $\chi(P(s)) = (-1)^s \sum P(S)$ for the Milnor element $P(s)$. Then we have

$$\chi(P(r)) \cdot \chi(P(s)) = (-1)^{r+s} \sum P(R) \cdot P(S).$$

Thus it follows from the Milnor product that an element $P(T)$ of dimension $2(r+s)(p-1)$ is a summnad of $[\chi(P(r))]\cdot[\chi(P(s))] \Leftrightarrow \sigma(T)\neq 0$ (mod-p) where $\sigma(T)$ is the number of pairs $(P(R),P(S))$ of Milnor elements which have dimensions $2r(p-1)$ and $2s(p-1)$ respectively $\overset{th1}{\Leftrightarrow}$ there is a number, which is not equal to zero in mod-p, of partitions of the canonical comb $C_b(T)$ whose first parts have weight $2s(p-1)$ and second part have weight $2r(p-1)$.

Product of n-times Milnor elements

In this section we'll characterize the Milnor element which appear as summand in a product $P(R_n)\cdot P(R_{n-1})\cdot\Lambda\cdot P(R_1)$. For $n\geq 2$ the ***n-partitioned comb K*** is one in which each tooth is divided into n parts, some possibly empty, by n-1 horizantal lines. That is, to each generalized tooth τ of length *l* and size p^{σ} is assigned an (n-1)-tuple of integers with $0\leq\pi_1(\tau)\leq\pi_2(\tau)\leq\Lambda\leq\pi_{n-1}(\tau)\leq l$. Let $\pi_0=0$ and $\pi_n=l$ for all τ. The ***i-th part of K***, $1\leq i\leq n$, is obtained by replacing all the p^*'s in each τ except those in rows $\sigma+\pi_{i-1}(\tau)$ through $\sigma+\pi_i(\tau)-1$ with 0's. Thus 2-partitions are the familiar partitions mentioned in the subsection compatability.

Example : A Picture of 5-partitioned bundled comb and its 4-th part are pictured below.

$$\begin{array}{ccccc} p^* & 0 & 0 & p^* & 0 \\ p^* & 0 & p^* & & 0 \\ p^* & p^* & p^* & & 0 \\ & p^* & & & 0 \\ & & & & p^* \end{array} \qquad \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ p^* & p^* \end{array}.$$

Now we can establish the following theorems, 3 (given in [9]) and 4, which can be proved by induction on *n*. For a given sequence *T* let *m(T)* be the number of n-partitions of $C_b(T)$ whose *i*-th parts are (bundled) combs for R_i for all $1\leq i\leq n$.

Theorem 3 : The Milnor basis element *P(T)* is a summand of a product $P(R_n)\cdot P(R_{n-1})\cdot\Lambda\cdot P(R_1)\Leftrightarrow m(T)\neq 0 \pmod p$.

Let $r_1,r_2,\mathrm{K}\,,r_n$ be non-negative integers and for a given sequence *T* let *k(T)* be the number of n-partitions $C_b(T)$ such that the *i*-th parts are (bundled) combs of weight $2r_i(p-1)$ for all $1\leq i\leq n$.

Theorem 4 : The Milnor basis element *P(T)* is a summand of a product $\chi(P(r_n))\cdot\chi(P(r_{n-1}))\cdot\mathrm{K}\cdot\chi(P(r_1))\Leftrightarrow k(T)\neq 0 \pmod p$.

Conclusion

İn this paper we state two important theorems, which gives whether a given Milnor element $P(T)$ is a summand in product of anti-automorphisms of the Milnor elements $P(r)$ and $P(s)$ (or $P(r_1),P(r_2),\mathrm{K}\,,P(r_n)$), $[\chi(P(r))]\cdot[\chi(P(s))]$ (or $\chi(P(r_n))\cdot\chi(P(r_{n-1}))\cdot\mathrm{K}\cdot\chi(P(r_1))$), without using Milnor product formula, about anti-automorphism of mod-p Steenrod algebra. Namely theorems 2 and 4.

References :

[1] Silverman, J.H**., Multiplication and the combinatorics in the Steenrod Algebra**, *Journal of Pure and Applied Algebra*. 111 (1996) 303-323

[2] Adem, J.,**The iteration of the Steenrod Squares in Algebraic Topology**,*Proc. Nat. Acad. Sci. U.S.A*. 38 (1952) 720-726.

[3] Cartan, H**., Sur l'iteration des Operations de Steenrod**, *Comm, Math. Helvet.* 29 (1955) 40-58.

[4] Serre, J.P**., Cohomologie Modulo 2 des complexes d'Eilenberg- Maclane**, *Comm. Math. Helvet*. 29 (1956) 198-232.

[5] Milnor, J., **The Steenrod Algebra and its Dual**, *Ann. Of Math*. 67 (1958) 150-171.

[6] Kraines, D., **On Excess in the Milnor Basis**, Bull. London Math. Soc. 3(1971)363-365.

[7] Milnor, J. and Moore, J., **On the Structure of Hopf Algebra**, *Ann. of Math*. 81 (1965) 211-264.

[8] Steenrod, N.E. ve Epstein, D .B.A., **Cohomology Operations**, Princeton University Press**,** (1962).

[9] Tanay, B**., The Bundled and partitioned Comb in the Mod-p Steenrod Algebra**, *Accepted to press in Int. Journal of Pure and Aplied Algebra.*