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Lower Bounds For Perron Root Of Positive Matrices

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Abstract: In this paper we define "greatest common divisor (or gcd)" symmetrization of a positive matrix and using this we obtain some lower bounds for Perron root of positive matrices.

Key Words: Positive matrix, Perron root

Pozitif Matrislerin Perron Kökleri İçin Alt Sınırlar

Özet: Bu çalışmada, pozitif matrisin en büyük ortak bölen simetrizasyonu tanımlandı ve bu simetrizasyon kullanılarak, pozitif matrislerin Perron kökleri için alt sınırlar elde edildi.

Anahtar Kelimeler: Pozitif matris, Perron kökü

Introduction

Definition 1.1. Let $A = (a_{ij}) \in M_n$. We say that $A \geq 0$ (A is nonnegative) if all its entries a_{ij} are real and nonnegative, where M_n denotes $n \times n$ matrices. We say that $A>0$ (A is positive) if all its entries a_{ij} are real and positive.

Definition 1.2. Let $A, B \in M_n$. We say that $A \geq B$ if $A-B \geq 0$.

Definition 1.3. The spectral radius $\rho(A)$ of a matrix $A = (a_{ij}) \in M_n$ is

$$\rho(A) \equiv \{ |\lambda| : \lambda \text{ is an eigenvalue of } A \}$$

Definition 1.4. Let A be square nonnegative matrix. Then a nonnegative eigenvalue r(A) which is not less than the absolute value of any eigenvalue of A is called Perron root.

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Let A be a (nonsymmetric) nonnegative matrix and let

$$\varepsilon = \frac{e^T S(A) e}{n}$$

where S(A) denotes the geometric symmetrization of A (see [8]) as $S(A) = (s_{ij})$, i.e., $s_{ij} = \sqrt{a_{ij} a_{ji}}$

and $e^T = (1,1,\ldots,1)$. Kolotilina (see [5]) showed that

$$\varepsilon \le r(A),$$

where r(A) is the Perron root of A.

Let A be a nonnegative n×n matrix and let

$$\gamma(A) = \min_{1 \le i \le n} \sum_{j=1}^{n} a_{ij} .$$

Yamamoto (see [9]) showed that

$$\gamma(A) \le r(A).$$

Let $A = (a_{ij})$ be a (nonsymmetric) positive n×n matrix, denoted A>0, and $a_{ij} \in Z^+$ where Z^+ denotes positive integers. We define "greatest common divisor (or gcd)" symmetrization of A, as

$[A] = (d_{ij})$, i.e., $d_{ij} = \gcd(a_{ij}, a_{ji})$. Clearly, if A is a positive symmetric matrix, then $[A] = A$.

The purpose of this paper is to obtain the following lower bound for the Perron root of a positive matrix:

$$r([A]) \le r(A) .$$

Result

Lemma 2.1. [3]. Let A, B $\in M_n$. If $0 \le A \le B$, then $\rho(A) \le \rho(B)$, where $\rho(A)$ denotes spectral radius of A and $\rho(B)$ denotes spectral radius of B.

Lemma 2.2. Let $A = (a_{ij})$ be a positive $n \times n$ matrix and let $a_{ij} \in Z^+$, then

$$r([A]) \le r(A), \tag{2.1}$$

where r([A]) denotes Perron root of "gcd" symmetrization matrix of A and r(A) denotes Perron root of A.

Proof. Clearly we have for all i, j (i, j = 1,2, ... , n)

$$\gcd(a_{ij}, a_{ji}) \le a_{ij} \tag{2.2}$$

and

$$\gcd(a_{ij}, a_{ji}) \le a_{ji} . \tag{2.3}$$

Considering (2.2), (2.3) and Lemma 2.1 we write

$$r([A]) \le r(A)$$

and

$$r([A]) \le r(A^T),$$

respectively, where A^T denotes transpose of A.

Theorem 2.1. Let $A = (a_{ij})$ be a positive $n \times n$ matrix, $a_{ij} \in Z^+$ and let

$$\mu_k = \left(r\left[A^{2^k}\right] \right)^{2^{-k}} .$$

Then

$$r(A) \geq \mu_k.$$

Proof. We note that from (2.1) it follows that

$$r\left(A^{2^k}\right) = r(A)^{2^k} \geq r\left[A^{2^k}\right]$$

which implies

$$r(A) \geq \mu_k.$$

Thus the therom is proved.

Lemma 2.3. Let A = (a_{ij}) be a positive n×n matrix and let $a_{ij} \in Z^+$. Then

$$\frac{e^T[A]e}{e^T e} \leq \frac{e^T A\, e}{e^T e} \tag{2.4}$$

or

$$\frac{e^T[A]e}{e^T e} \leq \frac{e^T A^T e}{e^T e} \tag{2.5}$$

where $e^T = (1,1, \ldots ,1)$.

Proof. To prove (2.4) and (2.5) it suffices that

$$e^T[A]\, e \leq e^T A\, e$$

or

$$e^T[A]\, e \leq e^T A^T e.$$

Indeed we have

$$e^T[A]\,e = \sum_{i,j=1}^{n} \gcd(a_{ij}, a_{ji}) \le \sum_{i,j=1}^{n} a_{ij} = e^T A\, e$$

or

$$e^T[A]\,e = \sum_{i,j=1}^{n} \gcd(a_{ij}, a_{ji}) \le \sum_{i,j=1}^{n} a_{ji} = e^T A^T e.$$

Thus the proof is complete.

Theorem 2.2. Let A = (a_{ij}) be a positive n×n matrix and let $a_{ij} \in Z^+$. Then

$$r\left([A]\right) \le r(S(A)) \le r(M(A)) ,$$

where S(A) denotes the “geometric” symmetrization and M(A) denotes the “arithmetic” symmetrization of A (see [4]) as M(A) = (m_{ij}), i.e.,

$$m_{ij} = \frac{a_{ij} + a_{ji}}{2} \qquad 1 \le i, j \le n.$$

Proof. Considering Lemma 2.1 and

$$\gcd(a_{ij} , a_{ji}) \le \sqrt{a_{ij} a_{ji}} \le \frac{a_{ij} + a_{ji}}{2}$$

the proof is immediately seen.

We end the paper with an example (see [4]). Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 5 \end{bmatrix}.$$

The Perron root of A is r(A) = 7.531. On the other hand since

$$[A] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

the Perron root of [A] is r[A] = 7.387.

Kolotilina (see [5]) showed ε is a better approximation to r(A) = 7.531 than the other lower bounds from the literature, such as those by Deutsch (see [1]), Deutsch and Wielandt (see [2]), and Szule (see [6,7]). The bound discussed in this paper yield γ(A)= 4, ε = 6.609, r[A] = 7.387. All values were rounded to four significant figures. Thus the lower bound r[A] provides a better approximation the Perron root than the lower bounds provided by other authors.

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