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Some Fractal-Fractional Integral Inequalities for Different Kinds of Convex Functions

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Abstract

The main objective of this work is to establish new upper bounds for different kinds of convex functions by using fractal-fractional integral operators with power law kernel. Furthermore, to enhance the paper, some new inequalities are obtained for product of different kinds of convex functions. The analysis used in the proofs is fairly elementary and based on the use of the well known Hölder inequality.

1. Introduction

Fractional analysis has been a field of rapid development with the definition of new integral and derivative operators in recent years, but has also closed a huge gap in terms of better identification and modeling of real-life problems. While the new fractional derivatives and integral operators continue to be examined in terms of singularity, local availability and convolution properties, another focus of researchers working in this field is to define more general operators that have applications in areas such as modeling, applied mathematics and mathematical biology.

The most frequently used derivatives of the fractional derivative in the literature are Riemann Liouville and Caputo fractional derivatives [1-3]. But as it is known, fundamental fractional derivative definitions like this include power kernel function in singular structure. Theoretically, this type of kernel functions that arise spontaneously creates difficulties in mathematical modeling for two reasons. The first of these is the computational difficulty due to singularity and the necessity of intensive numerical computations, which can often be overcome by the development of highly complex computer algorithms. The second difficulty is the inadequacy of kernel functions in the form of power functions in modeling phenomena that exhibit exponential behavior in nature.

In order to eliminate the weaknesses of fundamental derivatives, Caputo and Fabrizio replaced the kernel function of the Caputo fractional derivative with the exponential function in 2015, Atangana and Baleanu in 2016, replaced the exponential kernel function in the Caputo-Fabrizio fractional derivative with the Mittag-Leffler function, and obtained a more general definition [4,5].

These new operators, created by changing the kernel function, have been successfully used in heat transfer systems, problems such as groundwater flow in closed aquifers, wave motion on shallow water, electrical circuits, electromagnetic waves in dielectric medium.

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In 2017, Atangana defined a new fractional operator for modeling physical events that exhibit fractal behavior in the real world [6]. This new fractional derivative and integral operator, called fractal-fractional, have been created considering the nonlocality as well as the fractal effect. Since then, many authors have applied this fractional operator in different fields based on it.

In recent years, this issue has begun to be handled with the theory of inequalities, and classical integer order integral inequalities have been generalized with fractional integral operators. Many articles, papers and postgraduate thesis studies have been made on the fractional calculus [7-21].

The fact that the inequalities obtained by the proofs can be found more general with the help of the new fractional integrals defined in recent years has prompted us to study this subject.

2. Materials and Methods

In this section, we will give a brief discussion of some important definitions and properties releated to convex functions and fractal-fractional calculus that useful for this paper.

Definition 1: [22] The function $\Psi:[u,v]\to\mathbb{R}$ is said to be convex, if we have

$$\Psi\left(\tau z_1+(1-\tau)z_2\right)\le \tau\Psi(z_1)+(1-\tau)\Psi(z_2) \tag{1}$$

for all $z_1,z_2\in[u,v]$ and $\tau\in[0,1]$.

m-convexity was defined by Toader as follows:

Definition 2: [23] The function $\Psi:[0,v]\to\mathbb{R}$, $v>0$ is said to be m-convex, where $m\in[0,1]$, if we have

$$\Psi\left(\tau z_1+m(1-\tau)z_2\right)\le \tau\Psi(z_1)+m(1-\tau)\Psi(z_2) \tag{2}$$

for all $z_1,z_2\in[0,v]$ and $\tau\in[0,1]$.

Clearly, when we take $m=1$ in this definition, then f reduces to the ordinary convex on $[0,v]$.

s-convexity introduced by Breckner as a generalization of convex functions.

Definition 3: [22] The function $\Psi:[0,\infty)\to\mathbb{R}$ is said to be s-convex in the second sense, where $s\in(0,1]$, if we have

$$\Psi\left(\tau z_1+(1-\tau)z_2\right)\le \tau^s\Psi(z_1)+(1-\tau)^s\Psi(z_2) \tag{3}$$

for all $z_1,z_2\in[0,\infty)$ and $\tau\in[0,1]$.

Obviously, s-convexity means just convexity when $s=1$

Recently a new concept of differential and integral operators called fractal-fractional differential and integral operators were introduced by Atangana, as the convolution of the generalized Mittag-Lefler law, exponential law and power-law with fractal derivative [6]. These operators consist of two orders, firstly the fractional-order δ then the fractal dimension ω. The purpose of the new operators is to attract nonlocal problems in nature that also display fractal behavior.

The following definitions are discussed in detail in [6].

Definition 4: [6] Suppose that $\Psi(t)$ is continuous function and fractal differentiable on an open interval (u,v) with order ω then, δ order fractal-fractional derivative of function $\Psi(t)$, power-law kernel is given by:

$$ {}^{FFP}_{\ \ u}D_t^{\delta,\omega}\Psi(t)=\frac{1}{\Gamma(1-\delta)}\frac{d}{dt^{\omega}}\int_u^t\Psi(s)(t-s)^{-\delta}ds \tag{4}$$

where $0<\omega,\delta\le 1$ and

$$\frac{d\Psi(s)}{ds^{\omega}}=\lim_{t\to s}\frac{\Psi(t)-\Psi(s)}{t^{\omega}-s^{\omega}}.$$

Definition 5: [6] Suppose that $\Psi(t)$ is continuous function and fractal differentiable on an open interval (u,v) with order ω then, δ order fractal-fractional derivative of function $\Psi(t)$, exponential decay kernel is given by:

$$ {}^{FFE}_{\ \ u}D_t^{\delta,\omega}\Psi(t)=\frac{M(\delta)}{\Gamma(1-\delta)}\times\frac{d}{dt^{\omega}}\int_u^t\Psi(s)\exp\left[-\frac{\delta}{1-\delta}(t-s)\right]ds \tag{5}$$

where $0<\omega,\delta\le 1$ and $M(0)=M(1)=1$.

Definition 6: [6] Suppose that $\Psi(t)$ is continuous function and fractal differentiable on an open interval (u,v) with order ω then, δ order fractal-fractional derivative of function $\Psi(t)$, the generalized Mittag-Leffler kernel is given by:

$$ {}^{FFM}_{\ \ u}D_t^{\delta,\omega}\Psi(t)=\frac{AB(\delta)}{\Gamma(1-\delta)}\times\frac{d}{dt^{\omega}}\int_a^t\Psi(s)E_\delta\left[-\frac{\delta}{1-\delta}(t-s)^{\delta}\right]ds \tag{6}$$

where $0<\delta,\omega\le 1$ and $AB(\delta)=1-\delta+\dfrac{\delta}{\Gamma(\delta)}$.

The fractal-fractional integral operators associated with the derivatives in Eq. (4), (5), (6) are defined as follows, respectively.

Definition 7: [6] If $\Psi(t)$ is continuous in a closed interval $[u,v]$ then the fractal integral of Ψ with order δ is defined as:

$$ {}^{F}_{u}J_t^{\delta}\Psi(t)=\delta\int_u^t s^{\delta-1}\Psi(s)ds. \tag{7}$$

Definition 8: [6] Assuming that $\Psi(t)$ is a continuous function on (u,v), then δ order fractal-fractional integral of the function $\Psi(t)$ with power-law kernel is given by:

$$^{FFP}J_{u,t}^{\delta,\omega}\Psi(t)=\frac{\omega}{\Gamma(\delta)}\int_u^t (t-s)^{\delta-1}s^{\omega-1}\Psi(s)ds. \quad (8)$$

Definition 9: [6] Assuming that $\Psi(t)$ is a continuous function on (u,v), then δ order fractal-fractional integral of the function $\Psi(t)$ with an exponential decaying kernel is given by:

$$^{FFE}J_{u,t}^{\delta,\omega}\Psi(t)=\frac{\delta\omega}{M(\delta)}\int_u^t s^{\delta-1}\Psi(s)ds + \frac{\omega(1-\delta)t^{\omega-1}\Psi(t)}{M(\delta)}. \quad (9)$$

Definition 10: [6] Assuming that $\Psi(t)$ is a continuous function on (u,v), then δ order fractal-fractional integral of the function $\Psi(t)$ with generalized Mittag-Leffler kernel is given by:

$$^{FFM}J_{u,t}^{\delta,\omega}\Psi(t)=\frac{\delta\omega}{AB(\delta)}\int_u^t s^{\omega-1}(t-s)^{\delta-1}\Psi(s)ds + \frac{\omega(1-\delta)t^{\omega-1}\Psi(t)}{AB(\delta)}. \quad (10)$$

Remark: If the $\delta=\omega=1$, then the fractal-fractional integral operators in Eq. (8), (9) and (10) reduce to Riemann-Liouville, Caputo-Fabrizio and Atangana-Baleanu fractional integral operators respectively. Furthermore, if all fractional and fractal orders are equal to 1, the fractal-fractional integral operators reduce to the classical integral.

The purpose of this paper is to prove some fractional integral inequalities which provides the upper bounds via fractal-fractional integrals with power-law type kernel. To obtain the results, we use the different kinds of convex functions and some other features of the functions.

3. Main Results

Theorem 1: Suppose that $\Psi:[0,\infty)\to\mathbb{R}$ be a continuous function where $0\le u<t<\infty$ and $\Psi\in L_1[u,t]$. If $|\Psi|^q$ is an m-convex function, $m\in(0,1]$, then we have the following inequality for δ order fractal-fractional integral operators of the function $\Psi(t)$ with the power-law kernel:

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}{}^{FFP}J_{u,t}^{\delta,\omega}\Psi(t)\right| \le \left(\frac{t^{1+(\omega-1)p}-u^{1+(\omega-1)p}}{(t-u)(1+(\omega-1)p)}\right)^{\frac{1}{p}} \times\left(\frac{|\Psi(u)|^q(q(\delta-1)+1)+m\left|\Psi\left(\frac{t}{m}\right)\right|^q}{(q(\delta-1)+1)(q(\delta-1)+2)}\right)^{\frac{1}{q}} \quad (11)$$

where $p^{-1}+q^{-1}=1$ and $0<\delta,\omega\le 1$.

Proof: By using definition and changing variables can be written as

$$\frac{\Gamma(\delta)}{(t-u)^\delta\omega}{}^{FFP}J_{u,t}^{\delta,\omega}\Psi(t) =\int_0^1\tau^{\delta-1}(u\tau+(1-\tau)t)^{\omega-1}\Psi(u\tau+(1-\tau)t)d\tau. \quad (12)$$

By applying Hölder inequality, we have

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}{}^{FFP}J_{u,t}^{\delta,\omega}\Psi(t)\right| \le\left(\int_0^1(u\tau+(1-\tau)t)^{p(\omega-1)}d\tau\right)^{\frac{1}{p}} \times\left(\int_0^1\left(\tau^{\delta-1}\right)^q|\Psi(u\tau+(1-\tau)t)|^q d\tau\right)^{\frac{1}{q}}. \quad (13)$$

By using m-convexity of $|\Psi|^q$, we obtain

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}{}^{FFP}J_{u,t}^{\delta,\omega}\Psi(t)\right| \le\left(\int_0^1(u\tau+(1-\tau)t)^{p(\omega-1)}d\tau\right)^{\frac{1}{p}} \times\left(\int_0^1\left(\tau^{\delta-1}\right)^q\left(\tau|\Psi(u)|^q+m(1-\tau)\left|\Psi\left(\frac{t}{m}\right)\right|^q\right)d\tau\right)^{\frac{1}{q}}. \quad (14)$$

By calculating the above integrals and simplifying, the desired inequality is obtained.

Theorem 2. Suppose that $\Psi:(u,v)\subseteq[0,\infty)\to[0,\infty)$ be a continuous function and $\Psi\in L_1[u,v]$. If $|\Psi|^q$ is an s-convex function with $s\in(0,1]$, then we have the following inequality for δ order fractal-fractional integral operators of the function $\Psi(t)$ with the power-law kernel:

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}{}^{FFP}J_{u,t}^{\delta,\omega}\Psi(t)\right| \le\left(\frac{t^{1+(\omega-1)p}-u^{1+(\omega-1)p}}{(t-u)(1+(\omega-1)p)}\right)^{\frac{1}{p}}\times\left(\frac{|\Psi(u)|^q}{2+q(\delta+1)} \right. \quad (15)$$

$$\times\left(\frac{|\Psi(u)|^q}{2+q(\delta+1)}+\frac{|\Psi(t)|^q\Gamma(1+(\delta-1)q)\Gamma(1+s)}{\Gamma(2+s+(\delta-1)q)}\right)^{\frac{1}{q}}$$

where $p^{-1}+q^{-1}=1$, $q>1$ and $0<\delta,\omega\le 1$.

Proof: By means of Eq. (12) and Hölder integral inequality, we can write that

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}\,{}^{FFP}J_{u,t}^{\delta,\omega}\Psi(t)\right| \le\left(\int_0^1(u\tau+(1-\tau)t)^{p(\omega-1)}d\tau\right)^{\frac{1}{p}} \times\left(\int_0^1\left(\tau^{\delta-1}\right)^q|\Psi(u\tau+(1-\tau)t)|^q d\tau\right)^{\frac{1}{q}}. \tag{16}$$

Taking into account the s-convexity of $|\Psi|^q$, we have

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}\,{}^{FFP}J_{u,t}^{\delta,\omega}\Psi(t)\right| \le\left(\int_0^1(u\tau+(1-\tau)t)^{p(\omega-1)}d\tau\right)^{\frac{1}{p}} \times\left(\int_0^1\left(\tau^{\delta-1}\right)^q\left(\tau^s|\Psi(u)|^q+(1-\tau)^s|\Psi(t)|^q\right)d\tau\right)^{\frac{1}{q}}. \tag{17}$$

By computing the above integrals and simplifying, the statement is obtained.

Corollary 1: If we take $m=1$ in Eq. (11) and $s=1$ in Eq. (16), then we get the following inequality for δ order fractal-fractional integral operators of the function $\Psi(t)$ with the power-law kernel:

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}\,{}^{FFP}J_{u,t}^{\delta,\omega}\Psi(t)\right| \le\left(\frac{t^{1+(\omega-1)p}-u^{1+(\omega-1)p}}{(t-u)(1+(\omega-1)p)}\right)^{\frac{1}{p}} \times\left(\frac{|\Psi(u)|^q(q(\delta-1)+1)+|\Psi(t)|^q}{(q(\delta-1)+1)(q(\delta-1)+2)}\right)^{\frac{1}{q}} \tag{18}$$

where $p^{-1}+q^{-1}=1$, $q>1$ and $0<\delta,\omega\le 1$.

Theorem 3. Suppose that $\Psi,\Phi:[0,\infty)\to\mathbb{R}$ be functions with $0\le u<t<\infty$ and $\Psi,\ \Phi,\ \Psi\Phi\in L_1[u,t]$. If $|\Psi|^q$ is m_1-convex and $|\Phi|^q$ is m_2-convex function on $[u,t]$ for some fixed $m_1,m_2\in(0,1]$, then we have the following inequality for δ order fractal-fractional integral operators of the function $\Psi\Phi(t)$ with the power-law kernel:

$$\begin{aligned}\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}\,{}^{FFP}J_{u,t}^{\delta,\omega}\Psi\Phi(t)\right| &\le\left(\frac{t^{1+(\omega-1)p}-u^{1+(\omega-1)p}}{(t-u)(1+(\omega-1)p)}\right)^{\frac{1}{p}}\\ &\times\left[\frac{|\Psi(u)\Phi(u)|^q}{3+(\delta-1)q}\right.\\ &+\left(m_2\left|\Psi(u)\Phi\left(\frac{t}{m_2}\right)\right|^q+m_1\left|\Psi\left(\frac{t}{m_1}\right)\Phi(u)\right|^q\right)\\ &\times\frac{1}{(2+(\delta-1)q)(3+(\delta-1)q)}\\ &+\left|\Psi\left(\frac{t}{m_1}\right)\Phi\left(\frac{t}{m_2}\right)\right|^q\\ &\left.\times\frac{2m_1m_2}{(1+(\delta-1)q)(2+(\delta-1)q)(3+(\delta-1)q)}\right]^{\frac{1}{q}}\end{aligned} \tag{19}$$

where $p^{-1}+q^{-1}=1$, $q>1$ and $0<\delta,\omega\le 1$.

Proof: From the Eq. (12) and Hölder integral inequality, we get

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}\,{}^{FFP}J_{u,t}^{\delta,\omega}\Psi\Phi(t)\right| \le\left(\int_0^1(u\tau+(1-\tau)t)^{p(\omega-1)}d\tau\right)^{\frac{1}{p}} \times\left(\int_0^1\left(\tau^{\delta-1}\right)^q|\Psi(u\tau+(1-\tau)t)\Phi(u\tau+(1-\tau)t)|^q d\tau\right)^{\frac{1}{q}} \tag{20}$$

By using m-convexity of $|\Psi|^q$ and $|\Phi|^q$, we obtain

$$\begin{aligned}\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}\,{}^{FFP}J_{u,t}^{\delta,\omega}\Psi\Phi(t)\right| &\le\left(\int_0^1(u\tau+(1-\tau)t)^{p(\omega-1)}d\tau\right)^{\frac{1}{p}}\\ &\times\left(\int_0^1\left(\tau^{\delta-1}\right)^q\left|\left(\tau\Psi(u)+m_1(1-\tau)\Psi\left(\frac{t}{m_1}\right)\right)\right.\right.\\ &\left.\left.\times\left(\tau\Phi(u)+m_2(1-\tau)\Phi\left(\frac{t}{m_2}\right)\right)\right|^q d\tau\right)^{\frac{1}{q}}\end{aligned} \tag{21}$$

$$\leq\left(\int_0^1\left(u\tau+(1-\tau)t\right)^{p(\omega-1)}d\tau\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left(\tau^{\delta-1}\right)^q\tau^2\left|\Psi(u)\Phi(u)\right|^q d\tau\right.$$
$$+\int_0^1\left(\tau^{\delta-1}\right)^q m_2\tau(1-\tau)\left|\Psi(u)\Phi\left(\frac{t}{m_2}\right)\right|^q d\tau$$
$$+\int_0^1\left(\tau^{\delta-1}\right)^q m_1\tau(1-\tau)\left|\Psi\left(\frac{t}{m_1}\right)\Phi(u)\right|^q d\tau$$
$$\left.+\int_0^1\left(\tau^{\delta-1}\right)^q m_1m_2(1-\tau)^2\left|\Psi\left(\frac{t}{m_1}\right)\Phi\left(\frac{t}{m_2}\right)\right|^q d\tau\right)^{\frac{1}{q}}$$

By a simple computation, we get the desired result.

Theorem 4. Suppose that $\Psi,\Phi:(u,v)\subseteq[0,\infty)\to[0,\infty)$ be functions and $\Psi,\ \Phi,\ \Psi\Phi\in L_1[u,v]$. If $|\Psi|^q$ is s_1-convex and $|\Phi|^q$ is s_2-convex function on $[u,v]$ for some fixed $s_1,s_2\in(0,1]$, then we have the following inequality for δ order fractal-fractional integral operators of the function $\Psi\Phi(t)$ with the power-law kernel:

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}\,{}^{FFP}J_{u,t}^{\delta,\omega}\Psi\Phi(t)\right|$$
$$\leq\left(\frac{t^{1+(\omega-1)p}-u^{1+(\omega-1)p}}{(t-u)(1+(\omega-1)p)}\right)^{\frac{1}{p}}$$
$$\times\left[\frac{\left|\Psi(u)\Phi(u)\right|^q}{1+s_1+s_2+(\delta-1)q}+\frac{1}{\Gamma(2+s_1+s_2+(\delta-1)q)}\right. \tag{22}$$
$$\times\left(\left|\Psi(u)\Phi(t)\right|^q\left(\Gamma(1+s_2)\Gamma(1+s_1+(\delta-1)q)\right)\right.$$
$$+\left|\Psi(t)\Phi(u)\right|^q\left(\Gamma(1+s_1)\Gamma(1+s_2+(\delta-1)q)\right)$$
$$\left.\left.+\left|\Psi(t)\Phi(t)\right|^q\left(\Gamma(1+s_1+s_2)\Gamma(1+(\delta-1)q)\right)\right)\right]^{\frac{1}{q}}$$

where $p^{-1}+q^{-1}=1$, $q>1$ and $0<\delta,\omega\leq1$.

Proof: By means of Eq. (12) and Hölder integral inequality, we can get

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}\,{}^{FFP}J_{u,t}^{\delta,\omega}\Psi(t)\right|$$
$$\leq\left(\int_0^1\left(u\tau+(1-\tau)t\right)^{p(\omega-1)}d\tau\right)^{\frac{1}{p}} \tag{23}$$
$$\times\left(\int_0^1\left(\tau^{\delta-1}\right)^q\left|\Psi\left(u\tau+(1-\tau)t\right)\Phi\left(u\tau+(1-\tau)t\right)\right|^q d\tau\right)^{\frac{1}{q}}$$

Taking into account the s-convexity of $|\Psi|^q$ and $|\Phi|^q$, we have

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}\,{}^{FFP}J_{u,t}^{\delta,\omega}\Psi(t)\right|$$
$$\leq\left(\int_0^1\left(u\tau+(1-\tau)t\right)^{p(\omega-1)}d\tau\right)^{\frac{1}{p}} \tag{24}$$
$$\times\left(\int_0^1\left(\tau^{\delta-1}\right)^q\left|\left(\tau^{s_1}\Psi(u)+(1-\tau)^{s_1}\Psi(t)\right)\right.\right.$$
$$\left.\left.\times\left(\tau^{s_2}\Phi(u)+(1-\tau)^{s_2}\Phi(t)\right)\right|^q d\tau\right)^{\frac{1}{q}}$$

$$\leq\left(\int_0^1\left(u\tau+(1-\tau)t\right)^{p(\omega-1)}d\tau\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left(\tau^{\delta-1}\right)^q\tau^{s_1+s_2}\left|\Psi(u)\Phi(u)\right|^q d\tau\right.$$
$$+\int_0^1\left(\tau^{\delta-1}\right)^q\tau^{s_1}(1-\tau)^{s_2}\left|\Psi(u)\Phi(t)\right|^q d\tau$$
$$+\int_0^1\left(\tau^{\delta-1}\right)^q\tau^{s_2}(1-\tau)^{s_1}\left|\Psi(t)\Phi(u)\right|^q d\tau$$
$$\left.+\int_0^1\left(\tau^{\delta-1}\right)^q(1-\tau)^{s_1+s_2}\left|\Psi(t)\Phi(t)\right|^q d\tau\right)^{\frac{1}{q}}$$

By calculating the above integrals and simplifying, the desired inequality is obtained.

Corollary 2: If we take $m_1=m_2=1$ in Eq. (19) and $s_1=s_2=1$ in Eq. (22) then we get the following inequality for δ order fractal-fractional integral operators of the function $\Psi(t)$ with the power-law kernel:

$$\left|\frac{\Gamma(\delta)}{(t-u)^\delta\omega}\,{}^{FFP}J_{u,t}^{\delta,\omega}\Psi\Phi(t)\right|$$
$$\leq\left(\frac{t^{1+(\omega-1)p}-u^{1+(\omega-1)p}}{(t-u)(1+(\omega-1)p)}\right)^{\frac{1}{p}}$$
$$\times\left[\frac{\left|\Psi(u)\Phi(u)\right|^q}{3+(\delta-1)q}+\frac{\left|\Psi(u)\Phi(t)\right|^q+\left|\Psi(t)\Phi(u)\right|^q}{(2+(\delta-1)q)(3+(\delta-1)q)}\right. \tag{25}$$
$$\left.+\frac{2\left|\Psi(t)\Phi(t)\right|^q}{(1+(\delta-1)q)(2+(\delta-1)q)(3+(\delta-1)q)}\right]^{\frac{1}{q}}$$

4. Conclusions

In this paper, new upper bounds for different kinds of convex functions are given. To prove the main findings, fractal-fractional integral operators with power law kernel, the properties of the functions and Hölder's inequality are used. The method adopted for generating fractional inequalities is new and simple. Using the appropriate fractional integral operators, methods can be followed to develop further results for other classes of functions. In addition, the results are useful for fractional calculus and can be an inspiration for researchers working on this subject.

Declaration of Ethical Standards

The author of this article declares that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Conflict of Interest

The author declares that she has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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