on The Hadamard Products of Its Adjoint Matrix With a Square Matrix

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Abstract: In this paper for any matrix $A=(a_{ij})\in M_n(R)$ we defined $\varphi(A)=A\,o\,adjA$ and $\varphi_T(A)=A\,o\,(adjA)^T$, where T denotes transpose and o denotes Hadamard product. We obtained some properties of $\varphi(A)$ and $\varphi_T(A)$.

Key Words: Hadamard product, Adjoint matrix

Bir Kare Matris ile Onun Adjoint Matrisinin Hadamard Çarpımları Üzerine

Özet: Bu çalışmada reel elemanlı herhangi bir n×n A kare matrisi göz önüne alınarak $\varphi(A)=A\,o\,adjA$ ve $\varphi_T(A)=A\,o\,(adjA)^T$ matrisleri tanımlandı ve bu matrislerin bazı özellikleri elde edildi.

Anahtar Kelimeler: Hadamard Çarpımı, Adjoint Matris.

Introduction and the Main Results

Firstly we give the following definitions

Definition 1.[1] The Hadamard product of $A=(a_{ij})\in M_n$ and $B=(b_{ij})\in M_n$ is defined by $A o B=(a_{ij}b_{ij})\in M_n$.

Definition 2.[2] Let $A=(a_{ij})$ be an n×n matrix over any commutative ring. The permanent of A, written $per(A)$, is defined by

$$per(A)=\sum_{\sigma} a_{1\sigma(1)}a_{2\sigma(2)}\dots a_{n\sigma(n)},$$

where the summation extends over all one-to-one functions from $\{1,2,\dots,n\}$ to $\{1,2,\dots,n\}$.

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Definition 3.[3] For any matrix $A=(a_{ij})\in M_n$ the column matrix norm $\|\bullet\|_1$ is defined by

$$\|A\|_1=\max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|.$$

Definition 4.[3] For any matrix $A=(a_{ij})\in M_n$ the row matrix norm $\|\bullet\|_\infty$ is defined by

$$\|A\|_\infty=\max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|.$$

Definition 5.[3] For any $A=(a_{ij})\in M_n$, the Euclidean matrix norm is defined by

$$\|A\|_E=\left(\sum_{i,j=1}^{n}|a_{ij}|^2\right)^{1/2}.$$

Definition 6.[3] For any matrix $A=(a_{ij})\in M_n$ sum matrix norm $\|\bullet\|_t$ is defined by

$$\|A\|_t=\sum_{i,j=1}^{n}|a_{ij}|.$$

Now we present the main results.

Theorem 1. For any $A=(a_{ij})\in M_2(\mathbf{R})$

$$\det(\varphi(A))=\det(A)\operatorname{per}(A)$$

and

$$\det(\varphi_T(A))=\det(A)\operatorname{per}(A),$$

where $M_2(\mathbf{R})$ denotes 2×2 matrices with real entries, $\varphi(A)=A\circ\operatorname{adj}A$ and $\varphi_T(A)=A\circ(\operatorname{adj}A)^T$.

Proof. For any matrix

$$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$

since

$$\operatorname{adj}A=\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}$$

we write

$$\varphi(A)=A\circ\operatorname{adj}A=\begin{bmatrix}a_{11}a_{22} & -(a_{12})^2\\ -(a_{21})^2 & a_{11}a_{22}\end{bmatrix}.$$

If we compute the determinant of $\varphi(A)$, then we find

$$\begin{aligned}\det(\varphi(A)) &= (a_{11}a_{22})^2 - (a_{12}a_{21})^2 \\ &= (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{22} + a_{12}a_{21}) \\ &= \det(A)\operatorname{per}(A).\end{aligned}$$

Similarly it is easily seen that

$$\det(\varphi_T(A)) = \det(A)\operatorname{per}(A).$$

Thus the proof is complete.

Remark 1. Unfortunately Theorem 1 is wrong for $n \geq 3$. Let us give a counterexample. Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix}.$$

Since

$$\operatorname{adj} A = \begin{bmatrix} 6 & -2 & -4 \\ -2 & -2 & 2 \\ -2 & 2 & 0 \end{bmatrix}$$

we find

$$\varphi(A) = \begin{bmatrix} 6 & -4 & -12 \\ -2 & -4 & 2 \\ -4 & 4 & 0 \end{bmatrix}.$$

On the other hand, since $\det(\varphi(A)) = 272$, $\det(\varphi_T(A)) = 216$, $\det(A) = -4$ and $\operatorname{per} A = 40$, it follows that

$$\det(\varphi(A)) \neq \det(A)\, per(A)$$

and

$$\det(\varphi_T(A)) \neq \det(A)\operatorname{per}(A).$$

Theorem 2. For any matrix $A = (a_{ij}) \in M_2(\mathbf{R})$, the eigenvalues of $\varphi(A)$ are $\lambda_1 = \det A$, $\lambda_2 = \operatorname{per}(A)$. Also the eigenvalues of $\varphi_T(A)$ are $\lambda_1 = \det A$, $\lambda_2 = \operatorname{per}(A)$.

Proof. For any matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

since

$$\varphi(A) = \begin{bmatrix} a_{11}a_{22} & -(a_{12})^2 \\ -(a_{21})^2 & a_{11}a_{22} \end{bmatrix}$$

we have

$$\det(\lambda I-\varphi(A))=\det\begin{bmatrix}\lambda-a_{11}a_{22} & (a_{12})^2\\ (a_{21})^2 & \lambda-a_{11}a_{22}\end{bmatrix} \quad (1.1)$$

$$=\lambda^2-2a_{11}a_{22}\lambda+(a_{11}a_{22})^2-(a_{12}a_{21})^2$$

On the other hand the roots of the equation (1.1) are

$$\lambda_1=a_{11}a_{22}-a_{12}a_{21} \quad \text{and} \quad \lambda_2=a_{11}a_{22}+a_{12}a_{21}.$$

Moreover since

$$\det A=a_{11}a_{22}-a_{12}a_{21} \quad \text{and} \quad per(A)=a_{11}a_{22}+a_{12}a_{21}$$

we find

$$\lambda_1=\det A \quad \text{and} \quad \lambda_2=per(A).$$

Similarly it is easily seen that the eigenvalues of $\varphi_T(A)$ are $\lambda_1=\det A$ and $\lambda_2=per(A)$.
So the theorem is proved.

Remark 2. Unfortunately Theorem 2 is wrong for $n\geq 3$. But $\det A$ is always an eigenvalue of $\varphi_T(A)$ for $n\geq 3$. We state this fact as theorem .

Theorem 3. For $n\geq 3$, at least an eigenvalue of $\varphi_T(A)$ is equal to $\det A$.

Proof. We remark that

$$\det A=a_{i1}\alpha_{i1}+a_{i2}\alpha_{i2}+\ldots+a_{in}\alpha_{in}, \quad (i=1,2,\ldots,n) \quad (1.2)$$

or

$$\det A=a_{1j}\alpha_{1j}+a_{2j}\alpha_{2j}+\ldots+a_{nj}\alpha_{nj}, \quad (j=1,2,\ldots,n) \quad (1.3)$$

where α_{ij} denotes cofactor of a_{ij}, i.e., $\alpha_{ij}=(-1)^{i+j}\det(A_{ij})$.

Using the properties of determinants and considering (1.2) we have

$$\det(\lambda I-\varphi_T(A))=\det\begin{bmatrix}\lambda-a_{11}\alpha_{11} & -a_{12}\alpha_{12} & \ldots & -a_{1n}\alpha_{1n}\\ -a_{21}\alpha_{21} & \lambda-a_{22}\alpha_{22} & \ldots & -a_{2n}\alpha_{2n}\\ \vdots & \vdots & & \vdots\\ -a_{n1}\alpha_{n1} & -a_{n2}\alpha_{n2} & \ldots & \lambda-a_{nn}\alpha_{nn}\end{bmatrix}$$

$$= \det\begin{bmatrix} \lambda-\sum_{j=1}^{n} a_{1j}\alpha_{1j} & -a_{12}\alpha_{12} & \dots & -a_{1n}\alpha_{1n} \\ \lambda-\sum_{j=1}^{n} a_{2j}\alpha_{2j} & \lambda-a_{22}\alpha_{22} & \dots & -a_{2n}\alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda-\sum_{j=1}^{n} a_{nj}\alpha_{nj} & -a_{n2}\alpha_{n2} & \dots & \lambda-a_{nn}\alpha_{nn} \end{bmatrix}$$

$$= \det\begin{bmatrix} \lambda-\det A & -a_{12}\alpha_{12} & \dots & -a_{1n}\alpha_{1n} \\ \lambda-\det A & \lambda-a_{22}\alpha_{22} & \dots & -a_{2n}\alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda-\det A & -a_{n2}\alpha_{n2} & \dots & \lambda-a_{nn}\alpha_{nn} \end{bmatrix}$$

$$= (\lambda-\det A)\det\begin{bmatrix} 1 & -a_{12}\alpha_{12} & \dots & -a_{1n}\alpha_{1n} \\ 1 & \lambda-a_{22}\alpha_{22} & \dots & -a_{2n}\alpha_{2n} \\ \vdots & \vdots & & \vdots \\ 1 & -a_{n2}\alpha_{n2} & \dots & \lambda-a_{nn}\alpha_{nn} \end{bmatrix},$$

it follows that $\det A$ is eigenvalue of $\varphi_T(A)$.

Theorem 4. Let A be n×n real matrix, then all the row sums and column sums of $\varphi_T(A)$ are equal to $\det A$.

Proof. If $r_1, r_2, \dots, r_n$ denote the row sums of $\varphi_T(A)$, then we write

$$r_i = \sum_{j=1}^{n} a_{ij}\alpha_{ij} .$$

On the other hand considering (1.2) we have

$$\det A = r_i = \sum_{j=1}^{n} a_{ij}\alpha_{ij} \qquad (i = 1, 2, \dots, n) .$$

Similarly if $c_1, c_2, \dots, c_n$ denote the column sums of $\varphi_T(A)$, then we write

$$c_j = \sum_{i=1}^{n} a_{ij}\alpha_{ij}$$

Again considering (1.3) we have

$$\det A = c_j = \sum_{i=1}^{n} a_{ij}\,\alpha_{ij} \qquad (j = 1, 2, \ldots, n).$$

Thus the proof is complete.

Corollary 1. For $A = (a_{ij}) \in M_n(\mathbf{R})$,

$$e^T \varphi_T(A) e = n \det A,$$

where $e = (1, 1, \ldots, 1)^T$.

Proof. We write

$$e^T \varphi_T(A) e = \sum_{i,j=1}^{n} a_{ij}\,\alpha_{ij} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\,\alpha_{ij}\right)$$

$$= \sum_{i=1}^{n} \det A = n \det A$$

and the proof is complete.

Theorem 5. For any matrix $A = (a_{ij}) \in M_n(\mathbf{R})$, the following statements are true.

(i) $\| \varphi_T(A) \|_1 \geq | \det A |$

(ii) $\| \varphi_T(A) \|_\infty \geq | \det A |$

(iii) $\| \varphi_T(A) \|_E \geq | \det A |$

(iv) $\| \varphi_T(A) \|_t \geq n | \det A |$

Proof. (i) Considering the definition 3 and triangle inequality we have

$$\| \varphi_T(A) \|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} | a_{ij}\,\alpha_{ij} | \geq \max_{1 \leq j \leq n} \left| \sum_{i=1}^{n} a_{ij}\,\alpha_{ij} \right| = \max_{1 \leq j \leq n} | \det A | = | \det A |.$$

(ii) By the definition 4 and triangle inequality, we write??

EMBED Equation.3?? .

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(iii) Considering the definition 5 and triangle inequality

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it follows that??

EMBED Equation.3?? ??

(iv) Similarly by the definition 6 and triangle inequality we have??
 EMBED Equation.3??

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References

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