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VARIABLE STRUCTURE EXTREMUM PROBLEM WITH CONSTRAINTS FOR DELAY DISCRETE INCLUSIONS

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ABSTRACT: The paper considers a variable structure extremum problem with constraints for delay discrete inclusions. The necessary extremum conditions are obtained for the considered problem.

Keywords: Discrete Inclusion; Necessary Extremum Condition; Subdifferential; Lipschitz Condition; Tangent Cone; Hypertangent cone

JEL Classifications: C61

ÖZET: *Makalede gecikmeli diskret içermeler için kısıtlamalı değişken yapılı ekstremum problemi araştırılmaktadır. Ele aldığımız problem için gerekli ekstremum koşulları bulunmuştur.*

Anahtar Kelimeler: *Diskret İçerme; Gerek Ekstremum Koşulu, Altdiferansiyel, Lipschitz Koşulu, Teğet Koni, Hiperteğet Koni*

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1. Introduction

The extremum problems for discrete inclusions with delay are a generalization of the discrete problems of optimal control with delay. Application of nonsmooth analysis theory is of importance for investigation of such a problem. Since the definition of the subdifferential given by T. Rockafellar and F.Clarke is entirely a generalization of the smooth and convex problems, the application of the theory of nonsmooth analysis in this paper is advisable. As the necessary extremum conditions are formulated more naturally with the help of the subdifferential, the definition of subdifferential is of importance in the theory of extremum problems.

There exist a lot of papers devoted to the qualitative investigation of different problems of optimal control of discrete systems. The control problems of discrete systems, described by the different difference equations are of importance among various optimal control problems. In spite of this fact, there are many unstudied problems in the field of extremum problems for discrete inclusions. Recently, multivalued mappings became the subject of intensive study. Different properties of multivalued mappings and their connection with the theory of optimization were considered in (Aubin, Ekeland, 1984: 510; Borisovich, Gelman, Mishkins, Obukhovskiy, 1986: 103).

Note that main models of mathematical economics are reduced to extremum problems for discrete inclusions.

If the state of the control system is characterized by a system of more than two equations and an operator connecting the equations of the system (connection operator) exists, then such a control system is called a variable structure system.

This work generalizes some of the results, obtained in (Mirzayeva, Sadygov, 2005: 89-94; Mirzayeva, Sadygov, 2006: 106-112). An arbitrary order necessary extremum conditions are obtained for the nonconvex optimal control problem for discrete inclusions with delay in (Mirzayeva, Sadygov, 2006: 106-112). In this paper the extremum problem for delay discrete embedding is reduced to the mathematical programming problem using the method given in (Boltyanskiy, 1975: 3-55). Further, using Clarke's theory, necessary extremum conditions are obtained. Differently from the considered works (Mirzayeva, Sadygov, 2005: 89-94; Mirzayeva, Sadygov, 2006: 106-112; Mirzayeva, Sadygov, 2007: 67-72), in the paper the discrete extremum problem with constraints is considered. Note that the extremum problem for delay discrete inclusions, is a particular case of variable structure extremum problem for delay discrete inclusions. The known optimal control problems of a variable structure are obtained from variable structure extremum problems for delay discrete inclusions (Mirzayeva, 2007: 44-50). The optimal control problems of variable structure with discrete time delay are considered in (Mirzayeva, 2007: 44-50). Such problems are considered by many authors. Differently from the known works, in this paper a nonsmooth case is considered and the problem is reduced to the extremum problem for delay discrete inclusions with variable structure. Further, in the paper the necessary extremum condition is obtained for discrete systems with variable structure.

Variable structure extremum problem for delay discrete inclusions without constraints is considered in (Mirzayeva, Sadygov, 2007: 67-72). The necessary extremum conditions are obtained for one discrete systems class in (Mirzayeva, Sadygov, 2007: 67-72).

Note that optimal control problems of variable structure arise while investigating some chemical-technological processes, applied problems of economics and physics.

2. The formulation of the problem

Let X, Y be Banach spaces, $a_t : X^2 \to 2^X$, $t = 0,1,...,k-1$, $b_t : Y^2 \to 2^Y$, $t = k, k+1,...,m-1$ be the multivalued mappings, where 2^V denotes the set of all subsets of V . We denote $grF = \{(z,v) \in Z \times V : v \in F(z)\}$.

Let us consider the delay discrete inclusions with variable structure

$$
\begin{aligned}
&x_{t+1} \in a_t(x_{t-\Delta}, x_t), \quad t = 0,1,...,k-1 \\
&x_t = c(t) \quad at \quad t = -\Delta, -\Delta+1,...,-1,0 \\
&y_{t+1} \in b_t(y_{t-h}, y_t), \quad t = k, k+1,...,m-1 \\
&y_t = G(x_t) \quad at \quad t = k-h, k-h+1, k-h+2,...,k \\
&y_m \in C,
\end{aligned}
\tag{2.1}
$$

where $c(t) \in X$ at $t = -\Delta, -\Delta+1, ..., -1, 0$, $C \subset Y, G : X \to Y$ is mapping, k, m, Δ, h are fixed natural numbers. As a trajectory (solution) $(\{x_t\}, \{y_v\})$ of the discrete inclusion (2.1) we understand the process x_t, $t = 1, ..., k-1, k$, $y_v, v = k+1, ..., m$, for which (2.1) is satisfied.

Suppose that

$$\Delta < k-1,\ h < \min\{k-1, m-k-1\},\ g_i(\cdot, t) : X \to R,\ t = 1, ..., k,\ i = 1, ..., n$$
$$f_i(\cdot, t) : Y \to R,\ t = k+1, ..., m,\ i = 1, ..., n .$$

We denote $x = (x_1, ..., x_k),\ y = (y_{k+1}, ..., y_m)$.

Consider the minimization of the function

$$F_0(x, y) = \sum_{t=1}^{k} g_0(x_t, t) + \sum_{t=k+1}^{m} f_0(y_t, t) \tag{2.2}$$

on the trajectories of discrete inclusion (2.1) and with the following constraints

$$\begin{gathered} F_1(x, y) = \sum_{t=1}^{k} g_1(x_t, t) + \sum_{t=k+1}^{m} f_1(y_t, t) \le 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F_j(x, y) = \sum_{t=1}^{k} g_j(x_t, t) + \sum_{t=k+1}^{m} f_j(y_t, t) \le 0, \\ F_{j+1}(x, y) = \sum_{t=1}^{k} g_{j+1}(x_t, t) + \sum_{t=k+1}^{m} f_{j+1}(y_t, t) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F_n(x, y) = \sum_{t=1}^{k} g_n(x_t, t) + \sum_{t=k+1}^{m} f_n(y_t, t) = 0. \end{gathered} \tag{2.3}$$

We note that as a trajectory of (2.1) we take the pairs (x, y), for which (2.1) is satisfied. Denote by M the set of solutions of problem (2.1). To reduce the formulated problem to the mathematical programming problem we use the following notation. We denote $s = m - k$ and define the sets in $X^k \times Y^s$ as

$$M_0 = \{(x_1, ..., x_k, y_{k+1}, ..., y_m) \in X^k \times Y^s : x_1 \in a_0(c(-\Delta), c(0))\},$$
$$M_1 = \{(x_1, ..., x_k, y_{k+1}, ..., y_m) \in X^k \times Y^s : x_2 \in a_1(c(-\Delta+1), x_1)\},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$M_\Delta = \{(x_1, ..., x_k, y_{k+1}, ..., y_m) \in X^k \times Y^s : x_{\Delta+1} \in a_\Delta(c(0), x_\Delta)\},$$
$$M_{\Delta+1} = \{(x_1, ..., x_k, y_{k+1}, ..., y_m) \in X^k \times Y^s : x_{\Delta+2} \in a_{\Delta+1}(x_1, x_{\Delta+1})\},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$M_{k-1} = \{(x_1, ..., x_k, y_{k+1}, ..., y_m) \in X^k \times Y^s : x_k \in a_{k-1}(x_{k-1-\Delta}, x_{k-1})\},$$

$$M_k = \{(x_1,...,x_k,\ y_{k+1},...,y_m) \in X^k \times Y^s : y_{k+1} \in b_k(G(x_{k-h}), G(x_k))\},$$
$$M_{k+1} = \{(x_1,...,x_k,\ y_{k+1},...,y_m) \in X^k \times Y^s : y_{k+2} \in b_{k+1}(G(x_{k+1-h}), y_{k+1})\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$M_{k+h} = \{(x_1,...,x_k,\ y_{k+1},...,y_m) \in X^k \times Y^s : y_{k+h+1} \in b_{k+h}(G(x_k), y_{k+h})\},$$
$$M_{k+h+1} = \{(x_1,...,x_k,\ y_{k+1},...,y_m) \in X^k \times Y^s : y_{k+h+2} \in b_{k+h+1}(y_{k+1}, y_{k+h+1})\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$M_{m-1} = \{(x_1,...,x_k,\ y_{k+1},...,y_m) \in X^k \times Y^s : y_m \in b_{m-1}(y_{m-1-h}, y_{m-1})\},$$
$$M_m = \{(x_1,...,x_k,\ y_{k+1},...,y_m) \in X^k \times Y^s : y_m \in C\}.$$

It is clear that the formulated problem will be reduced to the minimization of the function $F_0(x,y)$ on the set $M = \bigcap_{t=0}^{m} M_t$ with the constraints

$$F_1(x,y) \le 0, \ldots, F_j(x,y) \le 0,\ F_{j+1}(x,y) = 0, \ldots, F_n(x,y) = 0.$$

3. The solution of the problem

Let Z be a Banach space, E be a nonempty subset of Z. Clarke's subdifferential of the function φ at the point z_0 is denoted as $\partial\varphi(z_0)$.

Suppose $z_0 \in E$. The set $T_E(z_0) = \{z \in Z : d_E^0(z_0; z) = 0\}$ is called tangent cone to E in z_0.

The set $N_E(z_0) = \{z^* \in Z^* : < z^*, z > \le 0 \text{ at } z \in T_E(z_0)\}$ is called normal cone to E in z_0.

Let $D \subset Z$. The set of all hypertangents to D at the point $\bar{z} \in D$ denote by $I_D(\bar{z})$. By the definition (see (Clarke, 1988: 279))

$$I_E(\bar{z}) = \{v \in Z : \exists \varepsilon > 0, \text{that } y + t\omega \in E \text{ at all } y \in (\bar{z} + \varepsilon B) \cap E,\ \ \omega \in v + \varepsilon B, t \in (0, \varepsilon)\}.$$

We note if $g_i(\cdot,t), i = 0,...,n$, satisfy the Lipschitz condition in the neighbourhood of x_t , where $t = 1,...,k$ and $f_i(\cdot,t), i = 0,...,n$, satisfy the Lipschitz condition in the neighbourhood of y_t , where $t = k+1,...,m$, then $F_i(x,y), i = 0,...,n$ satisfy the Lipschitz condition in the neighbourhood of (x,y). Denoted by Ω the set of solutions of problem (2.1), satisfying condition (2.3). The pairs $(\bar{x},\bar{y}) \in \Omega$ are called optimal , if $F_0(\bar{x},\bar{y}) \le F_0(x,y)$ at $(x,y) \in \Omega$.

We use Lagrange's generalized function for the nonsmooth problem of mathematical programming. Let $L((x,y),\lambda,r)=\sum_{i=0}^{n}\lambda_i F_i(x,y)+r|\lambda|d_M(x,y)$, where $\lambda=(\lambda_0,\lambda_1,\ldots,\lambda_n)$, $\lambda_i, r\in R$, $i=0,...,n$. The following corollary follows from Theorem 6.1.1 (Clarke, 1988: 279).

Corollary 1. Let $(\bar{x},\bar{y})\in\Omega$ minimize the functional F_0 on the set Ω, the functions $F_0,F_1,\ldots F_n$ satisfy the Lipschitz condition in the neighbourhood of $(\bar{x},\bar{y})$, M be a closed set. Then for sufficiently large r the numbers $\lambda_0\geq 0,\lambda_1\geq 0,\ldots,\lambda_j\geq 0$ and $\lambda_{j+1},\ldots,\lambda_n$ may be found, not all equal to zero simultaneously, such that $\sum_{i=1}^{j}\lambda_i F_i(\bar{x},\bar{y})=0$ and $0\in\partial L((\bar{x},\bar{y}),\lambda,r)$.

In the paper, the necessary extremum conditions are obtained, considering the additional conditions A, B of R.T. Rockafellar (Rockafellar, 1998: 733), the conditions C, D (Sadygov, 2007: 59). Similarly to subdifferential calculation the certain additional conditions are required for calculating normal cones. Such problems (the problems of calculation of normal cones) were studied by some authors as F.H. Clarke, R.T.Rockafellar, Dubovitsky-Milutin and others. These conditions are not equivalent, though they are considered for the same purpose.

Condition A. $T_{M_0}(\bar{z})\cap(\bigcap_{i=1}^{m} I_{M_i}(\bar{z}))\neq\varnothing.$

Condition B. $X=R^{n_1}$, $Y=R^{n_2}$, from $\omega_i^*\in N_{M_i}(\bar{z})$ and $\omega_0^*+\omega_1^*+\cdots+\omega_m^*=0$ it follows that $\omega_i^*=0$ at $i=0,...,m$.

Condition C. $X=R^{n_1}$, $Y=R^{n_2}$, $T_{M_l}(\bar{z})-\bigcap_{i=0}^{l-1}T_{M_i}(\bar{z})=R^{n_1k}\times R^{n_2s}$, $l=1,...,m$, and M_i are closed sets at $i=0,...,m$.

Condition D. $X=R^{n_1}$, $Y=R^{n_2}$, $T_{M_l}(\bar{z})-\bigcap_{i=l+1}^{m}T_{M_i}(\bar{z})=R^{n_1k}\times R^{n_2s}$, $l=0,...,m-1$, and M_i are closed sets at $i=0,...,m$.

Proposition 1. If one of the conditions A, B, C or D holds, then

$$N_M(\bar{z})\subset\sum_{i=0}^{m}N_{M_i}(\bar{z}).$$

Further we suppose that $b_t:Y^2\to C(Y)$, where $C(Y)$ denote the families of all nonempty closed subsets of Y. We denote $F_\lambda(\bar{x},\bar{y})=\sum_{i=0}^{n}\lambda_i F_i(x,y)$.

Theorem 1. Let $\bar{z}=(\bar{x},\bar{y})=(\bar{x}_1,\ldots,\bar{x}_k,\bar{y}_{k+1},\ldots\bar{y}_m)\in\Omega$ minimize the functional F_0 on the set Ω, gra_t at $t=0,...,k-1$, grb_t at $t=k,...,m-1$ and C be closed sets, $G:X\to Y$ continuous operator, the functions $g_i(\cdot,t):X\to R$, $i=0,...,n$

satisfy the Lipschitz condition in the neighbourhood of $\bar{x}_t$ at $t = 1,...,k$, the functions $f_i(\cdot,t)$, $i = 0,...,n$ satisfy the Lipschitz condition in the neighbourhood of $\bar{y}_t$ at $t = k+1,...,m$. If in addition one of the conditions A, B, C or D holds, then there exist vectors $\omega_t^* \in N_{M_t}(\bar{z}), t = 0,1,\ldots,m$ and the numbers $\lambda_0 \geq 0, \lambda_1 \geq 0, \ldots, \lambda_j \geq 0$ и $\lambda_{j+1},\ldots,\lambda_n$ may be found, not all equal to zero simultaneously, where $\sum_{i=1}^{j} \lambda_i F_i(\bar{x},\bar{y}) = 0$ and $\omega^* \in \partial F_\lambda(\bar{x},\bar{y})$ such that $\omega^* = -\sum_{i=0}^{m} \omega_i^*$.

Proof. Using Theorem 1.3.12 (Borisovich, Gelman, Mishkins, Obukhovskiy, 1986: 103) we have that M_s is closed at $s = k,...,k+h$. So the intersection of finitely many closed sets is a closed set, therefore according to this condition we obtain that M is closed. It is straightforward to check that the conditions of corollary 1 are satisfied. Then for enough large r the numbers $\lambda_0 \geq 0, \lambda_1 \geq 0, \ldots, \lambda_j \geq 0$ and $\lambda_{j+1},\ldots,\lambda_n$ may be found, not all equal to zero simultaneously such that $\sum_{i=1}^{j} \lambda_i F_i(\bar{x},\bar{y}) = 0$ and

$$0 \in \partial\left(\sum_{i=0}^{n} \lambda_i F_i + r|\lambda| d_M\right)(\bar{x},\bar{y}) \subset \partial \sum_{i=0}^{n} \lambda_i F_i(\bar{x},\bar{y}) + r|\lambda| \partial d_M(\bar{x},\bar{y}) \subset$$
$$\subset \partial \sum_{i=0}^{n} \lambda_i F_i(\bar{x},\bar{y}) + N_M(\bar{x},\bar{y}).$$

Since $N_M(\bar{x},\bar{y}) \subset \sum_{t=0}^{m} N_{M_t}(\bar{x},\bar{y})$, then we have that $0 \in \partial F_\lambda(\bar{x},\bar{y}) + \sum_{t=0}^{m} N_{M_t}(\bar{x},\bar{y})$. Therefore there exist $\omega_t^* \in N_{M_t}(\bar{z}), t = 0,1,\ldots,m$, and $\omega^* \in \partial F_\lambda(\bar{x},\bar{y})$ such that $\omega^* = -\sum_{t=0}^{m} \omega_t^*$. The theorem is proved.

Theorem 2. If $\bar{z} = (\bar{x},\bar{y}) = (\bar{x}_1,\ldots,\bar{x}_k,\bar{y}_{k+1},\ldots,\bar{y}_m) \in \Omega$ minimizes functional F_0 on the set Ω, gra_t at $t = 0,...,k-1$, grb_t at $t = k,...,m-1$ and C are closed sets, $G : X \to Y$ is continuous operator, the functions $g_i(\cdot,t) : X \to R, i = 0,...,n$, satisfy the Lipschitz condition in the neighbourhood of $\bar{x}_t$ at $t = 1,...,k$, the functions $f_i(\cdot,t), i = 0,...,n$, satisfy the Lipschitz condition in the neighbourhood of $\bar{y}_t$ at $t = k+1,...,m$ and in addition one of the conditions A, B, C or D holds, then the numbers $\lambda_0 \geq 0, \lambda_1 \geq 0, \ldots, \lambda_j \geq 0$ and $\lambda_{j+1},\ldots,\lambda_n$ may be found, not all equal to zero simultaneously, where $\sum_{i=1}^{j} \lambda_i F_i(\bar{x},\bar{y}) = 0$ and there vectors exist

$$x_{t0}^* \in \partial \sum_{i=0}^{n} \lambda_i g_i(\bar{x}_t, t) \text{ at } t = 1,\dots,k,\ x_{t0}^* \in \partial \sum_{i=0}^{n} \lambda_i f_i(\bar{y}_t, t) \text{ at } t = k+1,\dots,m,$$

$$x_1^*(0) \in N_{a_0(c(-\Delta),c(0))}(\bar{x}_1),\ \left(x_t^*(t), x_{t+1}^*(t)\right) \in N_{gra_t(c(-\Delta+t),\cdot)}(\bar{x}_t, \bar{x}_{t+1}),\ t = 1,\dots,\Delta,$$

$$\left(x_t^*(\Delta+t), x_{\Delta+t}^*(\Delta+t), x_{\Delta+t+1}^*(\Delta+t)\right) \in N_{gra_{\Delta+t}}\left(\bar{x}_t, \bar{x}_{\Delta+t}, \bar{x}_{\Delta+t+1}\right) \text{ at } t = 1,\dots,k-1-\Delta,$$

$$(x_{k-h}^*(k), x_k^*(k), y_{k+1}^*(k)) \in N_{grb_k(G(\cdot),G(\cdot))}(\bar{x}_{k-h}, \bar{x}_k, \bar{y}_{k+1}),$$

$$\left(x_{k+t-h}^*(k+t), y_{k+t}^*(k+t), y_{k+t+1}^*(k+t)\right) \in N_{grb_{k+t}(G(\cdot),\cdot)}(\bar{x}_{k+t-h}, \bar{y}_{k+t}, \bar{y}_{k+t+1}),\ t = 1,\dots,h$$

$$(y_{k+t}^*(k+h+t), y_{k+h+t}^*(k+h+t), y_{k+h+t+1}^*(k+h+t)) \in N_{grb_{k+h+t}}(\bar{y}_{k+t}, \bar{y}_{k+h+t}, \bar{y}_{k+h+t+1}),$$

$$t = 1, m-1-k-h,\quad y_m^*(m) \in N_C(\bar{y}_m)$$

such that in the case $h = \Delta$ the relations are fulfilled:

$$\begin{gathered} x_{t,0}^* + x_t^*(t-1) + x_t^*(t) + x_t^*(\Delta+t) = 0 \text{ at } t = 1,\dots,k; \\ x_{t,0}^* + y_t^*(t-1) + y_t^*(t) + y_t^*(t+\Delta) = 0 \text{ at } t = k+1,\dots,m-h-1; \\ x_{t,0}^* + y_t^*(t-1) + y_t^*(t) = 0 \text{ at } t = m-h,\dots,m; \end{gathered} \tag{3.1}$$

in the case $h < \Delta$ the relations are fulfilled:

$$\begin{gathered} x_{t,0}^* + x_t^*(t-1) + x_t^*(t) + x_t^*(\Delta+t) = 0 \text{ at } t = 1,\dots,k-1-\Delta; \\ x_{t,0}^* + x_t^*(t-1) + x_t^*(t) = 0 \text{ at } t = k-\Delta,\dots,k-h-1; \\ x_{t,0}^* + x_t^*(t-1) + x_t^*(t) + x_t^*(t+h) = 0 \text{ at } t = k-h;\dots,k; \\ x_{t,0}^* + y_t^*(t-1) + y_t^*(t) + y_t^*(t+h) = 0 \text{ at } t = k+1,\dots,m-h-1; \\ x_{t,0}^* + y_t^*(t-1) + y_t^*(t) = 0 \text{ at } t = m-h,\dots,m; \end{gathered} \tag{3.2}$$

in the case $h > \Delta$ the relations are fulfilled:

$$\begin{gathered} x_{t,0}^* + x_t^*(t-1) + x_t^*(t) + x_t^*(\Delta+t) = 0 \text{ at } t = 1,\dots,k-h-1; \\ x_{t,0}^* + x_t^*(t-1) + x_t^*(t) + x_t^*(t+\Delta) + x_t^*(t+h) = 0 \text{ at } t = k-h,\dots,k-1-\Delta; \\ x_{t,0}^* + x_t^*(t-1) + x_t^*(t) + x_t^*(t+h) = 0 \text{ at } t = k-\Delta,\dots,k; \\ x_{t,0}^* + y_t^*(t-1) + y_t^*(t) + y_t^*(t+h) = 0 \text{ at } t = k+1,\dots,m-h-1; \\ x_{t,0}^* + y_t^*(t-1) + y_t^*(t) = 0 \text{ at } t = m-h,\dots,m; \end{gathered} \tag{3.3}$$

Proof. Using the corollary of Theorem 2.4.5. (Clarke, 1988: 279), from the definition of $M_0, M_1, \dots, M_m$ we obtain that

$$T_{M_0}(\bar{z}) = \left\{(x_1,\dots,x_k,\ y_{k+1},\dots,y_m) \in X^k \times Y^s : x_1 \in T_{a_0(c(-\Delta),c(0))}(\bar{x}_1)\right\},$$

$$T_{M_t}(\bar{z}) = \left\{(x_1,\dots,x_k,\ y_{k+1},\dots,y_m) \in X^k \times Y^s : (x_t, x_{t+1}) \in T_{gr\,a_t(c(t-\Delta),\cdot)}(\bar{x}_t, \bar{x}_{t+1})\right\}$$

$$t=1,\dots,\Delta,$$
$$T_{M_t}(\bar z)=\left\{(x_1,\dots,x_k,\ y_{k+1},\dots,y_m)\in X^k\times Y^s:(x_{t-\Delta},x_t,x_{t+1})\in T_{gr\,a_t}(\bar x_{t-\Delta},\bar x_t,\bar x_{t+1})\right\}$$
$$\text{at } t=\Delta+1,\dots,k-1,$$
$$T_{M_k}(\bar z)=\left\{(x_1,\dots,x_k,\ y_{k+1},\dots,y_m)\in X^k\times Y^s:(x_{k-h},x_k,y_{k+1})\in\right.$$
$$\left.\in T_{gr\,b_k(G(\cdot),G(\cdot))}\left(\bar x_{k-h},\bar x_k,\bar y_{k+1}\right)\right\}$$
$$T_{M_t}(\bar z)=\left\{(x_1,\dots,x_k,y_{k+1},\dots,y_m)\in X^k\times Y^s:(x_{t-h},y_t,y_{t+1})\in T_{gr\,b_t(G(\cdot),\cdot)}(\bar x_{t-h},\bar y_t,\bar y_{t+1})\right\}$$
$$t=k+1,\dots,k+h,$$
$$T_{M_t}(\bar z)=\left\{(x_1,\dots,x_k,\ y_{k+1},\dots,y_m)\in X^k\times Y^s:(y_{t-h},y_t,y_{t+1})\in T_{gr\,b_t}(\bar y_{t-h},\bar y_t,\bar y_{t+1})\right\}$$
$$t=k+h+1,\dots,m-1,$$
$$T_{M_m}(\bar z)=\left\{(x_1,\dots,x_k,\ y_{k+1},\dots,y_m)\in X^k\times Y^s:y_m\in T_C(\bar y_m)\right\}$$

Therefore we have

$$N_{M_0}(\bar z)=\left\{(x_1^*,0,\dots,0)\in X^{*k}\times Y^{*s}:x_1^*\in N_{a_0(c(-\Delta),c(0))}(\bar x_1)\right\},$$
$$N_{M_t}(\bar z)=\left\{(x_1^*,x_2^*,\dots,x_k^*,0,\dots,0)\in X^{*k}\times Y^{*s}:(x_t^*,x_{t+1}^*)\in N_{gr\,a_t(c(t-\Delta),\cdot)}(\bar x_t,\bar x_{t+1}),\right.$$
$$\left.x_i^*=0 \text{ at } i\neq t,t+1\right\} \text{ at } t=1,\dots,\Delta,$$
$$N_{M_t}(\bar z)=\left\{(x_1^*,\dots,x_k^*,0,\dots,0)\in X^{*k}\times Y^{*s}:\left(x_{t-\Delta}^*,x_t^*,x_{t+1}^*\right)\in N_{gra_t}\left(\bar x_{t-\Delta},\bar x_t,\bar x_{t+1}\right),\right.$$
$$\left.x_i^*=0 \text{ at } i\neq t-\Delta,t,t+1\right\} \text{ at } t=\Delta+1,\dots,k-1,$$
$$N_{M_k}(\bar z)=\left\{(x_1^*,\dots,x_k^*,y_{k+1}^*,0,\dots,0)\in X^{*k}\times Y^{*s}:\left(x_{k-h}^*,x_k^*,y_{k+1}^*\right)\in\right.$$
$$\left.\in N_{grb_k(G(\cdot),G(\cdot))}\left(\bar x_{k-h},\bar x_k,\bar y_{k+1}\right),\ x_i^*=0 \text{ at } i\neq k-h,k\right\} \tag{3.4}$$
$$N_{M_t}(\bar z)=\left\{(x_1^*,\dots,x_k^*,y_{k+1}^*,\dots,y_m^*)\in X^{*k}\times Y^{*s}:(x_{t-h}^*,y_t^*,y_{t+1}^*)\in\right.$$
$$\left.\in N_{gr\,b_t(G(\cdot),\cdot)}(\bar x_{t-h},\bar y_t,\bar y_{t+1}),x_i^*=0 \text{ at } i\neq t-h,\ y_i^*=0 \text{ at } i\neq t,t+1\right\}$$
$$\text{at } t=k+1,\dots,k+h,$$
$$N_{M_t}(\bar z)=\left\{(x_1^*,\dots,x_k^*,y_{k+1}^*,\dots,y_m^*)\in X^{*k}\times Y^{*s}:(y_{t-h}^*,y_t^*,y_{t+1}^*)\in\right.$$
$$\left.\in N_{gr\,b_t}(\bar y_{t-h},\bar y_t,\bar y_{t+1}),y_i^*=0 \text{ at } i\neq t-h,t,t+1\right\}$$
$$\text{at } t=k+h+1,\dots,m-1,$$
$$N_{M_m}(\bar z)=\left\{(0,\dots,0,0,\dots,0,y_m^*)\in X^{*k}\times Y^{*s}:y_m^*\in N_C(\bar y_m)\right\}.$$

According to Theorem 1 the numbers $\lambda_0\geq 0,\lambda_1\geq 0,\dots,\lambda_j\geq 0$ and $\lambda_{j+1},\dots,\lambda_n$ may be found, not all equal to zero simultaneously, where $\sum_{i=1}^{j}\lambda_i F_i(\bar x,\bar y)=0$, such that

$$0 \in \partial\left(\sum_{i=0}^{n} \lambda_i F_i(\bar{x}, \bar{y})\right) + \sum_{t=0}^{m} N_{M_t}(\bar{x}, \bar{y}) = \partial\left(\lambda_0 \sum_{t=1}^{k} g_0(\bar{x}_t, t) + \lambda_0 \sum_{t=k+1}^{m} f_0(\bar{y}_t, t) + \lambda_1 \sum_{t=1}^{k} g_1(\bar{x}_t, t) + \right.$$

$$+ \lambda_1 \sum_{t=k+1}^{m} f_1(\bar{y}_t, t) + \ldots + \lambda_n \sum_{t=1}^{k} g_n(\bar{x}_t, t) + \left. \lambda_n \sum_{t=k+1}^{m} f_n(\bar{y}_t, t)\right) + \sum_{t=0}^{m} N_{M_t}(\bar{x}, \bar{y}) = \partial\left(\sum_{i=0}^{n} \lambda_i g_i(\bar{x}_1, 1) + \right.$$

$$+ \ldots + \sum_{i=0}^{n} \lambda_i g_i(\bar{x}_k, k) + \sum_{i=0}^{n} \lambda_i f_i(\bar{y}_{k+1}, k+1) + \ldots + \left. \sum_{i=0}^{n} \lambda_i f_i(\bar{y}_m, m)\right) + \sum_{t=0}^{m} N_{M_t}(\bar{x}, \bar{y}) = \quad (3.5)$$

$$= \prod_{t=1}^{k} \partial\left(\sum_{i=0}^{n} \lambda_i g_i(\bar{x}_t, t)\right) \times \prod_{t=k+1}^{m} \partial\left(\sum_{i=0}^{n} \lambda_i f_i(\bar{y}_t, t)\right) + \sum_{t=0}^{m} N_{M_t}(\bar{x}, \bar{y}).$$

Using the relations (3.4) and (3.5) it is straightforward to check the correctness of Theorem 2.

Theorem 3. Let $\bar{z} = (\bar{x}, \bar{y}) \in \Omega$ minimize the functional F_0 on the set Ω, $gr\, a_t$ at $t = 0,\ldots,k-1$, $gr\, b_t$ at $t = k,\ldots,m-1$ and C be closed sets, $G : X \to Y$ be continuous operator, the functions $g_i(\cdot, t): X \to R$, $i = 0,\ldots,n$ satisfy the Lipschitz condition in the neighbourhood of $\bar{x}_t$ at $t = 1,\ldots,k$, the functions $f_i(\cdot, t), i = 0,\ldots n$, satisfy the Lipschitz condition in the neighbourhood of $\bar{y}_t$ at $t = k+1,\ldots,m$, $I_{gr\, a_t(c(-\Delta+t),\, \cdot)}(\bar{x}_t, \bar{x}_{t+1})$ at $t = 1,\ldots,\Delta$, $I_{gr\, a_t}(\bar{x}_{t-\Delta}, \bar{x}_t, \bar{x}_{t+1})$ at $t = \Delta+1,\ldots,k-1$, $I_{grb_k(G(\cdot),G(\cdot))}(\bar{x}_{k-h}, \bar{x}_k, \bar{y}_{k+1})$, $I_{grb_t(G(\cdot),\cdot)}(\bar{x}_{t-h}, \bar{y}_t, \bar{y}_{t+1})$ at $t = k+1,\ldots,k+h$, $I_{grb_t}(\bar{y}_{t-h}, \bar{y}_t, \bar{y}_{t+1})$ at $t = k+h+1,\ldots,m-1$ and $I_C(\bar{y}_m)$ be nonempty sets. Then the numbers $\lambda_0 \geq 0, \lambda_1 \geq 0, \ldots, \lambda_j \geq 0$ and $\lambda_{j+1}, \ldots, \lambda_n$ may be found, not all equal to zero simultaneously, where $\sum_{i=1}^{j} \lambda_i F_i(\bar{x}, \bar{y}) = 0$, and there vectors exist

$$x_{t0}^* = \lambda z_{t0}^*, \text{ where } \lambda = 0,1,\ z_{t0}^* \in \partial \sum_{i=0}^{n} \lambda_i g_i(\bar{x}_t, t) \text{ at } t = 1,\ldots k,\ z_{t0}^* \in \partial \sum_{i=0}^{n} \lambda_i f_i(\bar{y}_t, t)$$

$$\text{at } t = k+1,\ldots,m,\ x_1^*(0) \in N_{a_0(c(-\Delta),c(0))}(\bar{x}_1),\ \left(x_t^*(t), x_{t+1}^*(t)\right) \in$$

$$\in N_{gra_t(c(-\Delta+t),\cdot)}(\bar{x}_t, \bar{x}_{t+1}) \text{ at } t = 1,\ldots,\Delta,\ \left(x_t^*(\Delta+t), x_{\Delta+t}^*(\Delta+t), x_{\Delta+t+1}^*(\Delta+t)\right) \in$$

$$\in N_{gra_{\Delta+t}}(\bar{x}_t, \bar{x}_{\Delta+t}, \bar{x}_{\Delta+t+1}) \text{ at } t = \Delta+1,\ldots,h-1,$$

$$\left(x_{k+t-h}^*(k+t), y_{k+t}^*(k+t), y_{k+t+1}^*(k+t)\right) \in N_{grb_{k+t}(G(\cdot),\cdot)}(\bar{x}_{k+t-h}, \bar{y}_{k+t}, \bar{y}_{k+t+1}),$$

$$t = 1,\ldots,h,\ (y_{k+t}^*(k+h+t), y_{k+h+t}^*(k+h+t), y_{k+h+t+1}^*(k+h+t)) \in$$

$$\in N_{grb_{k+h+t}}(\bar{y}_{k+t}, \bar{y}_{k+h+t}, \bar{y}_{k+h+t+1}),\ t = 1, m-1-k-h,\ \ y_m^*(m) \in N_C(\bar{y}_m)$$

such that in the case $h = \Delta$ the relations (3.1) are fulfilled, in the case $h < \Delta$ the relations (3.2) are fulfilled, in the case $h > \Delta$ the relations (3.3) are fulfilled.

Proof. It is straightforward to check that

$$I_{M_1}(\bar{z}) = \left\{(x_1,\ldots,x_k, y_{k+1},\ldots,y_m) \in X^k \times Y^s : (x_t, x_{t+1}) \in I_{gra_t(c(-\Delta+t),\cdot)}(\bar{x}_t, \bar{x}_{t+1})\right\}$$

$$\text{at } t = 1,\ldots,\Delta;$$

$$I_{M_t}(\bar{z}) = \{(x_1,...,x_k, y_{k+1},..., y_m) \in X^k \times Y^s : (x_{t-\Delta}, x_t, x_{t+1}) \in$$
$$\in I_{gra_t}(\bar{x}_{t-\Delta}, \bar{x}_t, \bar{x}_{t+1})\} \text{ at } t = \Delta+1,...,k-1;$$
$$I_{M_k}(\bar{z}) = \{(x_1,...,x_k, y_{k+1},..., y_m) \in X^k \times Y^s : (x_{k-h}, x_k, y_{k+1}) \in$$
$$\in I_{grb_k(G(\cdot),G(\cdot))}(\bar{x}_{k-h}, \bar{x}_k, \bar{y}_{k+1})\};$$
$$I_{M_t}(\bar{z}) = \{(x_1,...,x_k, y_{k+1},..., y_m) \in X^k \times Y^s : (x_{t-h}, y_t, y_{t+1}) \in I_{grb_t(G(\cdot),\cdot)}(\bar{x}_{t-h}, \bar{y}_t, \bar{y}_{t+1})\}$$
$$\text{at } t = k+1,...,k+h;$$
$$I_{M_t}(\bar{z}) = \{(x_1,...,x_k, y_{k+1},..., y_m) \in X^k \times Y^s : (y_{t-h}, y_t, y_{t+1}) \in I_{grb_{k+h+1}}(\bar{y}_{t-h}, \bar{y}_t, \bar{y}_{t+1})\}$$
$$\text{at } t = k+h+1,...,m-1;$$
$$I_{M_m}(\bar{z}) = \{(x_1,...,x_k, y_{k+1},..., y_m) \in X^k \times Y^s : y_m \in I_C(\bar{y}_m)\}$$

If $T_{M_0}(\bar{z}) \cap (\bigcap_{t=1}^{m} I_{M_t}(\bar{z})) = \varnothing$, then according to lemma 5.11 (Girsanov, 1970: 118) we can find the linear functionals $\omega_t^* \in N_{M_t}(\bar{z}),\ t = 0,1,...,m$, not all equal to zero, such that $\omega_0^* + \omega_1^* + ... + \omega_m^* = 0$. Then we obtain that at $\lambda = 0$ the statement of Theorem 3 is satisfied.

According to Theorem 3 we have that $I_{M_t}(\bar{z}) = \operatorname{int} T_{M_t}(\bar{z})$ at $t = 1,...,m$.

Therefore, if $T_{M_0}(\bar{z}) \cap (\bigcap_{t=1}^{m} I_{M_t}(\bar{z})) \neq \varnothing$, then the conditions of Theorem 2 with the condition A are satisfied. Then it follows from Theorem 2, that at $\lambda = 1$ the statement of Theorem 3 is satisfied. The theorem is proved.

Theorem 4. If $\bar{z} = (\bar{x}, \bar{y}) = (\bar{x}_1, \ldots, \bar{x}_k, \bar{y}_{k+1}, \ldots, \bar{y}_m) \in \Omega$, gra_t at $t = 0,...,k-1$, grb_t at $t = k,...,m-1$ and C are convex sets, $G : X \to Y$ is a linear operator, the functions $g_i(\cdot,t) : X \to R, i = 0,...,n$ are convex at $t = 1,...,k$, the functions $f_i(\cdot,t) : Y \to R, i = 0,...,n$ are convex at $t = k+1,...,m$, besides $j = n$ and $\lambda_0 = 1,\ \lambda_1 \geq 0, \ldots, \lambda_j \geq 0$ may be found, where $\sum_{i=1}^{j} \lambda_i F_i(\bar{x}, \bar{y}) = 0$, $x_{t0}^* \in \partial \sum_{i=0}^{n} \lambda_i g_i(\bar{x}_t, t)$ at $t = 1,...k$, $x_{t0}^* \in \partial \sum_{i=0}^{n} \lambda_i f_i(\bar{y}_t, t)$ at $t = k+1,...,m$,

and there the vectors exist

$$x_1^*(0) \in N_{a_0(c(-\Delta),c(0))}(\bar{x}_1),\ (x_t^*(t), x_{t+1}^*(t)) \in N_{gra_t(c(-\Delta+t),\cdot)}(\bar{x}_t, \bar{x}_{t+1}), \quad t = 1,...,\Delta,$$
$$(x_t^*(\Delta+t), x_{\Delta+t}^*(\Delta+t), x_{\Delta+t+1}^*(\Delta+t)) \in N_{gra_{\Delta+t}}(\bar{x}_t, \bar{x}_{\Delta+t}, \bar{x}_{\Delta+t+1}) \text{ at } t = 1,...,k-1-\Delta,$$
$$(x_{k-h}^*(k), x_k^*(k), y_{k+1}^*(k)) \in N_{grb_k(G(\cdot),G(\cdot))}(\bar{x}_{k-h}, \bar{x}_k, \bar{y}_{k+1}),$$
$$(x_{k+t-h}^*(k+t), y_{k+t}^*(k+t), y_{k+t+1}^*(k+t)) \in N_{grb_{k+t}(G(\cdot),\cdot)}(\bar{x}_{k+t-h}, \bar{y}_{k+t}, \bar{y}_{k+t+1}),$$

$$t = 1...h, \quad (y^*_{k+t}(k+h+t), y^*_{k+h+t}(k+h+t), y^*_{k+h+t+1}(k+h+t)) \in$$
$$\in N_{grb_{k+h+t}}(\bar{y}_{k+t}, \bar{y}_{k+h+t}, \bar{y}_{k+h+t+1}), \; t = 1,...,m-1-k-h, \; y^*_m(m) \in N_C(\bar{y}_m)$$

such that in the case $h = \Delta$ the relations are fulfilled:

$$\begin{gathered} x^*_{t,0} + x^*_t(t-1) + x^*_t(t) + x^*_t(\Delta + t) = 0 \text{ at } t = 1,...,k; \\ x^*_{t,0} + y^*_t(t-1) + y^*_t(t) + y^*_t(t+\Delta) = 0 \text{ at } t = k+1,...,m-h-1; \\ x^*_{t,0} + y^*_t(t-1) + y^*_t(t) = 0 \text{ at } t = m-h,...,m; \end{gathered} \tag{3.6}$$

in the case $h < \Delta$ the relations are fulfilled:

$$\begin{gathered} x^*_{t,0} + x^*_t(t-1) + x^*_t(t) + x^*_t(\Delta + t) = 0 \text{ at } t = 1,...,k-1-\Delta; \\ x^*_{t,0} + x^*_t(t-1) + x^*_t(t) = 0 \text{ at } t = k-\Delta,...,k-h-1; \\ x^*_{t,0} + x^*_t(t-1) + x^*_t(t) + x^*_t(t+h) = 0 \text{ at } t = k-h,...,k; \\ x^*_{t,0} + y^*_t(t-1) + y^*_t(t) + y^*_t(t+h) = 0 \text{ at } t = k+1,...,m-h-1; \\ x^*_{t,0} + y^*_t(t-1) + y^*_t(t) = 0 \text{ at } t = m-h,...m; \end{gathered} \tag{3.7}$$

in the case $h > \Delta$ the relations are fulfilled:

$$\begin{gathered} x^*_{t,0} + x^*_t(t-1) + x^*_t(t) + x^*_t(\Delta + t) = 0 \text{ at } t = 1,...,k-h-1; \\ x^*_{t,0} + x^*_t(t-1) + x^*_t(t) + x^*_t(t+\Delta) + x^*_t(t+h) = 0 \text{ at } t = k-h,...,k-1-\Delta; \\ x^*_{t,0} + x^*_t(t-1) + x^*_t(t) + x^*_t(t+h) = 0 \text{ at } t = k-\Delta,...,k; \\ x^*_{t,0} + y^*_t(t-1) + y^*_t(t) + y^*_t(t+h) = 0 \text{ at } t = k+1,...,m-h-1; \\ x^*_{t,0} + y^*_t(t-1) + y^*_t(t) = 0 \text{ at } t = m-h,...,m \end{gathered} \tag{3.8}$$

Then $\bar{z} = (\bar{x}, \bar{y}) = (\bar{x}_1, \ldots, \bar{x}_k, \bar{y}_{k+1}, \ldots \bar{y}_m) \in \Omega$ minimizes the functional F_0 on the set Ω .

Proof. Consider the case $h = \Delta$. Since $x^*_{t0} \in \partial \sum_{i=0}^{n} \lambda_i g_i(\bar{x}_t, t)$ at $t = 1,...,k$ and $x^*_{t0} \in \partial \sum_{i=0}^{n} \lambda_i f_i(\bar{y}_t, t)$ at $t = k+1,...,m$, then from the relation (3.6) we have

$$-x^*_t(t-1) - x^*_t(t) - x^*_t(h+t) \in \partial \sum_{i=0}^{n} \lambda_i g_i(\bar{x}_t, t) \text{ at } t = 1,...,k;$$
$$-y^*_t(t-1) - y^*_t(t) - y^*_t(t+h) \in \partial \sum_{i=0}^{n} \lambda_i f_i(\bar{y}_t, t) \text{ at } t = k+1,...,m-h-1;$$
$$-y^*_t(t-1) - y^*_t(t) \in \partial \sum_{i=0}^{n} \lambda_i f_i(\bar{y}_t, t) \text{ at } t = m-h,...,m;$$

where

$$x_1^*(0)\in N_{a_0(c(-\Delta),c(0))}(\bar{x}_1),\ \left(x_t^*(t),x_{t+1}^*(t)\right)\in N_{gra_t(c(-\Delta+t),\cdot)}(\bar{x}_t,\bar{x}_{t+1}),\quad t=1,...,h\ ,$$
$$\left(x_t^*(h+t),x_{\Delta+t}^*(h+t),x_{\Delta+t+1}^*(h+t)\right)\in N_{gra_{\Delta+t}}(\bar{x}_t,\bar{x}_{\Delta+t},\bar{x}_{\Delta+t+1}),\ t=1,...k-1-h,$$
$$(x_{k-h}^*(k),x_k^*(k),y_{k+1}^*(k))\in N_{grb_k(G(\cdot),G(\cdot))}(\bar{x}_{k-h},\bar{x}_k,\bar{y}_{k+1}),$$
$$\left(x_{k+t-h}^*(k+t),y_{k+t}^*(k+t),y_{k+t+1}^*(k+t)\right)\in N_{grb_{k+t}(G(\cdot),\cdot)}(\bar{x}_{k+t-h},\bar{y}_{k+t},\bar{y}_{k+t+1}),$$
$$t=1,...,h\ ,\ (y_{k+t}^*(k+h+t),y_{k+h+t}^*(k+h+t),y_{k+h+t+1}^*(k+h+t))\in$$
$$\in N_{grb_{k+h+t}}(\bar{y}_{k+t},\bar{y}_{k+h+t},\bar{y}_{k+h+t+1}),\ t=1,...,m-1-k-h,\quad y_m^*(m)\in N_C(\bar{y}_m).$$

Therefore

$$\sum_{i=0}^{n}\lambda_i g_i(x_t,t)-\sum_{i=0}^{n}\lambda_i g_i(\bar{x}_t,t)\ge\left\langle -x_t^*(t-1)-x_t^*(t)-x_t^*(h+t),x_t-\bar{x}_t\right\rangle \text{ at } t=1,...,k;$$
$$\sum_{i=0}^{n}\lambda_i f_i(y_t,t)-\sum_{i=0}^{n}\lambda_i f_i(\bar{y}_t,t)\ge\left\langle -y_t^*(t-1)-y_t^*(t)-y_t^*(t+h),\ y_t-\bar{y}_t\right\rangle$$
$$\text{at } t=k+1,...,m-h-1\ ;$$
$$\sum_{i=0}^{n}\lambda_i f_i(y_t,t)-\sum_{i=0}^{n}\lambda_i f_i(\bar{y}_t,t))\ge\left\langle -y_t^*(t-1)-y_t^*(t),\ y_t-\bar{y}_t\right\rangle \text{ at } t=m-h,...,m\ .$$

Hence follows

$$\sum_{t=1}^{k}(\sum_{i=0}^{n}\lambda_i g_i(x_t,t)-\sum_{i=0}^{n}\lambda_i g_i(\bar{x}_t,t))+\sum_{t=k+1}^{m-h-1}(\sum_{i=0}^{n}\lambda_i f_i(y_t,t)-\sum_{i=0}^{n}\lambda_i f_i(\bar{y}_t,t))+$$
$$+\sum_{t=m-h}^{m}\left(\sum_{i=0}^{n}\lambda_i f_i(y_t,t)-\sum_{i=0}^{n}\lambda_i f_i(\bar{y}_t,t)\right)\ge\sum_{t=1}^{k}\left\langle -x_t^*(t-1)-x_t^*(t)-x_t^*(h+t),x_t-\bar{x}_t\right\rangle+$$
$$+\sum_{t=k+1}^{m-h-1}\left(-y_t^*(t-1)-y_t^*(t)-y_t^*(t+h),\ y_t-\bar{y}_t\right)+\sum_{t=m-h}^{m}\left\langle -y_t^*(t-1)-y_t^*(t),\ y_t-\bar{y}_t\right\rangle.$$

Since according to the condition that $gr\,a_t, gr\,b_t, C$ are convex sets and $G: X\to Y$ is a linear operator, then we have

$$\left\langle x_1^*(0),x_1-\bar{x}_1\right\rangle\le 0 \text{ at } x_1\in a_0(c(-h),c(0)),$$
$$\left\langle (x_t^*(t),x_{t+1}^*(t)),(x_t,x_{t+1})-(\bar{x}_t,\bar{x}_{t+1})\right\rangle\le 0 \text{ at } (x_t,x_{t+1})\in gra_t(c(-h+t),\cdot)$$
$$\text{and } t=1,...,h,\ \left\langle\left(x_t^*(h+t),x_{\Delta+t}^*(h+t),x_{\Delta+t+1}^*(h+t)\right),(x_t,x_{\Delta+t},x_{\Delta+t+1})-\right.$$
$$\left.-(\bar{x}_t,\bar{x}_{\Delta+t},\bar{x}_{\Delta+t+1})\right\rangle\le 0 \text{ at } (x_t,x_{\Delta+t},x_{\Delta+t+1})\in gra_{\Delta+t} \text{ and } t=1,...,k-1-h\ ,$$
$$\left\langle (x_{k-h}^*(k),x_k^*(k),y_{k+1}^*(k)),(x_{k-h},x_k,y_{k+1})-(\bar{x}_{k-h},\bar{x}_k,\bar{y}_{k+1})\right\rangle\le 0 \text{ at}$$
$$(x_{k-h},x_k,y_{k+1})\in grb_k(G(.),G(.)),\ \left\langle\left(x_{k+t-h}^*(k+t),y_{k+t}^*(k+t),y_{k+t+1}^*(k+t)\right),\right.$$
$$\left.(x_{k+t-h},y_{k+t},y_{k+t+1})-(\bar{x}_{k+t-h},\bar{y}_{k+t},\bar{y}_{k+t+1})\right\rangle\le 0 \text{ at } t=1,...,h,$$
$$\left\langle (y_{k+t}^*(k+h+t),y_{k+h+t}^*(k+h+t),y_{k+h+t+1}^*(k+h+t)\right),(y_{k+t},y_{k+h+t},y_{k+h+t+1})-$$

$-(\bar{y}_{k+t},\bar{y}_{k+h+t},\bar{y}_{k+h+t+1})\rangle \le 0$ at $(y_{k+t},y_{k+h+t},y_{k+h+t+1}) \in grb_{k+h+t}$ and $t=1,...,m-1-k-h$, $\left\langle y_m^*(m),(y_m-\bar{y}_m)\right\rangle \le 0$ at $y_m \in C$.

If $(\{x_t\},\ \{y_v\}) \in M$, then hence we have

$$-\left\langle x_1^*(0),x_1-\bar{x}_1\right\rangle-\sum_{t=1}^{h}\left\langle (x_t^*(t),x_{t+1}^*(t)),(x_t,x_{t+1})-(\bar{x}_t,\bar{x}_{t+1})\right\rangle-$$
$$-\sum_{t=1}^{k-1-h}\left\langle \left(x_t^*(h+t),x_{h+t}^*(h+t),x_{h+t+1}^*(h+t)\right),\ (x_t,x_{h+t},x_{h+t+1})-(\bar{x}_t,\bar{x}_{h+t},\bar{x}_{h+t+1})\right\rangle-$$
$$-\left\langle (x_{k-h}^*(k),x_k^*(k),y_{k+1}^*(k)),(x_{k-h},x_k,y_{k+1})-(\bar{x}_{k-h},\bar{x}_k,\bar{y}_{k+1})\right\rangle-$$
$$-\sum_{t=1}^{h}\left\langle \left(x_{k+t-h}^*(k+t),y_{k+t}^*(k+t),y_{k+t+1}^*(k+t)\right),(x_{k+t-h},y_{k+t},y_{k+t+1})-\right.$$
$$\left.-(\bar{x}_{k+t-h},\bar{y}_{k+t},\bar{y}_{k+t+1})\right\rangle-\sum_{t=1}^{m-1-k-h}\left\langle \left(y_{k+t}^*(k+h+t),y_{k+h+t}^*(k+h+t),\right.\right.$$
$$\left.\left.y_{k+h+t+1}^*(k+h+t)\right),(y_{k+t},y_{k+h+t},y_{k+h+t+1})-(\bar{y}_{k+t},\bar{y}_{k+h+t},\bar{y}_{k+h+t+1})\right\rangle+$$
$$+\left\langle -y_m^*(m),(y_m-\bar{y}_m)\right\rangle \ge 0.$$

Hence it follows

$$\sum_{t=1}^{k}\left\langle -x_t^*(t-1)-x_t^*(t)-x_t^*(h+t),x_t-\bar{x}_t\right\rangle+\sum_{t=k+1}^{m-h-1}\left\langle -y_t^*(t-1)-y_t^*(t)-y_t^*(t+h),\right.$$
$$\left.y_t-\bar{y}_t\right\rangle+\sum_{t=m-h}^{m}\left\langle -y_t^*(t-1)-y_t^*(t),\ y_t-\bar{y}_t\right\rangle \ge 0.$$

Then we obtain

$$\sum_{t=1}^{k}\Big(\sum_{i=0}^{n}\lambda_i g_i(x_t,t)-\sum_{i=0}^{n}\lambda_i g_i(\bar{x}_t,t)\Big)+\sum_{t=k+1}^{m-h-1}\Big(\sum_{i=0}^{n}\lambda_i f_i(y_t,t)-\sum_{i=0}^{n}\lambda_i f_i(\bar{y}_t,t)\Big)+$$
$$+\sum_{t=m-h}^{m}\Big(\sum_{i=0}^{n}\lambda_i f_i(y_t,t)-\sum_{i=0}^{n}\lambda_i f_i(\bar{y}_t,t)\Big)\ge\sum_{t=1}^{k}\left\langle -x_t^*(t-1)-x_t^*(t)-x_t^*(h+t),x_t-\bar{x}_t\right\rangle+$$
$$+\sum_{t=k+1}^{m-h-1}\left\langle -y_t^*(t-1)-y_t^*(t)-y_t^*(t+h),\ y_t-\bar{y}_t\right\rangle+$$
$$+\sum_{t=m-h}^{m}\left\langle -y_t^*(t-1)-y_t^*(t),\ y_t-\bar{y}_t\right\rangle \ge 0$$

i.e.

$$\sum_{t=1}^{k}\Big(\sum_{i=0}^{n}\lambda_i g_i(x_t,t)-\sum_{i=0}^{n}\lambda_i g_i(\bar{x}_t,t)\Big)+\sum_{t=k+1}^{m-h-1}\Big(\sum_{i=0}^{n}\lambda_i f_i(y_t,t)-\sum_{i=0}^{n}\lambda_i f_i(\bar{y}_t,t)\Big)+$$
$$+\sum_{t=m-h}^{m}\Big(\sum_{i=0}^{n}\lambda_i f_i(y_t,t)-\sum_{i=0}^{n}\lambda_i f_i(\bar{y}_t,t)\Big)\ge 0$$

at $(\{x_t\},\ \{y_v\}) \in M$.

It is clear that

$$\Omega = \{(x, y) \in M = \bigcap_{t=0}^{m} M_t : F_1(x, y) \le 0, \ldots, F_n(x, y) \le 0 \}.$$

Since $\lambda_0 = 1$, $\sum_{i=1}^{n} \lambda_i F_i(\bar{x}, \bar{y}) = 0$ and $\sum_{i=1}^{n} \lambda_i F_i(x, y) \le 0$ at $(\{x_t\},\ \{y_v\}) \in \Omega$, then

$$\sum_{t=1}^{k} (g_0(x_t, t) - g_0(\bar{x}_t, t)) + \sum_{t=k+1}^{m-h-1} (f_0(y_t, t) - f_0(\bar{y}_t, t)) + \sum_{t=m-h}^{m} (f_0(y_t, t) - f_0(\bar{y}_t, t)) \ge 0$$

at $(\{x_t\},\ \{y_v\}) \in \Omega$. Hence it follows that $\bar{z} = (\bar{x}, \bar{y}) = (\bar{x}_1, \ldots, \bar{x}_k, \bar{y}_{k+1}, \ldots, \bar{y}_m)$ is an optimal solution of problem (2.1)-(2.3).

The correctness of Theorem 4 is checked analogously in the cases $\Delta > h$ and $\Delta < h$, using (3.4) and (3.5). The theorem is proved.

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