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On The Bounds of Norms of Circulant Cauchy-Toeplitz Matrices

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Abstract: *In this study, we have found bounds for the spectral and ℓ_p norm of circulant Cauchy-Toeplitz matrices in the form*

$$T_n = [1/(g+(i-j)h)]_{i,j=1}^{n} \equiv \left([1/(g+kh)]_{i,j=1}^{n}\right)$$

where $k = 0,1,\dots,n-1$.

Keywords: Circulant matrix, Cauchy-Toeplitz matrix, norm.

Circulant Cauchy-Toeplitz Matrislerin Normlarının Sınırları Üzerine

Özet: Bu çalışmada $T_n = [1/(g+(i-j)h)]_{i,j=1}^{n} \equiv \left([1/(g+kh)]_{i,j=1}^{n}\right)$ formunda tanımladığımız circulant Cauchy-Toeplitz matrislerinin spektral ve ℓ_p normları için sınırlar bulduk.

Anahtar Kelimeler: **Circulant matris, Cauchy-Toeplitz matris, norm.**

1. Introduction

Let $A_n = [1/(x_i - y_j)]_{i,j=1}^{n}$ be a Cauchy matrix and $T_n = [t_{j-i}]_{i,j=1}^{n}$ be a Toeplitz matrix. In generally Cauchy-Toeplitz matrices are being defined as

$$T_n = \left[\frac{1}{g+(j-i)h}\right]_{i,j=1}^{n} \tag{1.1}$$

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where $h \neq 0$, *g* and *h* are some numbers and quotient *g/h* is not integer.

Closely related to Toeplitz matrices are the so-called circulant matrices. An $(n \times n)$ matrix *C* is called a circulant matrix if it is of the form

$$C_n = \begin{bmatrix} c_0 & c_1 & c_2 & . & . & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & . & . & c_{n-3} & c_{n-2} \\ . & . & . & . & . & . & . \\ c_2 & c_3 & c_4 & . & . & c_0 & c_1 \\ c_1 & c_2 & c_3 & . & . & c_{n-1} & c_0 \end{bmatrix}.$$

For each *i,j*=1,...,*n* and *k*=0,1,2,..,*n*-1, all the elements (*i,j*) such that $j - i \equiv k (\mathrm{mod}\, n)$ have the same value c_k; these elements form the so-called *k*th stripe of *C*. Obviously, a circulant matrix is determined by its first row (or column). It is clear that every circulant matrix is a Toeplitz matrix, but the converse is not necessarily true [2].

If we replace $j - i \equiv k (\mathrm{mod}\, n)$ by (*j-i*) in (1.1) then the matrix T_n as follows

$$T_n = \left[\frac{1}{g+kh}\right]_{n \times n}, \quad k=0,1,2,..,n-1 \tag{1.2}$$

i.e.

$$T_n = \begin{bmatrix} 1/g & 1/(g+h) & 1/(g+2h) & . & . & . & 1/(g+(n-1)h) \\ 1/(g+(n-1)h) & 1/g & 1/(g+h) & . & . & . & 1/(g+(n-2)h) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1/(g+h) & 1/(g+2h) & 1/(g+3h) & . & . & . & 1/g \end{bmatrix}_{n \times n}.$$

We called this matrix as circulant Cauchy-Toeplitz matrix.

Let *A* be any $n \times n$ matrix. The ℓ_p norms of the matrix *A* are defined as

$$\|A\|_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p\right)^{1/p} \quad (1 \leq p < \infty). \tag{1.3}$$

If $p = \infty$, then

$$\|A\|_{\infty} = \lim_{n\to\infty} \|A\|_{p} = \max_{i,j} |a_{ij}|.$$

The well known Euclidean norm of matrix A is

$$\|A\|_{E} = \left[\sum_{i,j=1}^{n} |a_{ij}|^{2} \right]^{1/2}. \quad (1.4)$$

and also the spectral norm of matrix A is

$$\|A\|_{2} = \sqrt{\max_{1\le i\le n} \lambda_{i}} \quad (1.5)$$

where λ_i is eigenvalue of $A^{H}A$ and A^{H} is conjugate transpose of matrix A. Between $\|A\|_{E}$ and $\|A\|_{2}$ norms is valid as follows [5]:

$$\frac{1}{\sqrt{n}} \|A\|_{E} \le \|A\|_{2}. \quad (1.6)$$

A function Ψ is called as psi (or digamma) function if

$$\Psi(x) = \frac{d}{dx}\{\log[\Gamma(x)]\}$$

where

$$\Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt.$$

It is called as Polygamma function the n th derivatives of psi function [4] i.e.

$$\Psi(n,x) = \frac{d}{dx^{n}} Psi(x)$$

$$= \frac{d}{dx^{n}} \left[\frac{d}{dx} \ln[\Gamma(x)] \right]$$

Where if n=0 then $\Psi(0,x) = Psi(x) = \frac{d}{dx}\{\ln[\Gamma(x)]\}$. On the other hand if a>0 and b is any number and n is positive integer, then

$$\lim_{n\to\infty} \Psi(a, n+b) = 0. \quad (1.7)$$

In this study, we have found bounds for norms of circulant Cauchy-Toeplitz matrices. Where $\mathbb{Z}^+$ and $\mathbb{R}^+$ will represent the sets of positive integers and positive real numbers, respectively.

2. Norms of Circulant Cauchy-Toeplitz Matrices

Theorem 2.1. Let the matrix T_n be as in (1.2). Then

$$n^{-1/p}\|T_n\|_p \le \left\{\frac{(-1)^p}{(p-1)!h^p}\Psi(p-1,g/h)\right\}^{1/p}$$

is valid for the ℓ_p norm of the matrix T_n where $2 \le p < \infty$ and $g,h \in \mathbb{R}^+$.

Proof. From (1.3) we have

$$\|A\|_p = \left[\sum_{i,j=1}^{n}|a_{ij}|^p\right]^{1/p} = \left[\sum_{i=1}^{n}\left(|a_{1i}|^p\|e_i\|^p + |a_{2i}|^p\|e_i\|^p + \ldots + |a_{ni}|^p\|e_i\|^p\right)\right]^{1/p}$$

where $e_i\,(1 \le i \le n)$ is the basis of $\mathbb{R}^n$. For the matrix T_n

$$\|T_n\|_p^p = \sum_{s=1}^{n}\frac{n}{[g+(s-1)h]^p}. \qquad (2.1)$$

If we divide by n two hand side of the (2.1) then

$$\frac{1}{n}\|T_n\|_p^p = \sum_{s=1}^{n}\frac{1}{[g+(s-1)h]^p}.$$

If we evaluate the right handside of this equality, we have

$$\sum_{s=1}^{n}\frac{1}{(g+(s-1)h)^p} = \frac{(-1)^{p-1}}{(p-1)!h^p}\left[\Psi(p-1,n+g/h) - \Psi(p-1,g/h)\right].$$

Hence from (1.7), we obtain

$$\lim_{n\to\infty}\left\{\sum_{s=1}^{n}\frac{1}{(g+(s-1)h)^p}\right\} = \frac{(-1)^p}{(p-1)!h^p}\Psi(p-1,g/h).$$

Consequently we can write as follows

$$\frac{1}{n}\|T_n\|_p^p \le \left\{ \frac{(-1)^p}{(p-1)!h^p} \Psi(p-1, g/h) \right\} \tag{2.2}$$

If we take *1/p th* power of inequality (2.2) then the proof is completed.

Theorem 2.2. Let the matrix T_n be as in (1.2). then

$$\sqrt{\frac{1}{h^2}\Psi(1, g/h)} \le \|T_n\|_2$$

is valid for the spectral norm of the matrix T_n where $h \in \mathbb{Z}^+$, $g \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, *h*>*g*, *n*>1.

Proof. From (1.4) we have

$$\|T_n\|_E^2 = \left\{ \sum_{s=1}^{n} \frac{n}{(g+(s-1)h)^2} \right\}. \tag{2.3}$$

If we divide by *n* two hand side of the (2.3) then

$$\frac{1}{n}\|T_n\|_E^2 = \left\{ \sum_{s=1}^{n} \frac{1}{(g+(s-1)h)^2} \right\}.$$

From the properties of polygamma functions the right handside of this equality is wrote as the following:

$$\sum_{s=1}^{n} \frac{1}{(g+(s-1)h)^2} = \frac{1}{h^2}\left[\Psi(1, g/h) - \Psi(1, n+g/h)\right]$$

Hence from (1.7),

$$\lim_{n\to\infty} \left\{ \sum_{s=1}^{n} \frac{1}{(g+(s-1)h)^2} \right\} = \frac{1}{h^2}\Psi(1, g/h).$$

Hence

$$\frac{1}{n}\|T_n\|_E^2 = \left\{ \frac{1}{h^2}\Psi(1, g/h) \right\}$$

and

$$\frac{1}{\sqrt{n}}\|T_n\|_E = \sqrt{\frac{1}{h^2}\Psi(1, g/h)}.$$

Thus from (1.6) we have

$$\sqrt{\frac{1}{h^2}\Psi(1, g/h)} \le \|T_n\|_2. \tag{2.4}$$

3. Numerical Results

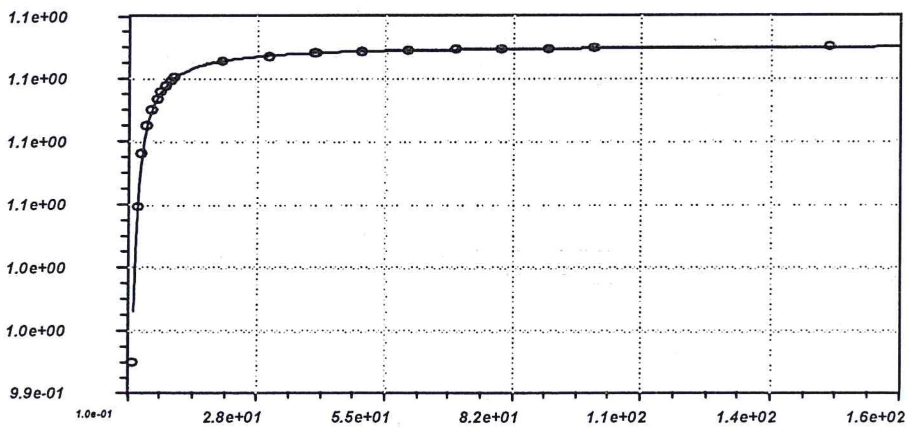
We shall give numerical values, graphs and functions for theorem 2.1. and theorem 2.2.

Example 3.1. Let *g*, *h* and *p* be 1,2,2 respectively. Using these values in theorem 2.1, we get the following values in table 3.1.

n	$n^{-1/p}\|T_n\|_p$	$\left[\frac{(-1)^p\Psi(p-1,g/h)}{(p-1)!h^p}\right]^{1/p}$	n	$n^{-1/p}\|T_n\|_p$	$\left[\frac{(-1)^p\Psi(p-1,g/h)}{(p-1)!h^p}\right]^{1/p}$
1	1	1.110720735	20	1.105080609	1.110720735
2	1.054092553	"	30	1.106963408	"
3	1.072898463	"	40	1.107903821	"
4	1.082367440	"	50	1.108467733	"
5	1.088055584	"	60	1.108843532	"
6	1.091846792	"	70	1.109111889	"
7	1.094553140	"	80	1.109313116	"
8	1.096581515	"	90	1.109469604	"
9	1.098158107	"	100	1.109594778	"
10	1.099418624	"	150	1.109970220	"

Table 3.1.

The graph which corresponds to n and $n^{-1/p}\|T_n\|_p$ values in the table 3.1. is as follows.



Where horizontal axis and vertical axis denote n and $n^{-1/p}\|T_n\|_p$, respectively. The function corresponding to the above graph is as follows.

$$f(n)=\frac{ab+cn^d}{b+n^d}$$

where a=-0.046957818, b=0.10554744, c=1.1105646 and d=1.0387502. If we take limit as $n\to\infty$, we get

$$\lim_{n\to\infty} f(n)=c=1.1105646.$$

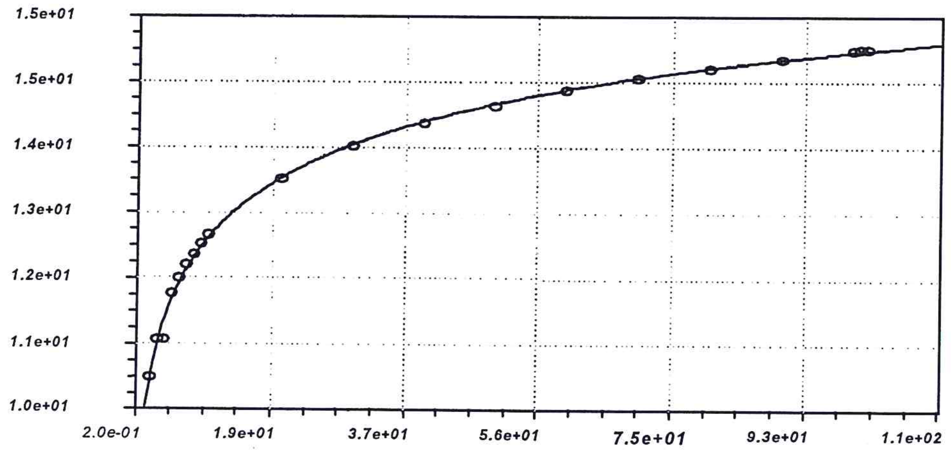
Consequently, we can write $n^{-1/p}\|T_n\|_p \le \lim_{n\to\infty} f(n)=c$.

Example 3.2. Let g and h be 0.1 and 1 respectively. Using these values in theorem 2.2, we get the following values in table 3.2.

n	$\sqrt{\frac{\Psi(1,g/h)}{h^2}}$	$\|T_n\|_2$	n	$\sqrt{\frac{\Psi(1,g/h)}{h^2}}$	$\|T_n\|_2$
2	10.07140999	10.90909091	20	10.07140999	13.39939292
3	"	11.38528139	30	"	13.81157685
4	"	11.38528139	40	"	14.10261063
5	"	11.95176447	50	"	14.32776271
6	"	12.14784290	60	"	14.51142225
7	"	12.31177733	70	"	14.66652811
8	"	12.45262240	80	"	14.80077561
9	"	12.57607919	90	"	14.91911545
10	"	12.68596930	100	"	15.02492131

Table 3.2.

The graph which corresponds to n and $n^{-1/p}\|T_n\|_p$ values in the table 3.2. is as follows.



Where horizontal axis and vertical axis denote n and $\|T_n\|_2$, respectively. The function corresponding to the above graph is as follows.

$$f(n) = \frac{ab + cn^d}{b + n^d}$$

where a=3.6122232, b=1.4853264, c=20.019447 and d=0.26394827. If we take limit as $n \to \infty$, we get

$$\lim_{n \to \infty} f(n) = c = 20.019447 .$$

Consequently, we can write $\|T_n\|_2 \leq \lim_{n \to \infty} f(n) = c$.

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