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On Two Efficient Numerical Schemes for Nonlinear Burgers' Type Equations

Bilge İNAN[1,*]

[1]*İskenderun Technical University, Faculty of Engineering and Natural Sciences, Department of Engineering Basic Sciences, 31200, Hatay, Türkiye*
bilge.inan@iste.edu.tr, ORCID: 0000-0002-6339-5172

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Abstract

In this work, we investigate two finite difference schemes to solve nonlinear Burgers' type equations. In the first stage, we define the numerical methods to solve the equations. Secondly, numerical solutions are obtained and compared with the exact solutions. In comparison with other defined results in the literature, it is deduced in a conclusive way that the methods are reliable and convenient alternative methods for solving nonlinear Burgers' type equations.

Keywords: The Burgers' type equations; The generalized Burgers-Huxley equation; Finite difference method; Logarithmic finite difference method.

Lineer Olmayan Burgers Tip Denklemler İçin Etkili İki Nümerik Yöntem Üzerine

Öz

Bu çalışmada, iki sonlu fark yöntemi kullanarak lineer olmayan Burgers tipi denklemlerin çözümleri incelenmiştir. İlk aşamada, denklemleri çözmek için nümerik yöntemler tanımlanmıştır. Daha sonra, nümerik çözümler elde edilmiş ve tam çözümlerle karşılaştırılmıştır. Literatürde tanımlanmış diğer sonuçlarla karşılaştırıldığında, yöntemlerin lineer olmayan Burgers tipi denklemlerin çözümü için güvenilir ve uygun alternatif yöntemler olduğu sonucu kesin bir şekilde elde edilmiştir.

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Anahtar Kelimeler: Burgers tip denklem; Genelleştirilmiş Burgers-Huxley denklemi; Sonlu fark yöntemi; Logaritmik sonlu fark yöntemi.

1. Introduction

In this manuscript, we consider the following initial-boundary-value problem:

$$\frac{\partial u}{\partial t} + \alpha u^{\delta} \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u\left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right), \; 0 \leq x \leq 1, \; t \geq 0 \tag{1}$$

with the following initial condition

$$u(x, 0) = f(x) \tag{2}$$

and the boundary conditions

$$u(0, t) = g_1(t) \text{ and } u(1, t) = g_2(t) \tag{3}$$

where $\alpha,\ \beta,\ \gamma$ and δ are parameters that $\beta \geq 0,\ \delta > 0,\ \gamma \epsilon (0,1)$. The equation is called the generalized Burgers-Huxley equation.

Nonlinear partial differential equations are generally obtained while modeling the problems in various fields like physics, chemistry, biology, mathematics and engineering. The generalized Burgers-Huxley equation which is a study in this paper is one of the nonlinear partial differential equations.

In the literature, many numerical methods have been defined for numerical solutions to the generalized Burgers-Huxley equation. Hashim et al. [1] used the Adomian decomposition method for solving the equation. Javidi [2, 3] applied the collocation method to solve the generalized Burgers-Huxley equation. Spectral collocation method for solving the equation was applied by Darvishi et al. [4]. Batiha et al. [5] presented the variational iteration method for the generalized Burgers-Huxley equation. Numerical solution of the equation was obtained by Sari and Gürarslan [6] using a polynomial differential quadrature method. A numerical method called Kansa's approach based on the collocation method using Radial basis functions was presented by Khattak [7] for the solution of the equation. The spectral collocation method uses Chebyshev polynomials for spatial derivatives and fourth order Runge-Kutta method for time integration to solve the equation applied by Javidi and Golbabai [8]. The differential transform method was used for the solution of the generalized Burgers-Huxley equation by Biazar and Mohammadi [9]. Bratsos [10] defined a fourth order finite-difference method in a two-time level recurrence relation for the generalized Burgers-Huxley equation. Çelik [11] solved the equation. El-Kady et al. [12] applied cardinal Legendre and Chebyshev basis functions with the Galerkin method for the solution of the equation. Al-Rozbayani [13] used the discrete Adomian decomposition method for the solution of the generalized Burgers-Huxley equation.

When $\alpha = 0$, Eqn. (1) is reduced to the generalized Huxley equation. The generalized Huxley equation describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals.

In literature, various numerical methods have been proposed by researchers for the numerical solutions of the generalized Huxley equation. The Adomian decomposition method for the numerical solution of the equation proposed by Hashim et al. [14]. The equation was solved numerically using the variational iteration method by Batiha and coworkers [15]. Hashemi et al. [16] applied the homotopy perturbation method and the Adomian decomposition method to solve the generalized Huxley equation. Tenth-order finite difference methods to obtain approximation solution of the generalized Huxley equation proposed by Sari et al. [17]. Hemida and Mohamed [18, 19] used the homotopy analysis method for solving the generalized Huxley equation. İnan [20] used an implicit exponential finite difference method for the numerical solution of the equation. Also, numerical solutions of the generalized Huxley and generalized Burgers Huxley equations were obtained by using explicit exponential finite difference methods by İnan [21]. Also, there are various numerical methods for the solutions of the Burgers type equations in the literature [32, 33].

On the other hand, the logarithmic finite difference method has been used by some authors. This method was obtained by getting inspired by the exponential finite difference method and used by El Morsy and El-Azab for the first time [22] in 2012 and they used the logarithmic finite difference method for the solution of the KdVB equation. Srivastava and coworkers [23, 24] proposed an implicit logarithmic finite difference method for numerical solutions of the one and two-dimensional coupled viscous Burgers' equations. Also, Srivastava et al. [25] used the method for the numerical solution of the two-dimensional unsteady nonlinear coupled viscous generalized Burgers' equation. Çelikten et al. [26] solved Burgers' equation with explicit logarithmic finite difference method. İnan, defined an explicit logarithmic method to the solutions of generalized Huxley equation and generalized Burgers-Huxley equation and these works were presented orally at congresses and published in the congresses' abstract books [27, 28]. Macías-Díaz and İnan presented a structural and numerical analysis of an implicit logarithmic method for diffusion equation [29]. Macías-Díaz solved the classical Fisher' s equation and the Hodgkin-Huxley model using the explicit logarithmic finite difference method. Also, Macías-Díaz showed the existence and the uniqueness of the numerical solutions obtained by the explicit logarithmic method and proved that the numerical model preserves the positivity, the boundedness, and the monotonicity of the solutions under suitable conditions and presented that the logarithmic scheme is stable and convergent [30]. Macías-Díaz and Hendy investigated stability and convergence of implicit logarithmic finite difference method for diffusion equations [31].

In this paper, the generalized Huxley and generalized Burgers-Huxley equations are solved by two different logarithmic finite difference methods which are explicit and implicit logarithmic finite difference methods. These methods are abbreviated as E-LOGFDM and I-LOGFDM and used respectively in the next part of the paper. To examine the effectiveness of the methods while solving the equations, we consider some examples. Additionally, obtained numerical solutions compared with the exact solutions and other numerical results. So, it concluded that the methods ensure high accuracy for the solution of the nonlinear generalized Huxley and Burgers-Huxley equations. In this paper, MATLAB R2015a was used for obtaining graphs and Fortran was used for computing.

2. Implementation of Logarithmic Finite Difference Methods

2. 1. Explicit Logarithmic Finite Difference Method

If we rearrange Eqn. (1);

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-\alpha u^\delta\frac{\partial u}{\partial x}+\beta u\left(1-u^\delta\right)\left(u^\delta-\gamma\right), \tag{4}$$

$F(u)$ denote any continuous and differentiable function, multiplying equation Eqn. (1) by a derivative of F, we have

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}= F'(u)\left(\frac{\partial^2 u}{\partial x^2}-\alpha u^\delta\frac{\partial u}{\partial x}+\beta u\left(1-u^\delta\right)\left(u^\delta-\gamma\right)\right) \tag{5}$$

and

$$\frac{\partial F}{\partial t}= F'(u)\left(\frac{\partial^2 u}{\partial x^2}-\alpha u^\delta\frac{\partial u}{\partial x}+\beta u\left(1-u^\delta\right)\left(u^\delta-\gamma\right)\right) \tag{6}$$

If we use forward difference approximation for $\frac{\partial F}{\partial t}$ then the following equation is obtained

$$\frac{F\left(U_i^{n+1}\right)-F\left(U_i^n\right)}{k}= F'(u)\left(\frac{\partial^2 u}{\partial x^2}-\alpha u^\delta\frac{\partial u}{\partial x}+\beta u\left(1-u^\delta\right)\left(u^\delta-\gamma\right)\right). \tag{7}$$

Where if we let $F(u) = F'(u) = e^u$, then we get explicit logarithmic finite difference method

$$F\left(U_i^{n+1}\right) = F(U_i^n)\left[1 + k\left(\frac{\partial^2 u}{\partial x^2} - \alpha u^\delta \frac{\partial u}{\partial x} + \beta u\left(1 - u^\delta\right)\left(u^\delta - \gamma\right)\right)\right] \tag{8}$$

and

$$U_i^{n+1} = U_i^n + \log_e\left[1 + k\left(\frac{\partial^2 u}{\partial x^2} - \alpha u^\delta \frac{\partial u}{\partial x} + \beta u\left(1 - u^\delta\right)\left(u^\delta - \gamma\right)\right)\right]. \tag{9}$$

If Eqn. (9) is arranged and the finite difference approximations are written in the equation, the following equations have been obtained;

$$\begin{cases} U_i^{n+1} = U_i^n + \log_e[1 + k\Phi_i^n] \\ \Phi_i^n = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} - \alpha(U_i^n)^\delta \frac{U_{i+1}^n - U_{i-1}^n}{2h} + \beta U_i^n\left(1 - (U_i^n)^\delta\right)\left((U_i^n)^\delta - \gamma\right). \end{cases} \tag{10}$$

Eqn. (10) is the explicit logarithmic finite difference method for the solution of the generalized Burgers-Huxley equation. When $\alpha = 0$, the method (10) is the turned into to explicit logarithmic finite difference method for solution of the generalized Huxley equation. The explicit logarithmic finite difference method for Eqn. (1) takes linear form defined by Eqn. (10) were lying in the interval $1 \leq N \leq N - 1$.

2. 2. Implicit Logarithmic Finite Difference Method

Eqn. (9) is rearranged and considered finite difference approximations for the equation, we obtain the following implicit logarithmic finite difference method to the solution of the generalized Burgers-Huxley equation;

$$\begin{cases} U_i^{n+1} = U_i^n + \log_e\left[1 + k\Phi_i^{n+1}\right] \\ \Phi_i^{n+1} = \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} - \alpha\left(U_i^{n+1}\right)^\delta \frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2h} + \beta U_i^{n+1}\left(1 - \left(U_i^{n+1}\right)^\delta\right)\left(\left(U_i^{n+1}\right)^\delta - \gamma\right). \end{cases} \tag{11}$$

Eqn. (11) is the implicit logarithmic finite difference method for the solution of the generalized Burgers-Huxley equation. When $\alpha = 0$, the method (11) is turned into to implicit logarithmic finite difference method for the solution of the generalized Huxley equation. Eqn. (11) is a nonlinear difference equations system. Let us regard the nonlinear system of equations in the form

$$\mathrm{F}(\mathrm{V}) = 0, \tag{12}$$

where $\mathrm{F} = [f_1, f_2, \dots, f_{N-1}]^T$ and $\mathrm{V} = [U_1^{n+1}, U_2^{n+1}, \dots, U_{N-1}^{n+1}]^T$. Newton's method is applied to Eqn. (10) results in the following iteration:

1. Set $\mathrm{V}^{(0)}$, an initial guess.

2. Solved $\mathrm{V}^{(m+1)=}\mathrm{V}^{(m)} - J\left(\mathrm{V}^{(m)}\right)^{-1}\mathrm{F}\left(\mathrm{V}^{(m)}\right)$ for $m = 0,1,2,\dots$

where $J\left(\mathrm{V}^{(m)}\right)$ is the Jacobian matrix and the matrix is evaluated analytically. According to the nature of iteration methods, the solution at the previous time-step is considered as the initial estimate. The stopped criteria for Newton's iteration at each time-step is taken as $\left\|\mathrm{F}\left(\mathrm{V}^{(m)}\right)\right\|_\infty \leq 10^{-4}$. The convergence is generally confirmed in two or three iterations.

While solving the problems the solution domain is discretized into the nodes set (x_i, t_n) in which $x_i = a + ih, (i = 0,1,2,\dots,N)$ and $t_n = nk, (n = 0,1,2,\dots)$, $h = \Delta x = \frac{b-a}{N}$ is the spatial mesh size and $k = \Delta t$ is the time step. Also, where U_i^n denotes the logarithmic finite difference approximation and $u(x,t)$ denotes the exact solution.

3. Stability Analysis

In this section, to investigate the stability of the method, the Fourier method is used. For the sake of examining the stability, the nonlinear term *is* accepted constant. So, stability can be discussed in the linearized sense. The stability analysis is ground on the von Neumann theory in which the growth factor of typical Fourier mode is defined as:

$$U_i^n = \xi^n e^{I\theta ih},\ I^2 = -1 \tag{13}$$

where h is the spatial mesh size and k is the time step, is determined from a linearization of the numerical scheme, so all the U_i^n are equal to local constant d, so that $u^\delta = (\epsilon d)^\delta$. Substituting Eqn. (13) in Eqn. (11) and $\xi^{n+1} = g\xi^n$ gives

$$g = \frac{1-\frac{2k}{h^2}+\beta k\left(1-(\epsilon d)^\delta\right)\left((\epsilon d)^\delta-\gamma\right)}{1+\frac{2k}{h^2}cos\frac{\theta h}{2}+\alpha(\epsilon d)^\delta\frac{k}{h}isin\theta h} \leq 1 \tag{14}$$

where h and k are usually a small quantity, and d represents the single speed and will usually be around unity. Hence, $|g| \leq 1$ will always be ensured for any problem. As a result, because $|g| \leq 1$, implicit logarithmic finite difference method is unconditionally stable. Also, the explicit logarithmic finite difference method is stable for $\frac{k}{h^2} \leq 0.5$.

4. Numerical Results

In this section, numerical solutions are presented obtained by logarithmic finite difference methods of the generalized Huxley and generalized Burgers-Huxley equations. The absolute error which is defined by the following form to measure the accuracy of the present method is used:

$$|u(x_i, t_n) - U(x_i, t_n)|.$$

Example 1. The generalized Huxley equation of the form;

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u^\delta)(u^\delta - \gamma), \ 0 \le x \le 1, \ t \ge 0 \tag{15}$$

with the initial condition

$$u(x, 0) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} tanh[\sigma\gamma x]\right)^{\frac{1}{\delta}}. \tag{16}$$

The exact solution of Eqn. (15) was derived by Wang [1] following for

$$u(x, t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} tanh\left[\sigma\gamma\left(x + \left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\} t\right)\right]\right)^{\frac{1}{\delta}} \tag{17}$$

where $\sigma = \delta\rho/4(1 + \delta)$, $\rho = \sqrt{4\beta(1 + \delta)}$ and β, γ and $\delta$are parameters that $\beta \ge 0$, $\gamma \epsilon (0,1)$.

Case 1. Table 1 presents numerical and exact solutions for various values of x, t and with $\delta = 1$, $\beta = 1$, $\gamma = 10^{-3}$. Also, numerical solutions were obtained by explicit and implicit logarithmic finite difference methods compared with exact solutions and those were obtained by another method in Table 1.

Case 2. In Table 2, absolute errors for various values of x, t and δ with $\beta = 10^{-2}$, $\gamma = 10^{-3}$ displayed.

Case 3. Table 3 presents absolute errors for various values of x, t and β with $\delta = 1$, $\gamma = 10^{-4}$.

Case 4. Table 4 shows absolute errors for various values of x, t and γ with $\beta = 10$, $\delta = 2$.

Table 1: Comparison of the solutions for $\delta = 1$

x	t	[16]	E-LOGFDM	I- LOGFDM	Exact
0.1	0.05	5.00005184E-04	5.000199E-04	5.000192E-04	5.000302E-04
	0.1	4.99992690E-04	5.000276E-04	5.000222E-04	5.000427E-04
	1	4.99767803E-04	5.002451E-04	5.002274E-04	5.002676E-04
0.5	0.05	5.00075895E-04	5.000777E-04	5.000586E-04	5.001009E-04
	0.1	5.00063401E-04	5.000749E-04	5.000285E-04	5.001134E-04
	1	4.99838513E-04	5.002758E-04	5.001874E-04	5.003383E-04
0.9	0.05	5.00146605E-04	5.001613E-04	5.000284E-04	5.001716E-04
	0.1	5.00134111E-04	5.001690E-04	5.000235E-04	5.001841E-04
	1	4.99909224E-04	5.003864E-04	5.002273E-04	5.004090E-04

Figure 1 displays absolute errors for $\delta = 1, 5, 10$, β=1, $\gamma = 10^{-3}$ at $t = 5$. Figure 2 shows absolute errors for β=1, 10, 100, $\delta = 2$, $\gamma = 10^{-3}$ at $t = 5$. Figure 3 demonstrates absolute errors for, $\gamma = 10^{-3}, 10^{-4}, 10^{-5}$ for β=1, $\delta = 2$ at $t = 5$.

Table 2: Absolute errors for various values of *x*, *t* and *δ*

		E-LOGFDM			I-LOGFDM		
x	t	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 1$	$\delta = 2$	$\delta = 3$
0.1	0.5	2.234E-10	9.995E-09	1.729E-07	1.966E-09	7.363E-08	8.082E-07
	1	2.248E-10	1.006E-08	1.739E-07	1.992E-09	7.460E-08	8.183E-07
	10	2.249E-10	1.006E-08	1.739E-07	1.993E-09	7.460E-08	8.182E-07
0.5	0.5	6.198E-10	2.773E-08	4.797E-07	9.378E-09	3.475E-07	3.672E-06
	1	6.246E-10	2.794E-08	4.832E-07	9.463E-09	3.507E-07	3.705E-06
	10	6.247E-10	2.794E-08	4.831E-07	9.463E-09	3.507E-07	3.704E-06
0.9	0.5	2.234E-10	9.994E-09	1.729E-07	1.611E-08	5.900E-07	5.963E-06
	1	2.249E-10	1.006E-08	1.740E-07	1.613E-08	5.910E-07	5.973E-06
	10	2.249E-10	1.006E-08	1.739E-07	1.613E-08	5.910E-07	5.972E-06

Table 3: Absolute errors for various values of *x*, *t* and *β*

		E-LOGFDM			I-LOGFDM		
x	t	$\beta = 1$	$\beta = 10$	$\beta = 100$	$\beta = 1$	$\beta = 10$	$\beta = 100$
0.1	0.5	2.235E-10	2.235E-09	2.235E-08	3.978E-10	2.786E-09	2.410E-08
	1	2.249E-10	2.250E-09	2.250E-08	4.017E-10	2.809E-09	2.426E-08
	10	2.249E-10	2.249E-09	2.244E-08	4.017E-10	2.808E-09	2.421E-08
0.5	0.5	6.201E-10	6.203E-09	6.203E-08	1.496E-09	8.972E-09	7.079E-08
	1	6.248E-10	6.249E-09	6.249E-08	1.509E-09	9.044E-09	7.133E-08
	10	6.248E-10	6.249E-09	6.235E-08	1.507E-09	9.044E-09	7.116E-08
0.9	0.5	2.235E-10	2.235E-09	2.235E-08	1.812E-09	7.259E-09	3.824E-08
	1	2.249E-10	2.250E-09	2.250E-08	1.816E-09	7.281E-09	3.841E-08
	10	2.249E-10	2.250E-09	2.244E-08	1.816E-09	7.280E-09	3.831E-08

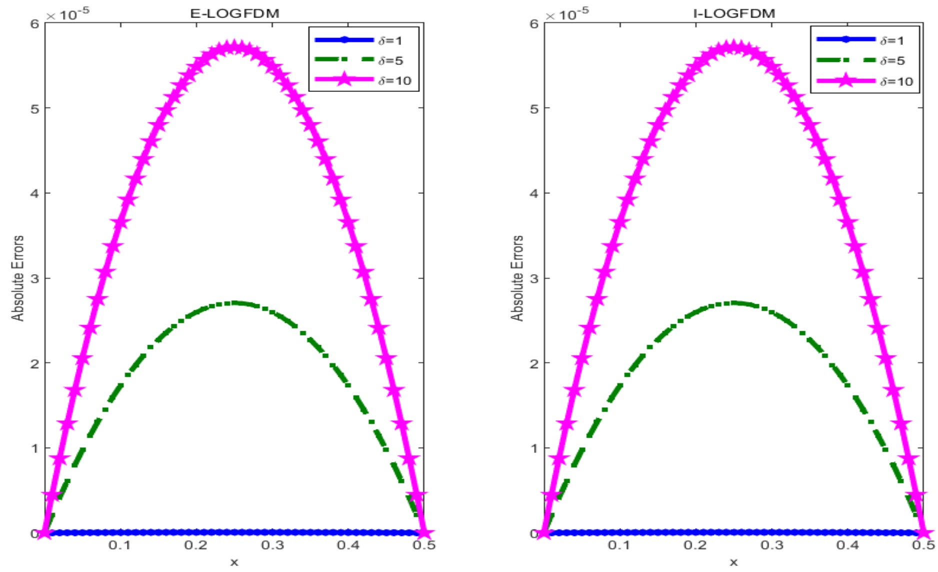

Figure 1: Absolute errors for different values of δ

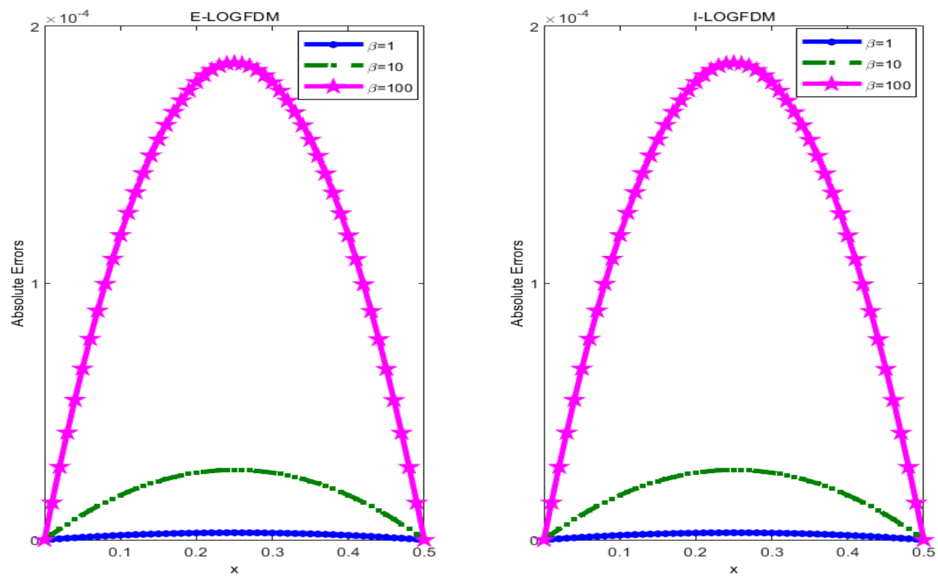

Figure 2: Absolute errors for different values of β

From Figs. 1, 2 and 3 and all of the computed results, it can be seen that the values of the errors are quite small. Also, it is observed that when δ, β and γ increase, the accuracy of the results decreases. From comparisons of the numerical solutions with the exact solutions and the others, it is concluded that the proposed method achieved highly accurate solutions. All of the computational works for Cases 1-4 and Figs. 1-3 are performed with $h = 10^{-2}$, $k = 10^{-5}$ and $h = 2x10^{-2}$, $k = 10^{-4}$, respectively.

Table 4: Absolute errors for various values of x, t and γ

		E-LOGFDM			I-LOGFDM		
x	t	$\gamma = 10^{-3}$	$\gamma = 10^{-4}$	$\gamma = 10^{-5}$	$\gamma = 10^{-3}$	$\gamma = 10^{-4}$	$\gamma = 10^{-5}$
0.1	0.5	9.975E-06	3.160E-07	9.998E-09	1.198E-05	3.797E-07	1.201E-08
	1	1.001E-05	3.180E-07	1.006E-08	1.205E-05	3.826E-07	1.210E-08
	10	9.504E-06	3.166E-07	1.006E-08	1.143E-05	3.808E-07	1.210E-08
0.5	0.5	2.768E-05	8.771E-07	2.774E-08	3.777E-09	1.197E-06	3.785E-08
	1	2.781E-05	8.834E-07	2.795E-08	3.798E-09	1.206E-06	3.815E-08
	10	2.640E-05	8.794E-07	2.793E-08	3.605E-09	1.200E-06	3.814E-08
0.9	0.5	9.973E-06	3.161E-07	9.998E-09	2.827E-09	8.960E-07	2.834E-08
	1	1.001E-05	3.180E-07	1.006E-08	2.829E-09	8.987E-07	2.843E-08
	10	9.501E-06	3.166E-07	1.006E-08	2.684E-09	8.946E-07	2.842E-08

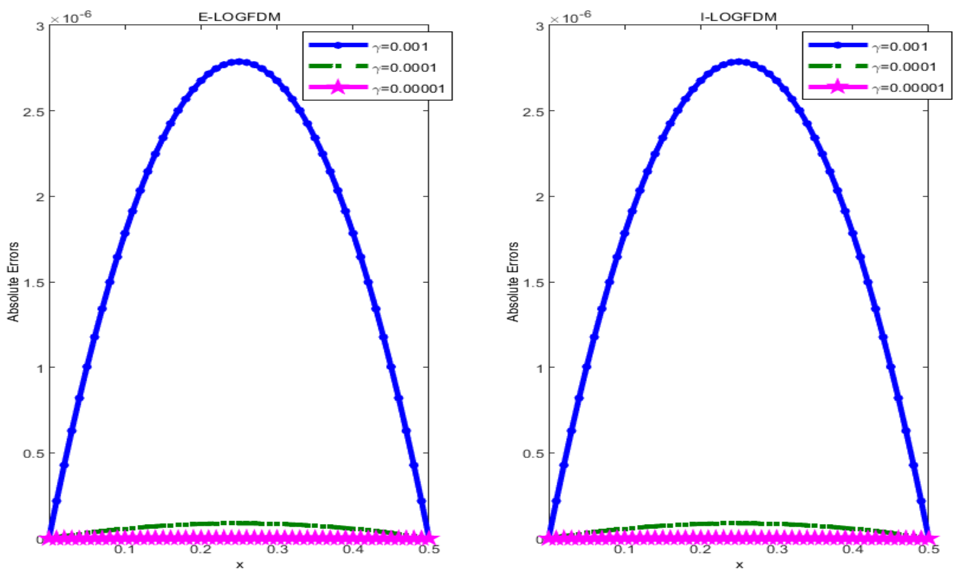

Figure 3: Absolute errors for different values of γ

Example 2. In this example, we consider the following generalized Burgers-Huxley equation,

$$\frac{\partial u}{\partial t} + \alpha u^{\delta} \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u\left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right), \ 0 \leq x \leq 1, \ t \geq 0 \quad (18)$$

with the following initial condition taken from

$$u(x, 0) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} tanh[A_1 x]\right)^{\frac{1}{\delta}} \quad (19)$$

and the boundary conditions

$$u(0,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} tanh[-A_1 A_2 t]\right)^{\frac{1}{\delta}} \tag{20}$$

and

$$u(1,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} tanh[A_1(1 - A_2 t)]\right)^{\frac{1}{\delta}}. \tag{21}$$

The exact solution of Eqn. (18) is

$$u(x,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} tanh[A_1(x - A_2 t)]\right)^{\frac{1}{\delta}}, \tag{22}$$

where

$$A_1 = \frac{-\alpha\delta + \delta\sqrt{\alpha^2 + 4\beta(1+\delta)}}{4(1+\delta)}\gamma, A_2 = \frac{\gamma\alpha}{1+\delta} - \frac{(1+\delta-\gamma)\left(-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)}\right)}{2(1+\delta)}. \tag{23}$$

Case 5. Table 5 presents numerical and exact solutions for various values of x, t and with $\delta = 1$, $\alpha = 1, \beta = 1$, $\gamma = 10^{-3}$.

Case 6. Absolute errors for various values of x, t and δ with $\alpha = 0.1, \beta = 10^{-3}$, $\gamma = 10^{-4}$ shown in Table 6.

Case 7. Table 7 shows absolute errors for various values of x, t and β with $\alpha = \delta = 1$, $\gamma = 10^{-4}$.

Case 8. Absolute errors for various values of x, t and γ with $\alpha = 1$, $\beta = 10$, $\delta = 2$ presented in Table 8.

Case 9. Table 9 displays absolute errors for various values of x, t and α with $\beta = \delta = 1$, $\gamma = 10^{-4}$.

Case 10. Table 10 shows comparisons of the present method with Batiha et al. [5], Biazar&Mohammadi [7] and Al-Rozbayani [11] for $\alpha = \beta = \delta = 1$ and $\gamma = 10^{-3}$.

Table 5: Comparison of the solutions for $\delta = 1$

			E-LOGFDM		I-LOGFDM	
x	t	Exact Solution	Numerical Solution	Absolute Error	Numerical Solution	Absolute Error
0.1	0.05	0.000500037	0.000500022	1.545E-08	0.000500022	1.545E-08
	0.1	0.000500062	0.000500040	2.259E-08	0.000500040	2.259E-08
	1	0.000500512	0.000500478	3.373E-08	0.000500478	3.373E-08
0.5	0.05	0.000500087	0.000500053	3.470E-08	0.000500053	3.470E-08

	0.1	0.000500112	0.000500055	5.766E-08	0.000500055	5.766E-08
	1	0.000500562	0.000500468	9.370E-08	0.000500471	9.370E-08
0.9	0.05	0.000500137	0.000500122	1.545E-08	0.000500122	1.545E-08
	0.1	0.000500162	0.000500140	2.259E-08	0.000500140	2.259E-08
	1	0.000500612	0.000500578	3.373E-08	0.000500578	3.373E-08

Table 6: Absolute errors for various values of x, t and δ

		E-LOGFDM			I-LOGFDM		
x	t	$\delta = 1$	$\delta = 4$	$\delta = 8$	$\delta = 1$	$\delta = 4$	$\delta = 8$
0.1	1	5.551E-13	1.456E-09	1.989E-08	4.470E-13	1.456E-9	1.989E-8
	5	5.996E-13	1.457E-09	1.989E-08	4.431E-13	1.456E-9	1.989E-8
	10	5.996E-13	1.457E-09	1.989E-08	4.419E-13	1.456E-9	1.989E-8
0.5	1	1.459E-12	4.046E-09	5.525E-08	1.235E-12	4.045E-9	5.525E-8
	5	1.666E-12	4.046E-09	5.525E-08	1.232E-12	4.045E-9	5.524E-8
	10	1.666E-12	4.046E-09	5.524E-08	1.226E-12	4.045E-9	5.525E-8
0.9	1	5.551E-13	1.457E-09	1.989E-08	4.421E-13	1.456E-9	1.989E-8
	5	5.996E-13	1.457E-09	1.989E-08	4.421E-13	1.456E-9	1.989E-8
	10	5.996E-13	1.457E-09	1.989E-08	4.430E-13	1.456E-9	1.989E-8

Absolute errors displayed by Fig. 4 for $\delta = 1, 2, 3$, $\alpha = \beta = 1$, $\gamma = 10^{-3}$, $h = 0.02$ and $k = 10^{-4}$ at $t = 5$. Figure 5 displays absolute errors for $\beta = 1, 5, 10$, $\alpha = 1$, $\delta = 2, , \gamma = 10^{-3}$. Figure 6 presents absolute errors for $\gamma = 10^{-3}, 10^{-4}, 10^{-5}$, $\alpha = \beta = 1$, $\delta = 2$. Figure 7 shows absolute errors for $\alpha = 0.1, 1, 10$, $\delta = 2, \beta = 1$, $\gamma = 10^{-3}$.

Table 7: Absolute errors for various values of x, t and β

		E-LOGFDM			I-LOGFDM		
x	t	$\beta = 1$	$\beta = 10$	$\beta = 100$	$\beta = 1$	$\beta = 10$	$\beta = 100$
0.1	0.05	1.546E-10	2.165E-09	2.436E-08	1.546E-10	2.165E-09	2.436E-08
	0.1	2.260E-10	3.165E-09	3.562E-08	2.261E-10	3.165E-09	3.562E-08
	1	3.375E-10	4.724E-09	5.316E-08	3.379E-10	4.725E-09	5.316E-08
0.5	0.05	3.471E-10	4.861E-09	5.471E-08	3.470E-10	4.861E-09	5.470E-08
	0.1	5.766E-10	8.076E-09	9.089E-08	5.770E-10	8.076E-09	9.089E-08
	1	9.370E-10	1.312E-08	1.477E-07	9.382E-10	1.312E-08	1.477E-07
0.9	0.05	1.546E-10	2.165E-09	2.436E-08	1.546E-10	2.165E-09	2.436E-08
	0.1	2.260E-10	3.165E-09	3.562E-08	2.261E-10	3.165E-09	3.562E-08
	1	3.375E-10	4.724E-09	5.316E-08	3.379E-10	4.725E-09	5.316E-08

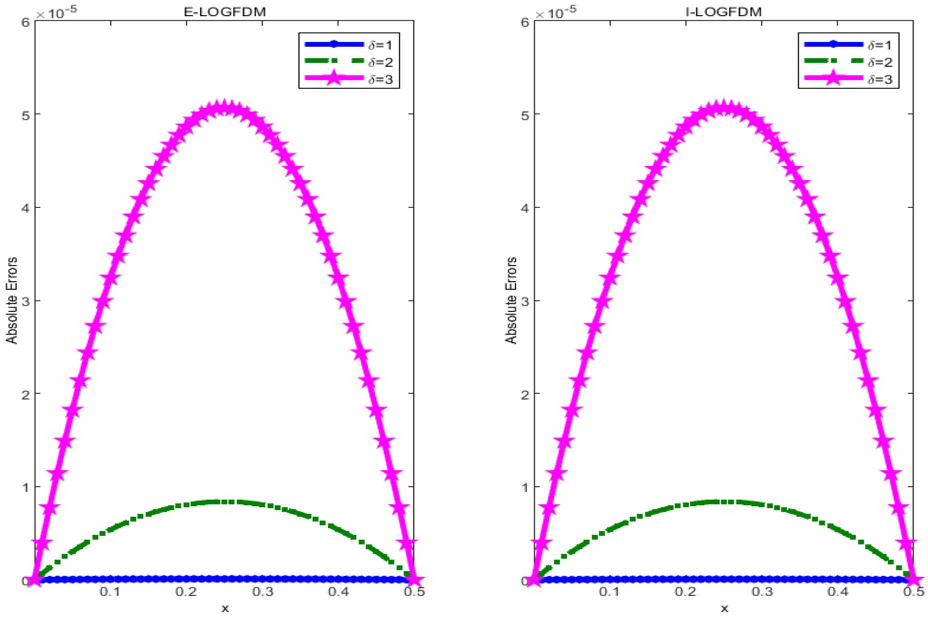

Figure 4: Absolute errors for different values of δ

As it can be observed from Tables 5-10 that the obtained results have excellent conform with the exact solutions. All numerical results for Cases 5-10 are obtained with the space step $h = 10^{-2}$ and the time step $k = 10^{-5}$. All numerical results shown in Figs. 5-7 are obtained for $h = 0.02$ and $k = 10^{-5}$ at $t = 5$. From all of the computed results and figures can be observed that the values of the errors are very small. Also, it is observed from all computations that the accuracy of the numerical results decreases when δ, β and γ increase. However, the accuracy of the results increases when α increased.

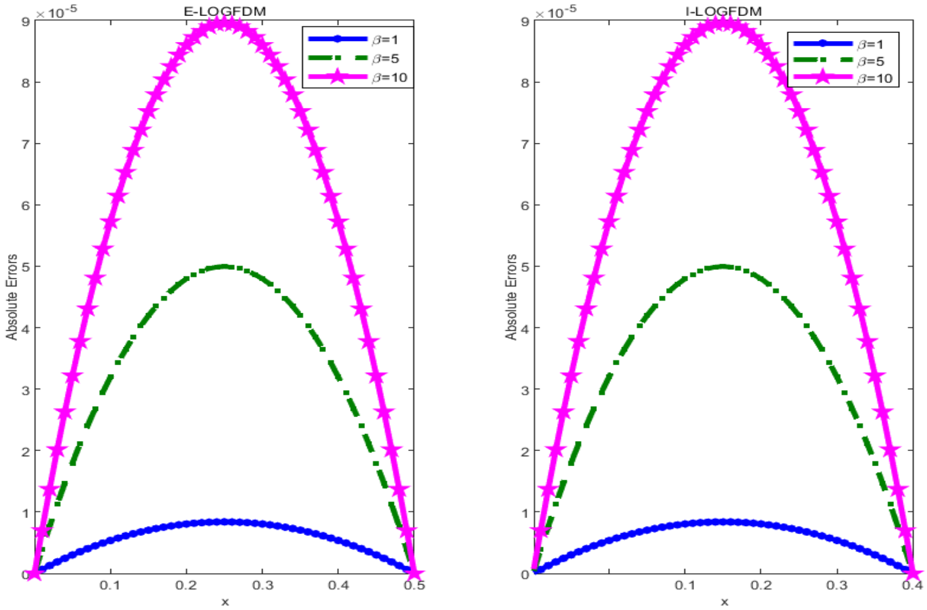

Figure 5: Absolute errors for different values of β

Table 8: Absolute errors for various values of x, t and γ

		E-LOGFDM			I-LOGFDM		
x	t	$\gamma = 10^{-3}$	$\gamma = 10^{-4}$	$\gamma = 10^{-5}$	$\gamma = 10^{-3}$	$\gamma = 10^{-4}$	$\gamma = 10^{-5}$
0.1	0.05	1.956E-05	6.195E-07	1.959E-08	1.956E-05	6.194E-07	1.959E-08
	0.1	2.856E-05	9.055E-07	2.864E-08	2.856E-05	9.055E-07	2.864E-08
	1	4.118E-05	1.348E-06	4.275E-08	4.118E-05	1.348E-06	4.275E-08
0.5	0.05	4.393E-05	1.391E-06	4.400E-08	4.392E-05	1.391E-06	4.399E-08
	0.1	7.290E-05	2.311E-06	7.309E-08	7.290E-05	2.311E-06	7.309E-08
	1	1.144E-04	3.743E-06	1.187E-07	1.144E-04	3.743E-06	1.187E-07
0.9	0.05	1.955E-05	6.194E-07	1.959E-08	1.955E-05	6.194E-07	1.959E-08
	0.1	2.856E-05	9.055E-07	2.864E-08	2.855E-05	9.055E-07	2.864E-08
	1	4.117E-05	1.347E-06	4.275E-08	4.117E-05	1.348E-06	4.275E-08

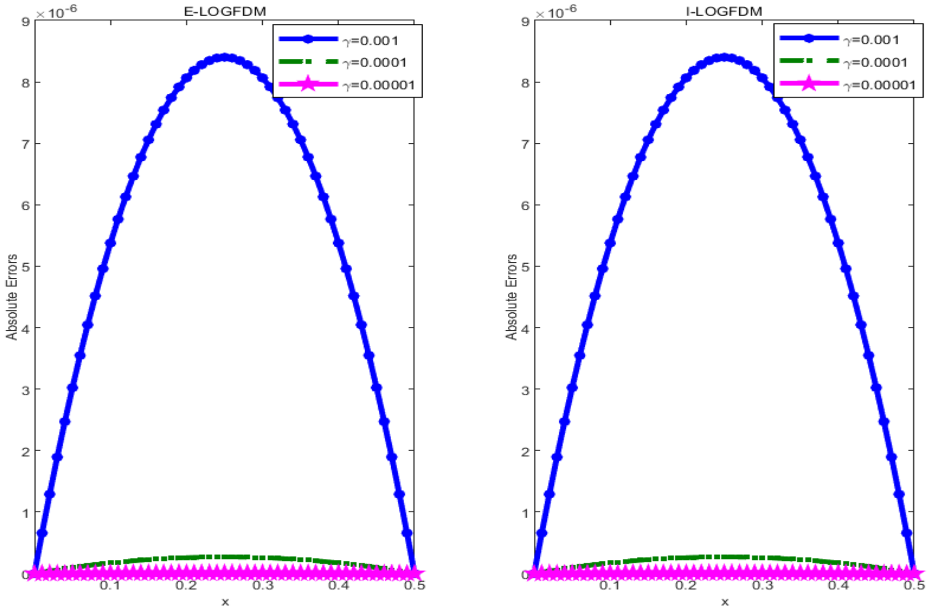

Figure 6: Absolute errors for different values of γ

Table 9: Absolute errors for various values of x, t and α

		E-LOGFDM			I-LOGFDM		
x	t	$\alpha = 0.1$	$\alpha = 1$	$\alpha = 5$	$\alpha = 0.1$	$\alpha = 1$	$\alpha = 5$
0.1	0.05	2.436E-10	1.546E-10	6.579E-11	2.437E-10	1.547E-10	6.576E-11
	0.1	3.561E-10	2.260E-10	9.618E-11	3.563E-10	2.261E-10	9.618E-11
	1	5.317E-10	3.375E-10	1.438E-10	5.321E-10	3.379E-10	1.442E-10
0.5	0.05	5.470E-10	3.471E-10	1.477E-10	5.470E-10	3.470E-10	1.476E-10
	0.1	9.087E-10	5.766E-10	2.454E-10	9.092E-10	5.770E-10	2.453E-10
	1	1.4769E-9	9.370E-10	3.993E-10	1.478E-09	9.382E-10	4.005E-10
0.9	0.05	2.436E-10	1.546E-10	6.579E-11	2.437E-10	1.546E-10	6.576E-11
	0.1	3.561E-10	2.260E-10	9.618E-11	3.563E-10	2.261E-10	9.618E-11
	1	5.317E-10	3.375E-10	1.438E-10	5.321E-10	3.379E-10	1.442E-10

Table 10: Comparisons of the absolute errors for $\delta = 1$

x	t	E-LOGFDM	I-LOGFDM	[5]	[9]	[13]
0.1	0.05	1.545E-08	1.545E-08	1.87405E-08	1.87406E-08	1.87406E-08
	0.1	2.259E-08	2.259E-08	3.74813E-08	3.74813E-08	3.74812E-08
	1	3.373E-08	3.373E-08	3.74812E-07	3.74813E-07	3.74812E-07
0.5	0.05	3.470E-08	3.470E-08	1.87405E-08	1.87406E-08	1.87406E-08
	0.1	5.766E-08	5.766E-08	1.37481E-08	3.74813E-08	3.74812E-08
	1	9.370E-08	9.370E-08	3.74813E-07	3.74813E-07	3.74812E-07
0.9	0.05	1.545E-08	1.545E-08	1.87405E-08	1.87406E-08	1.87406E-08
	0.1	2.259E-08	2.259E-08	3.74813E-08	3.74813E-08	3.74812E-08
	1	3.373E-08	3.373E-08	3.74813E-07	3.74813E-07	3.74812E-07

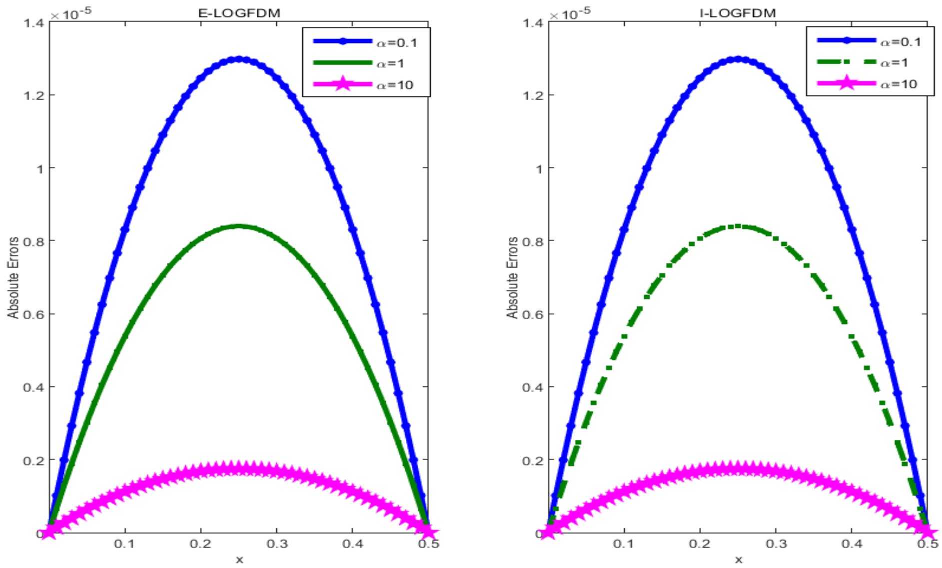

Figure 7: Absolute errors for different values of α

5. Conclusion

In this manuscript, we have designed explicit and implicit logarithmic finite difference methods has been proposed the generalized Huxley and Burgers-Huxley equations. The numerical solutions for different two test problems are presented through tables. The numerical results show that the solution using methods gives high accuracy and to obtain high accuracy results no restrictions for parameters are needed. As can be seen from comparisons and obtained results that the logarithmic finite difference methods are effective and reliable methods for solving a wide range of engineering problems.

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