On Semi δ-Continuous Functions

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Abstract: In this paper , we give properties of semi δ-continuous and semi δ-open functions defined by Y.Beceren and Ş.Yüksel [1].

Semi δ-Sürekli Fonksiyonlar Üzerine

Özet: Bu makalede, [1]' de Y. Beceren ve Ş. Yüksel tarafından tanımlanan semi δ-sürekli ve semi δ-açık fonksiyonların bazı özellikleri verilmiştir.

Anahtar Kelimeler: **Semi açık küme, δ-küme, semi sürekli fonksiyon, semi açık fonksiyon, semi δ-sürekli fonksiyon, semi δ-açık fonksiyon**

1. Introduction

Throughout this paper X will always denote topological spaces on which no separations axiom are assumed unless stated explicitly. No mapping is assumed to be continuous unless stated . Let S be a subset of a topological space X. The closure of S in X and interior of S in X will be denoted by Cl(S) and Int (S), respectively.

Definition 1.1. [3] Let S be a subset of a space X. The set S is said to be a semi open if there exists an open subset O of X such that $O \subset S \subset Cl(O)$.

Lemma 1.1. [3] A subset S of a space X is semi open if and only if $S \subset Cl(Int(S))$.

Definition 1.2. [2] A subset S of a space X is said to be δ-set in X if $Int(Cl(S)) \subset Cl(Int(S))$.

In 1991, it shown in [2] that a semi open set is a δ-set, but not converse in general.

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Definition 1.3. [3] A mapping f : $X \to Y$ is said to be semi continuous (resp. semi δ-continuous [1]) if for each open subset V of Y, $f^{-1}(V)$ is semi open set (resp. δ-set) in X.

Definition 1.4. [3] A mapping f : $X \to Y$ is said to be semi open (resp. semi δ-open [1]) if for each open subset U of X, f(U) is semi open set (resp. δ-set) in Y.

Remark 1.1. [2] Obviously every semi continuous mapping (semi open mapping) is semi δ-continuous (semi δ-open), but the converse is not necessarily true as is shown by the following example.

Example 1.1. Let X and Y be the set of real numbers with usual topology. Let the mapping f: $X \to Y$ be defined as follows f(x)=x , if $x \neq 0$ and $x \neq 1$; f(0)=1, f(1)=0. Then f is one-to-one semi δ-continuous , semi δ-open, but neither semi continuous nor semi open.

Theorem 1. 1.[3] A mapping $f:X \to Y$ is semi continuous if and only if for any point $x \in X$ and any open set V of Y containing f(x), there exists $U \in SO(X)$ such that $x \in U$ and $f(U) \subset V$.

However, we have the following theorem.

Theorem 1.2. If $f:X \to Y$ is semi δ-continuous, then for any point $x \in X$ and any open set V of Y containing f(x), there exists $U \in \delta(X)$ such that $x \in U$ and $f(U) \subset V$.

Proof. Let $f(x) \in V$. Then $x \in f^{-1}(V) \in \delta(X)$ since f is semi δ-continuous. Let $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subset V$.

Theorem 1.3. Let $h : X \to X_1 x X_2$ be semi δ-continuous where X, X_1 and X_2 are topological spaces. Let $f_i : X \to X_i$ as follows: for $x \in X$, $h(x) = (x_1, x_2)$. Let $f_i(x) = x_i$. Then $f_i : X \to X_i$ is semi δ-continuous for i=1,2.

Proof. We shall show only that $f_1 : X \to X_1$ is semi δ-continuous. Let O_1 be open in X_1. Then $O_1 x X_2$ is open in $X_1 x X_2$ and $h^{-1}(O_1 x X_2)$ is δ-set in X. But $f_1^{-1}(O_1) = h^{-1}(O_1 x X_2)$ and thus $f_1 : X \to X_1$ is semi δ-continuous.

The following theorem is a generalization of Theorem 1.3.

Theorem 1.4. Let $\{X_\alpha | \alpha \in I\}$ be any family of topological spaces. If $f:X \to \prod X_\alpha$ is a semi δ-continuous , then $p_\alpha o f : X \to X_\alpha$ is semi δ-continuous for each $\alpha \in I$, where p_α is the projection of $\prod X_\beta$ into X_α.

Proof. We shall consider a fixed $\alpha \in I$. Suppose U_α is an arbitrary open set in X_α. Then $p_\alpha^{-1}(U_\alpha)$ is open in $\prod X_\alpha$. Since f is semi δ-continuous, we have

$$f^{-1}[\, p_\alpha^{-1}(U_\alpha)\,] = (p_\alpha o f)^{-1}(U_\alpha) \in \delta(X).$$

Therefore, $p_\alpha o f$ is semi δ-continuous.

N. Levine [4] showed that if $f:X \to Y$ is an open and semi continuous , then $f^{-1}(B) \in SO(X)$ for every $B \in SO(Y)$.

We have the following theorem from this theorem.

Theorem 1.5. If $f:X \to Y$ is an open and semi δ-continuous, then $f^{-1}(B) \in \delta(X)$ for every $B \in SO(Y)$.

Proof. For an arbitrary $B \in SO(Y)$, there exists an open set V in Y such that $V \subset B \subset ClV$. Since f is open, we have $f^{-1}(V) \subset f^{-1}(B) \subset f^{-1}(ClV) \subset Clf^{-1}(V)$ [4,(i), p. I 3]. Since f is semi δ-continuous and V is open in Y, $f^{-1}(V) \in \delta(X)$. Therefore by Theorem 1.3 of [2], we obtain $f^{-1}(B) \in \delta(X)$.

Corollary 1.1. Let X,Y and Z be topological spaces. If $f:X\rightarrow Y$ is an open and semi δ-continuous and $g:Y\rightarrow Z$ is a semi δ-continuous, then $g\circ f:X\rightarrow Z$ is semi δ-continuous.

Theorem 1.6. Let $S\subset Y\subset X$. If Y is an open subset of X and S is a δ-set in X, then the set S is a δ-set inY.

Proof. Let S be a δ-set of space X. Then $Int(Cl(S))\subset Cl(Int(S))$. Hence

$$Int_Y(Cl_Y(S))=Int(Cl(S))\cap Y\subset Cl(Int(S))= Cl_Y(Int_Y(S))$$

where $Cl_Y(S)=Cl(S)\cap Y$. Thus, the set S is a δ-set inY.

Theorem 1.7. Let $f:X\rightarrow Y$ be semi δ-continuous mapping and let S be an open subset of X. Then $f_{/S}: S\rightarrow f(S)$ defined by $f_{/S}(x)=f(x)$, for all $x\in S$ is semi δ-continuous.

Proof. Let W be any open subset in f(S). Then there exists an open subset V in Y such that $W=V\cap f(S)$. Consequently $(f_{/S})^{-1}(W) = f^{-1}(W) \cap S = f^{-1}(V \cap f(S)) \cap S$. From this, we have $(f_{/S})^{-1}(W)=S\cap f^{-1}(V)$. Since f is semi δ-continuous, then $f^{-1}(V)$ is δ-set in X and also $S\cap f^{-1}(V)$ is δ-set in X by [2, Proposition 2.1]. Hence $(f_{/S})^{-1}(W)=S\cap f^{-1}(V)$ is δ-set in S by Theorem 1.6.

Theorem 1.8. Let $f : X \rightarrow Y$ be one-to-one semi δ-open and let $S\subset X$ be such that f(S) is open in Y. Then $f_{/S}:S\rightarrow f(S)$ defined by $f_{/S}(x)=f(x)$, for all $x\in S$ is semi δ-open.

Proof. Let U be any open set in S. Then there exists an open subset V in X such that $U=S\cap V$. Thus, $f_{/S}(U)=f(U)=f(S\cap V)=f(S)\cap f)(V)$. Since f(V) is δ-set by the semi δ-open of f, it follows that $f_{/S}(U)$ is δ-set in the subspace f(S) showing $f_{/S}: S\rightarrow f(S)$ is semi δ-open.

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