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The Harnack Inequalities for The Solutions of an Elliptic Type Equation

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Abstract: *In this study, by using the well-known Harnack inequalities of the harmonic functions, some Harnack type inequalities are given for the solution of an elliptic type equation, which has variable coefficients.*

Key Words: **Elliptic equation, Harnack inequality, Harmonic function**

Eliptik Türden Bir Denklemin Çözümleri İçin Harnack Eşitsizlikleri

Özet: *Bu çalışmada harmonik fonksiyonlar için bilinen Harnack eşitsizliklerinden yararlanarak, değişken katsayılı eliptik tipten bir denklemin çözümleri için Harnack tipi eşitsizlikler elde edilmiştir.*

Anahtar Kelimeler: **Eliptik denklem, Harnack eşitsizliği, Harmonik fonksiyon**

Introduction

Let , in xoy-plane, $u^*(x,y)$ be a nonnegative harmonic function in a disk *D* of radius *a* with center *M*. Then for any $P \in D$, the following Harnack inequality

$$\frac{a-\rho}{a+\rho}u^*(M) \le u^*(P) \le \frac{a+\rho}{a-\rho}u^*(M) \tag{1}$$

is hold between the values of $u^*(x,y)$ at the point *P* and at the center *M*. (Figure 1) [3,4,5].

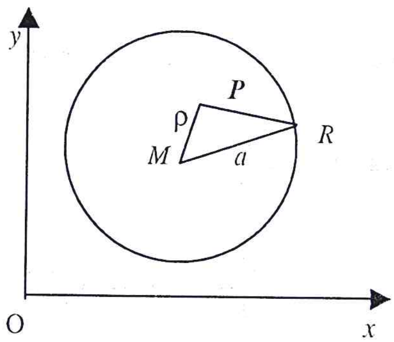

Figure 1.

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It should be noted that the Harnack Inequality is hold also for n – dimensional case with the inequality

$$\frac{a-\rho}{(a+\rho)^{n-1}}a^{n-2}u^*(M)\le u^*(P)\le\frac{a+\rho}{(a-\rho)^{n-1}}a^{n-2}u^*(M) \tag{2}$$

where *M* is the center of the n-dimensional ball B^n of radius *a*, $P\in B^n$ is a point at distance $\rho<a$ from the center, and *u** is a non-negative harmonic function in B^n.

More generally, let *u** be a non-negative harmonic function defined in a domain $D\subset R^n$ and *S* be a closed bounded set contained in *D*. Then there is a positive constant *A* depending on *S* and *D* but not on *u** such that for every pair of points *P* and *Q* in *S*, we have

$$Au^*(Q)\le u^*(P)\le A^{-1}u^*(Q) \tag{3}$$

Harnack Type Inequalities

In this study, we obtain Harnack type inequalities for the solutions of the class of equation

$$Lu=\sum_{i=1}^{n-1}\frac{\partial^2 u}{\partial x_i^2}+\sum_{j=1}^{p}\left(\frac{1}{m_j^2}y_j^{2-2m_j}\frac{\partial^2 u}{\partial y_j^2}-\frac{1}{m_j}\left(2-\frac{1}{m_j}-\frac{1}{p}\right)y_j^{1-2m_j}\frac{\partial u}{\partial y_j}\right)=0 \tag{4}$$

where $m_i\in Z^+$, $(i=1,2,\dots,n)$ are arbitrary constants .

Thus, we can give the following theorem.

Theorem 1. Let the function $u(x_1,x_2,\dots,x_{n-1},y_1,\dots,y_p)$ be a nonnegative solution of the equation (4) in the domain $D=\{x_1^2+\dots+x_{n-1}^2+y_1^{2m_1}+\dots+y_p^{2m_p}<R^2\}$.Then the following inequality holds.

$$\frac{R-r}{(R+r)^{n-1}}R^{n-2}u(O)\le u(P)\le\frac{R+r}{(R-r)^{n-1}}R^{n-2}u(O) \tag{5}$$

where $P(x_1,\dots,x_{n-1},y_1,\dots,y_p)$ is a point at distance $r<R$ from the center *O* of the ball *D*.

Proof. If we let $x_n^2=\sum_{j=1}^{p}y_j^{2m_j}$ in (4), then

$$\frac{\partial x_n}{\partial y_j}=m_j\frac{1}{x_n}y_j^{2m_j-1}$$

$$\frac{\partial^2 x_n}{\partial y_j^2}=-\frac{1}{x_n^3}m_j^2y_j^{4m_j-2}+m_j(2m_j-1)\frac{1}{x_n}y_j^{2m_j-2}$$

and hence

$$\frac{\partial u}{\partial y_j}=m_j\frac{1}{x_n}y_j^{2m_j-1}\frac{\partial u}{\partial x_n}$$

and

$$\frac{\partial^2 u}{\partial y_j^2}=\left(-\frac{1}{x_n^3}m_j^2y_j^{4m_j-2}+m_j(2m_j-1)\frac{1}{x_n}y_j^{2m_j-2}\right)\frac{\partial u}{\partial x_n}+m_j^2\frac{1}{x_n^2}y_j^{4m_j-4}\frac{\partial^2 u}{\partial x_n^2}$$

substituting these values in (4) we obtain

$$Lu=\sum_{i=1}^{n-1}\frac{\partial^2 u}{\partial x_i^2}$$

$$+\sum_{j=1}^{p}\left\{\frac{1}{m_j^2}y_j^{2-2m_j}\left(m_j^2\frac{1}{x_n^2}y_j^{4m_j-4}\frac{\partial^2 u}{\partial x_n^2}-\frac{1}{x_n^3}m_j^2y_j^{4m_j-2}\frac{\partial u}{\partial x_n}+m_j(2m_j-1)\frac{1}{x_n}y_j^{2m_j-2}\frac{\partial u}{\partial x_n}\right)\right.$$

$$\left.-\frac{1}{m_j}(2-\frac{1}{m_j}-\frac{1}{p})y_j^{1-2m_j}m_j\frac{1}{x_n}y_j^{2m_j-1}\frac{\partial u}{\partial x_n}\right\}=0$$

or

$$Lu=\sum_{i=1}^{n-1}\frac{\partial^2 u}{\partial x_i^2}$$

$$+\sum_{j=1}^{p}\left\{\frac{1}{x_n^2}y_j^{2m_j}\frac{\partial^2 u}{\partial x_n^2}-\frac{1}{x_n^3}y_j^{2m_j}\frac{\partial u}{\partial x_n}+(2-\frac{1}{m_j})\frac{1}{x_n}\frac{\partial u}{\partial x_n}-(2-\frac{1}{m_j}-\frac{1}{p})\frac{1}{x_n}\frac{\partial u}{\partial x_n}\right\}=0$$

By making the necessary simplifications, we get

$$Lu=\sum_{i=1}^{n-1}\frac{\partial^2 u}{\partial x_i^2}+\frac{\partial^2 u}{\partial x_n^2}=0 \qquad (6)$$

which is the Laplace equation of n-dimension. Since $u(x_1,\ldots,x_{n-1},y_1,\ldots,y_p)$ is a non-negative solution of (4), then the function

$$u(x_1,x_2,\ldots,x_{n-1},y_1,\ldots,y_p)=u^*\left(x_1,\ldots,x_{n-1},\pm\sqrt{y_1^{2m_1}+\ldots+y_p^{2m_p}}\right)=u^*(x_1,\ldots,x_n) \qquad (7)$$

is a non-negative solution of (6). Thus u^* satisfies the inequality

$$\frac{R-r}{(R+r)^{n-1}}R^{n-2}u^*(O)\le u^*(P)\le\frac{R+r}{(R-r)^{n-1}}R^{n-2}u^*(O) \qquad (8)$$

and hence u satisfies the inequality (5).

Remark 1. The Harnack inequality given by (8) can be applied to the solutions, which are bounded from below or above. For if *u* is bounded from below by a constant *m* in *D*, then the function $v = u - m$ satisfies the equation *Lv=0* and is non-negative and hence the Harnack inequality (8) is valid for it. Similarly, if *u* is bounded from above by a constant *M*, then the non-negative function $w = M - u$ also satisfies the Harnack inequality.

Remark 2. An analogous result of (3) holds for non-negative solutions (or for the solutions bounded from above or below) of the equation (4) .That is, if *u* is a non-negative solution contained in $D \subset R^{n+p-1}$, then there is a positive constant *A* depending on *S* but not on *u* such that for every pair of points *P* and *Q* in *S*, we have

$$Au(Q) \leq u(P) \leq A^{-1}u(Q)$$

Remark 3. If *u* is any solution of (4), bounded from below or above in all of n+*p*-1-dimensional space, then since (5) holds also for $R \to \infty$, we have *u*=*u*(0), a constant.

Example. Let in (4) *n*=2 and *p*=2. Then,

$$\begin{aligned} Lu = \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{m_1^2} y_1^{2-2m_1} \frac{\partial^2 u}{\partial y_1^2} - \frac{1}{m_1}\left(2 - \frac{1}{m_1} - \frac{1}{2}\right) y_1^{1-2m_1} \frac{\partial u}{\partial y_1} \\ + \frac{1}{m_2^2} y_2^{2-2m_2} \frac{\partial^2 u}{\partial y_2^2} - \frac{1}{m_2}\left(2 - \frac{1}{m_2} - \frac{1}{2}\right) y_2^{1-2m_2} \frac{\partial u}{\partial y_2} = 0 \end{aligned} \quad (10)$$

Now, $u = x_1\sqrt{y_1^{2m_1} + y_2^{2m_2}}$ is a solution of (10). On the other hand, in the domain $x_1^2 + y_1^{2m_1} + y_2^{2m_2} \leq 1$, $m = -1/2$ is a lower bound for *u* since the minimum value of the function $u = x_1 x_n$ in the disk $x_1^2 + x_n^2 \leq 1$ is $-\frac{1}{2}$. Hence, by Remark 1 and Theorem 1, the function *v=u*+1/2 satisfies the Harnack inequality. In this case, in (8), *R=1, n=2* and so letting $u^* = v$, we obtain

$$\frac{1-r}{1+r} v(0,0,0) \leq v(P) \leq \frac{1+r}{1-r} v(0,0,0)$$

or since $v(0,0,0) = u(0,0,0) + \frac{1}{2} = \frac{1}{2}$, we have

$$\frac{1}{2}\frac{1-r}{1+r} \leq v(P) \leq \frac{1}{2}\frac{1+r}{1-r}$$

where *P* is any point of the distance *r* from the origin in the domain $x_1^2 + y_1^{2m_1} + y_2^{2m_2} \leq 1$. From the last inequality, we get

$$\frac{1}{2}\frac{1-r}{1+r} \leq x_1\sqrt{y_1^{2m_1} + y_2^{2m_2}} + \frac{1}{2} \leq \frac{1}{2}\frac{1+r}{1-r}$$

or

$$\frac{1-r}{1+r} \leq 2x_1\sqrt{y_1^{2m_1} + y_2^{2m_2}} + 1 \leq \frac{1+r}{1-r}$$

Hence, for a bounded solution of (10) in the domain $x_1^2 + y_1^{2m_1} + y_2^{2m_2} \leq 1$, the Harnack inequality

$$\frac{-r}{1+r} \leq x_1\sqrt{y_1^{2m_1}+y_2^{2m_2}} \leq \frac{r}{1-r} \quad ; \quad r \leq 1$$

is hold. For the special case *r=1* we have

$$-\frac{1}{2} \leq x_1\sqrt{y_1^{2m_1}+y_2^{2m_2}} \leq \infty$$

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