An Analysis of a Mathematical Model on Advection-Dispersion of Contamination in a Medium of one Dimensional Underground Water Flow

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Özet: Bu çalışmada bir boyutlu yeraltı su akımı ortamında kirliliğin yayılması durumunda ortaya çıkan *sınır değer problemi Laplace Yöntemi* ile çözülmüş ve *Laplace Yöntemi* ile problem çözmede oldukça kullanışlı olan bir integrasyon formülü kanıtlanmıştır.

Anahtar Kelimeler: Kirlilik, Yeraltı Su Akımı , Sınır Değer Problemi, Laplace Yöntemi

Bir Boyutlu Yeraltı Su Akım Ortamında Kirliliğin Yayılmasının Bir Matematik Modelinin Çözümlenmesi Üzerine

Abstract: In this study, the solution of the *boundary value problem* of advection-dispersion equation arising in contamination problems in a medium of one dimensional underground water flow has been solved by using *Laplace Method* and an integration formula that is rather useful in solving problems via *Laplace Method* has been proven.
Key Words: Contamination, Underground Water Flow, Boundary Value Problem, Laplace Method

Introduction

The equations controlling underground water flow is a subject appearing in many problems of *hydrogeology* (See [1, 2 and 3]). Techniques of analysis in almost every science and engineering field are based on understanding the physical processes. *The mathematical models of flow equation* are generally studied in media of steady-state saturated flow, transient saturated flow, and transient unsaturated flow. Thus, the need comes out for the solving of the problems concerning *hydrogeology*.

This study covers the advection and dispersion of non-reactive dissolved constituents in an isotropic and homogeneous one dimensional flow media.

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The Advection-Dispersion of The Contamination in One Dimensional Flow Media

The governing equation with *boundary and initial conditions* can be defined as follows.

Definition:

Let " t " denotes time $t > 0$, through distance x and let the function of two variables, $c(x, t)$ denote *the advection-dispersion of contamination* made dimensionless. Then the expression

$$\frac{\partial c(x,t)}{\partial t} + V\frac{\partial c(x,t)}{\partial x} = D\frac{\partial^2 c(x,t)}{\partial x^2} - kc(x,t), \qquad x \geq 0 \tag{1}$$

is called *the advection-dispersion of one dimensional contamination* by time (See [1, 2 and 3]), where,

V represents constant flow rate of water (L/T)
D represents dispersion coefficient of the homogeneous isotropic flow media (L^2/T)
k represents chemical degradation coefficient (1/T)
c represents concentration (M/L^3)

Boundary Conditions:

$$c(0, t) = c_0, \quad t \geq 0 \tag{2}$$

$$c(\infty, t) = 0, \quad t \geq 0 \tag{3}$$

c_0: constant

Initial Condition:

$$c(x, 0) = 0, \quad 0 \leq x < \infty \tag{4}$$

Laplace Method, Integration Formula

Laplace Method is a useful method in solving *differential equations with partial derivative* and takes place very often in literature(See [4, 5]).

Solution of equation (1) by using *Laplace Method* in *boundary and initial conditions* of (2), (3), (4). Using the *Laplace transformation* rules we write

$$L\{\frac{\partial c(x,t)}{\partial t}\} = sc(x,s) \tag{5}$$

$$L\{\frac{\partial c(x,t)}{\partial x}\} = \frac{dc(x,s)}{dx} \tag{6}$$

$$L\{\frac{\partial^2 c(x,t)}{\partial x^2}\} = \frac{d^2 c(x,s)}{dx^2} \tag{7}$$

and *boundary conditions*,

$$L\{c(0, t)\} = \frac{c_0}{s} \tag{8}$$

$$L\{c(\infty, t)\} = 0 \tag{9}$$

can be expressed. When (5), (6) and (7) are written and arranged in equation (1), with the boundary conditions (8) and (9), the second degree linear differential equation is obtained:

$$D\frac{d^2 c}{dx^2} - V\frac{dc}{dx} - (k+s)c = 0\,, \tag{10}$$

$$c(0, s) = \frac{c_0}{s}, \qquad c(\infty, s) = 0. \tag{11}$$

Theorem (Integration Formula). r, it has been any parameter, for an integral value between r and infinity:

$$\frac{4}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{(-\lambda^2-\frac{a^2}{\lambda^2})} d\lambda = e^{2a} erfc(r+\frac{a}{r}) + e^{-2a} erfc(r-\frac{a}{r}) \tag{12}$$

verifies the equation(See [4, 5]).

Proof .

The expression of this equation is not made in Churchill sec. 51. In order to prove equation (12), the following operations are applied:

$$\frac{4}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{(-\lambda^2-\frac{a^2}{\lambda^2})} d\lambda = \frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2} e^{-\frac{a^2}{\lambda^2}} d\lambda + \frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2} e^{-\frac{a^2}{\lambda^2}} d\lambda$$

and $\frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2} e^{\frac{a^2}{\lambda^2}} \frac{a}{\lambda^2} d\lambda$ added and subtracted,

$$= \frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2} e^{-\frac{a^2}{\lambda^2}} d\lambda + \frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2} e^{-\frac{a^2}{\lambda^2}} d\lambda + \frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2} e^{\frac{a^2}{\lambda^2}} \frac{a}{\lambda^2} d\lambda - \frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2} e^{\frac{a^2}{\lambda^2}} \frac{a}{\lambda^2} d\lambda$$

is obtained. When the last equation is arranged;

$$= \frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2} e^{-\frac{a^2}{\lambda^2}} (1+\frac{a}{\lambda^2}) d\lambda + \frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2} e^{-\frac{a^2}{\lambda^2}} (1-\frac{a}{\lambda^2}) d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2-\frac{a^2}{\lambda^2}} \, d(\lambda+\frac{a}{\lambda}) + \frac{2}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-\lambda^2-\frac{a^2}{\lambda^2}} d(\lambda-\frac{a}{\lambda})$$

$$= \frac{2}{\sqrt{\pi}} e^{2a}\int_{\tau}^{\infty} e^{-(\lambda+\frac{a}{\lambda})^2} d(\lambda+\frac{a}{\lambda}) + \frac{2}{\sqrt{\pi}} e^{-2a}\int_{\tau}^{\infty} e^{-(\lambda-\frac{a}{\lambda})^2} d(\lambda-\frac{a}{\lambda})$$

is found. If $\tau = r + \frac{a}{r}$ transformation and complement of error function

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_{x}^{\infty} e^{-\tau^2} d\tau$$

are written, thus integration formula

$$= \frac{2}{\sqrt{\pi}} e^{2a}\int_{r+\frac{a}{r}}^{\infty} e^{-\tau^2} d\tau + \frac{2}{\sqrt{\pi}} e^{-2a}\int_{r-\frac{a}{r}}^{\infty} e^{-\tau^2} d\tau$$

$$= e^{2a} erfc(r+\frac{a}{r}) + e^{-2a} erfc(r-\frac{a}{r})$$

is proven.

When the method of undetermined coefficients is applied in equation (10) and boundary conditions of (11) are also considered,

$$c(x,s) = \frac{c_0}{s} e^{\frac{Vx}{2D}} e^{-x\sqrt{\frac{(\frac{V^2}{4D}+k)+s}{D}}} \qquad (13)$$

is obtained as a solution. In order to apply *the convolution property*

$$L^{-1}\{f(s)g(s)\} = F(t)*G(t) = \int_0^t F(\tau)G(t-\tau)d\tau$$

on this equation, let us define

$$f(s) = e^{-x\sqrt{\frac{(\frac{V^2}{4D}+k)+s}{D}}}, \qquad g(s) = \frac{c_0}{s}e^{\frac{Vx}{2D}}$$

By using *inverse Laplace transformation* properties and the formula

$$L^{-1}\{e^{-x\sqrt{\frac{s+h}{l}}}\} = \frac{xe^{-ht}}{2\sqrt{\pi l t^3}}e^{-\frac{x^2}{4lt}} \quad \text{(See [4]),}$$

where

$$h = \frac{V^2}{4D}+k, \qquad l = D,$$

$$F(t) = L^{-1}\{ e^{-x\sqrt{\frac{(\frac{V^2}{4D}+k)+s}{D}}}\} = \frac{xe^{-(\frac{V^2}{4D}+k)t}}{2\sqrt{\pi D t^3}}e^{-\frac{x^2}{4Dt}}$$

$$G(t) = L^{-1}\{\frac{c_0}{s}e^{\frac{Vx}{2D}}\} = c_0 e^{\frac{Vx}{2D}}$$

are obtained. Then,

$$c(x,t) = \frac{c_0 x}{2\sqrt{\pi D}}e^{\frac{Vx}{2D}}\int_0^t e^{-(\frac{V^2}{4D}+k)\tau}e^{-\frac{x^2}{4D\tau}}\frac{d\tau}{\tau^{3/2}} \tag{14}$$

is obtained. If the integration limits are considered as well and replaced and arranged in equation (14) :

$$\lambda = \frac{x}{2\sqrt{D\tau}}, \quad \tau = \frac{x^2}{4D\lambda^2}, \quad -\frac{4\sqrt{D}}{x}d\lambda = \frac{d\tau}{\tau^{3/2}}$$

$$c(x,t) = \frac{2c_0}{\sqrt{\pi}}e^{\frac{Vx}{2D}}\int_{\frac{x}{2\sqrt{Dt}}}^{\infty} e^{-(\frac{V^2}{4D}+k)\frac{x^2}{4D\lambda^2}}e^{-\lambda^2}d\lambda \tag{15}$$

can be found. When equation (15) is equalised with equation (12), it can be seen that

$$a = \sqrt{\frac{V^2}{4D} + k}\,\frac{x}{2\sqrt{D}}\ ,\quad r = \frac{x}{2\sqrt{Dt}}\ .$$

Then, the expressions in equation (12) turns into

$$r \pm \frac{a}{r} = \frac{x \pm t\sqrt{V^2 + 4kD}}{2\sqrt{Dt}}\ , \qquad e^{\pm 2a} = e^{\pm\frac{x\sqrt{V^2+4kD}}{2D}}\ .$$

As a result, the solution of problem (10) with conditions of (11) can be obtained as follows;

$$c(x,t) = \frac{c_0}{2} e^{\frac{Vx}{2D}} \left[e^{(\frac{x\sqrt{V^2+4kD}}{2D})} erfc(\frac{x + t\sqrt{V^2+4kD}}{2\sqrt{Dt}}) + e^{(\frac{-x\sqrt{V^2+4kD}}{2D})} erfc(\frac{x - t\sqrt{V^2+4kD}}{2\sqrt{Dt}}) \right]. \tag{16}$$

in a special case for *k = 0*, equation (16) can be written as

$$c(x,t) = \frac{c_0}{2} \left[e^{\frac{Vx}{2D}} erfc(\frac{x+Vt}{2\sqrt{Dt}}) + erfc(\frac{x-Vt}{2\sqrt{Dt}}) \right]$$ (See [1]).

Conclusion

In this study, partial differential equation with constant coefficient which appear in hydrological problems were taken from literature and were solved by *Laplace transformation*. Problems occurring in advection-dispersion of the contamination in a medium of variable coefficient and one dimensional underground water flow, of which the solutions are complicated, will be dealt with later in this chapter.

References

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