On the Upper Bounds for Permanents

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Abstract: In this paper, considering λ_1, λ_2 and λ_∞ operator norms, we obtained some upper bounds for permanents.

Key Words: λ_1, λ_2 and λ_∞ operator norms, permanent

Permanentlerin Üst Sınırları Üzerine

Özet: Bu çalışmada, λ_1, λ_2 ve λ_∞ operatör normları gözönüne alınarak permanentler için bazı üst sınırlar elde edilmiştir.

Anahtar Kelimeler: λ_1, λ_2 ve λ_∞ operatör normları, permanent

Introduction and the Statemens of Results

Definition 1. [1] The permanent of a real n×n matrix $A = (a_{ij})$ is defined by

$$\operatorname{per}(A) = \sum_{\sigma\in S_n}\prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the symmetric group of order n.

Definition 2. ([2]) The λ_1 operator norm of an n×n matrix $A = (a_{ij}) \in \mathbf{C}_{n\times n}$ is defined

$$\|A\|_1 = \max\left\{\|Ax\|_1 : x \in \mathbf{C}^n, \|x\|_1 = 1\right\},$$

where $x = (x_1, x_2, \ldots, x_n)^T$, (T denoting the transpoze) and

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

Definition 3. ([2]) The λ_2 operator norm of an n×n matrix $A = (a_{ij}) \in \mathbf{C}_{n\times n}$ is defined

$$\|A\|_2 = \max\left\{\|Ax\|_2 : x \in \mathbf{C}^n, \|x\|_2 = 1\right\},$$

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where $x = (x_1, x_2, \ldots, x_n)^T$ and

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}.$$

Definition 4. ([2]) The λ_∞ operator norm of an n×n matrix $A = (a_{ij}) \in \mathbf{C}_{n\times n}$ is defined

$$\|A\|_\infty = \max\left\{ \|Ax\|_\infty : x \in \mathbf{C}^n, \|x\|_\infty = 1 \right\},$$

where $x = (x_1, x_2, \ldots, x_n)^T$ and

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

Lemma 1. Let $a_1, a_2, \ldots, a_n$ be the columns of $A = (a_{ij}) \in \mathbf{C}_{n\times n}$. Then

$$|\operatorname{per}(A)| \le \left(\sqrt{n}\right)^n \|a_1\|_2 \|a_2\|_2 \ldots \|a_n\|_2,$$

where

$$\|a_j\|_2 = \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}, \quad 1 \le j \le n.$$

Proof. We make use of the inequality (see e.g. [1, p.113])

$$\operatorname{per}(A) \le \prod_{i=1}^{n} c_i$$

where $c_1, c_2, \ldots, c_n$ are column sums of A and $A = (a_{ij})_{n\times n}$ is a nonnegative matrix. Since

$$|\operatorname{per}(A)| \le \operatorname{per}(|A|)$$

by the triangle inequality, any such bound can be used to produce an upper bound for the permanents of complex matrices. For example from the inequality (1), we obtain

$$|\operatorname{per}(A)| \le \prod_{j=1}^{n} q_j, \tag{2}$$

where

$$q_j = \sum_{i=1}^{n} |a_{ij}|, \quad j = 1, 2, \ldots, n.$$

By the Cauchy-Schwarz Inequality, we have

$$q_j = \sum_{i=1}^{n} |a_{ij}| \le \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} 1 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} \sqrt{n} = \|a_j\|_2 \sqrt{n} .$$

So from inequality (2) we obtain

$$|\operatorname{per}(A)| \le \left(\sqrt{n}\right)^n \|a_1\|_2 \|a_2\|_2 \ldots \|a_n\|_2$$

and the proof is complete.

Theorem 1. Let

$$\|A\|_2 = \max\left\{ \|Ax\|_2 : x \in \mathbf{C}^n, \|x\|_2 = 1 \right\}$$

be λ_2 operator norm of $A \in \mathbf{C}_{n\times n}$. Then

$$|\operatorname{per}(A)| \le n^{\frac{n}{2}} \|A\|_2^n .$$

Proof. Denote the columns of A by $a_1, a_2, \ldots, a_n$ and let $e_1, e_2, \ldots, e_n$ be the standart basis of $\mathbf{C}^n$. Then we have

$$a_j = A e_j , \quad 1 \le j \le n . \tag{3}$$

So considering Lemma 1 we have

$$\begin{aligned} |\operatorname{per}(A)| &\le n^{\frac{n}{2}} \|a_1\|_2 \|a_2\|_2 \ldots \|a_n\|_2 \\ &\le n^{\frac{n}{2}} \left(\max_{1\le j\le n} \|a_j\|_2 \right)^n \\ &= n^{\frac{n}{2}} \left(\max_{1\le j\le n} \|Ae_j\|_2 \right)^n \\ &\le n^{\frac{n}{2}} \left(\max_{\|x\|_2=1} \|Ax\|_2 \right)^n \\ &= n^{\frac{n}{2}} \|A\|_2^n \end{aligned}$$

and thus the theorem is proved.

Lemma 2. Let $a_1, a_2, \ldots, a_n$ be the columns of $A \in \mathbf{C}_{n\times n}$. Then

$$\left|\operatorname{per}(A)\right| \le \prod_{j=1}^{n} \left\|a_j\right\|_1 ,$$

where

$$\left\|a_j\right\|_1 = \sum_{i=1}^{n} \left|a_{ij}\right| , \quad j = 1,2,\ldots,n .$$

Proof. The proof of Lemma is immediately seen from (2).

Theorem 2. Let

$$\|A\|_1 = \max\left\{ \|Ax\|_1 : x \in \mathbf{C}^n,\ \|x\|_1 = 1 \right\}$$

be λ_1 operator norm of $A \in \mathbf{C}_{n\times n}$. Then

$$\left|\operatorname{per}(A)\right| \le \|A\|_1^n .$$

Proof. Considering Lemma 2 and the equality (3), we have

$$\left|\operatorname{per}(A)\right| \le \|a_1\|_1 \|a_2\|_1 \ldots \|a_n\|_1$$

$$\le \|Ae_1\|_1 \|Ae_2\|_1 \ldots \|Ae_n\|_1$$

$$\le \left(\max_{1\le j\le n} \|Ae_j\|_1 \right)^n$$

$$\le \left(\max_{\|x\|_1 = 1} \|Ax\|_1 \right)^n$$

$$= \|A\|_1^n .$$

Thus the theorem is proved.

Lemma 3. Let $a_1, a_2, \ldots, a_n$ be the columns of $A = (a_{ij}) \in \mathbf{C}_{n\times n}$. Then

$$\left\|a_j\right\|_1 \le n \left\|a_j\right\|_\infty ,$$

where

$$\left\|a_j\right\|_1 = \sum_{i=1}^{n} \left|a_{ij}\right| , \quad j = 1,2,\ldots,n ,$$

and

$$\|a_j\|_\infty = \max_{1\le i\le n} |a_{ij}| \ , \ 1\le j\le n .$$

Proof. For all j, $1\le j\le n$, we have

$$\|a_j\|_1 = \sum_{i=1}^{n}|a_{ij}| = |a_{1j}| + |a_{2j}| + \Lambda + |a_{nj}|$$

$$\le n \max_{1\le i\le n}|a_{ij}|$$

$$= n \|a_j\|_\infty .$$

Thus the proof is complete.

Theorem 4. Let a_1, a_2, K, a_n be the columns of $A=(a_{ij})\in \mathbf{C}_{n\times n}$. Then

$$|per(A)| \le n^n \prod_{j=1}^{n}\|a_j\|_\infty \ .$$

Proof. Considering Lemma 2 and Lemma 3 the proof is easily seen.

Theorem 5. Let

$$\|A\|_\infty = \max\left\{\|Ax\|_\infty : x\in \mathbf{C}^n, \|x\|_\infty = 1 \right\}$$

be λ_∞ operator norm of A .Then

$$|per(A)| \le n^n \|A\|_\infty^n \ .$$

Proof. From Theorem 4 and equality (3), we have

$$|\,per(A)\,| \le n^n \|a_1\|_\infty \|a_2\|_\infty K \|a_n\|_\infty$$

$$\leq n^n \left(\max_{1 \leq j \leq n} \|a_j\|_\infty \right)^n$$

$$= n^n \left(\max_{1 \leq j \leq n} \|Ae_j\|_\infty \right)^n$$

$$\leq n^n \left(\max_{\|x\|_\infty = 1} \|Ax\|_\infty \right)^n$$

$$= n^n \ \|A\|_\infty^n \ ,$$

and thus the theorem is proved.

REFERENCES

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