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The Lyapunov Exponents of Thirring Instantons

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Abstract
Recently, nonlinear differential equations corresponding to pure spinor instanton solutions have been obtained by using Heisenberg ansatz in the 2D Thirring Model, which is used as a subject model in Quantum field theory. In addition, the evolution of spinor type instanton solutions in phase space was investigated according to the change in the constant parameter β. Spinor instanton dynamics is a special case in which nonlinear terms play an important role. Chaos describes certain nonlinear dynamical systems that depend very precisely on initial conditions. Lyapunov exponents are an important method for measuring stability and deterministic chaos in dynamical systems. Lyapunov exponents characterize and quantify the dynamics of small perturbations of a state or orbit in state space. In this study, The chaotic behavior of spinor type instanton solutions is analyzed by numerical study of the time evolution of the Lyapunov exponents. Moreover, the Lyapunov spectrum of spinor type instanton solutions with respect to varying the parameter are plotted. As a result of the Lyapunov Spectrum, it was determined that the spinor type instanton solutions exhibit chaotic behavior at parameter value $\beta = 2$. Periodic and quasi periodic behaviors were detected when the parameter values were $\beta<2$. In cases of $\beta>2$, weak chaotic behaviors were observed. This study demonstrates that Thirring Instantons, which are spinor type instanton solutions, exhibit chaotic properties.

1. Introduction

Instantons are classical solutions that spontaneously break conformal symmetry with zero energy in finite and non-zero actions. In quantum field theories, instantons are defined as tunneling processes between vacuums with different topological structures [1]. This property plays a significant role in understanding the problems of quark confinement within particles.

The Thirring Model was proposed by Walter Thirring in 1958 as a test model for quantum field theories [2]. The model is a 2-dimensional conformal invariant pure fermionic a model [2]. In addition, the model is an important known model for fermions in (1 + 1) space-time dimensions with a nonlinear term [3]. Akdeniz–Smailagic found a class of pure spinor type instanton solutions of the Thirring Model by breaking of the conformal symmetry i.e. $\langle 0|\bar{\psi}\bar{\psi}|0\rangle \neq 0$ in 1979 [3]. Later, it was shown that the spinor type instanton solutions are stable [4]. A decade ago, nonlinear differential equations corresponding to pure spinor instanton solutions were obtained by Heisenberg ansatz in the Thirring Model [5]. Moreover, the evolution of these spinor-type instanton solutions in phase space and the role of coupling constant in this evolution were examined [5]. A few years ago, the stability of the spinor type instanton solutions was investigated by the scale index method [6]. Recently, the chaotic behavior of the spinor type instanton solutions was investigated using the General Alignment Index (GALI) method [7].

Chaos describes certain nonlinear dynamical systems that depend very precisely on initial

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conditions [8]. Instability is one of the most fundamental properties of nonlinear dynamical systems. Generally, Lyapunov exponents are characterized by exponential deviation rates of infinitesimal irregularities given an orbit. Lyapunov exponents are also known to characterize properties of chaotic systems other than instability, such as metric entropy and attractive size. Also, for large systems, the extent of chaos is defined on the basis of the spectrum of Lyapunov exponents [9].

Lyapunov exponents (LEs) are one of the most common methods for determining the degree of sensitivity of the evolution of a dynamical system to initial conditions [10]. LEs measure how the distance between orbitals with two slightly different initial conditions grows or shrinks over time Having a positive largest Lyapunov exponent (LLE) is generally considered an indication that the system is chaotic [11]. The LLE value is an indicator of chaoticity, but numerical computation can take a lot of time before they manifest themselves, especially for orbits that adhere to regular orbits for a long time. Since the chaotic behavior LLE is defined as the $t \to \infty$ limit of the system, the time required for the system to converge to the limit value may be excessively long [12]. LLE provides more information than characterizing a trajectory as regular or chaotic, because it also quantifies the concept of chaoticity by providing a characteristic time scale for the dynamical system studied. In a detailed analysis of the time evolution of the LLE, a sloping decrease of the power-law $\lambda_1 \propto t^{-1}$ indicates regular state and any deviation from this law signifies chaos or weak chaos [12]. Also, staying constant to a positive value according to increasing time durations indicates that the system is in a chaotic or weakly chaotic state.

In this study, chaotic behavior of spinor type instanton solutions [5] is investigated by Lyapunov exponents. The chaotic behavior is analyzed by performing a numerical study of the time evolution of the Lyapunov exponents (LEs). Lyapunov exponent (LE) spectrum of system of two nonlinear ordinary differential equations corresponding spinor type instanton solutions with varying the parameters is plotted. Additionally, the behaviors in phase space and Lyapunov exponents of the spinor type instanton solutions are demonstrated comparatively.

2. Material and Method

2.1. Thirring Model

The Thirring model [2] is described by the two-dimensional pure fermionic, conformal invariant Lagrange equation.

$$L = i\bar{\psi}\sigma_\mu\partial_\mu\psi + \frac{g}{2}(\bar{\psi}\psi)^2 \quad (1)$$

Here, the positive coupling constant is g [3].The equation of motion is,

$$i\sigma_\mu\partial_\mu\psi + g(\bar{\psi}\psi)\psi = 0 \quad (2)$$

The Euclidean configuration of the Heisenberg approximation [13],

$$\psi = \left[ix_\mu\gamma_\mu\chi(s) + \varphi(s)\right]C \quad (3)$$

is given by the equation. Here C is an arbitrarily chosen spinor constant, $\chi(s)$ and $\varphi(s)$ of $s = x^2 + t^2 (x_1 \equiv x, x_2 \equiv t)$ are the real functions. Substituting equation (3) for equation (2), we get

$$\chi(s) + s\frac{d\chi(s)}{ds} + \alpha[s\chi(s)^2 + \varphi(s)^2]\varphi(s) = 0 \quad (4a)$$

$$\frac{d\varphi(s)}{ds} - \alpha[s\chi(s)^2 + \varphi(s)^2]\chi(s) = 0 \quad (4b)$$

The dimensionless form of nonlinear simple differential equation system pair (4a) and (4b),

$$2\frac{dp(t)}{dt} + \frac{1}{2}p(t) - \alpha AB(p(t)^2 + q(t)^2)q(t) = 0 \quad (5a)$$

$$2\frac{dq(t)}{dt} - \frac{1}{2}q(t) - \alpha AB(p(t)^2 + q(t)^2)p(t) = 0 \quad (5b)$$

Here, p and q are dimensionless functions of t and A, B are constants [5]. The solution of this system of equations for $\beta=\alpha(AB)=1$ is the Thirring instantons [5]. The evolution of Thirring instantons has been investigated in phase space as seen Figure 1 [5].

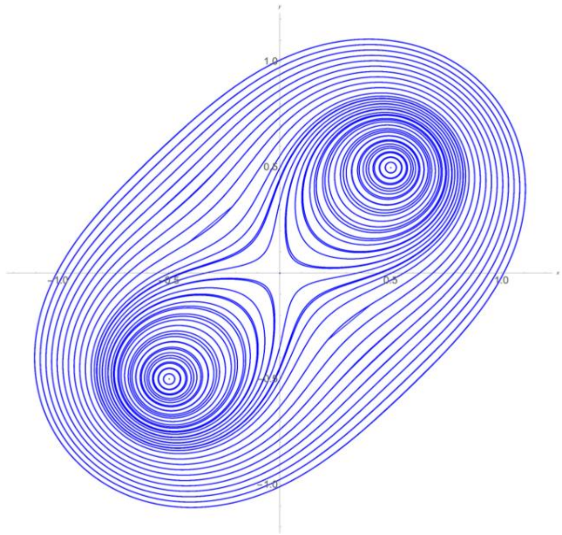

Figure 1. Phase diagram corresponding to the solutions of Thirring instantons [5].

A while ago, the stability of spinor type instanton solutions with different $\beta=\alpha(AB)$ values has been investigated by the scale index method [6]. It was determined that spinor type instanton solutions are unstable according to the scale index method for the value of $\beta=2$ as seen Figure 2 [6].

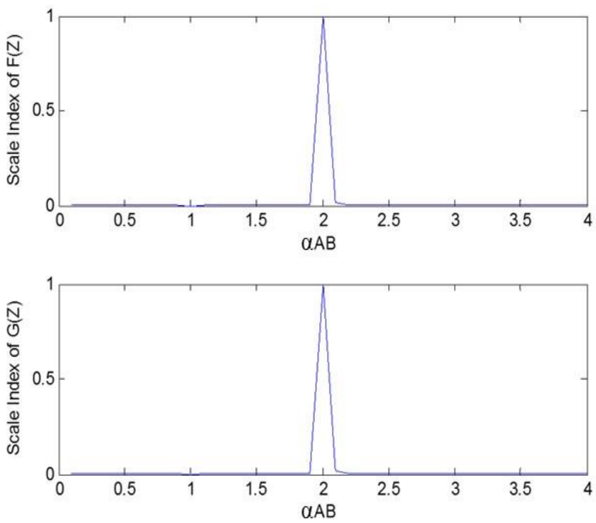

Figure 2. Scale index parameters of F(Z) and G(Z) versus αAB for the initial points: -1/2,-1/2 [6]. (In this study, Scale index parameters of *F(Z)* and *G(Z)* are *q* and *p* respectively)

2.2 Computation of Lyapunov Exponents

Lyapunov exponents are the rate of convergence or divergence of two adjacent orbits over time [14]. If the orbitals diverge slowly, they are periodic systems, and if this divergence is exponential, they are chaotic systems. The positive value of the Lyapunov exponent corresponds to the chaos, the zero value to the periodicity, and the negative value to the stable equilibrium state. In Ref. [15], a numerical technique was developed for calculating all LEs based on the time evolution of many deviation vectors kept linearly independent by a Gram-Schmidt orthonormalization procedure. In this study, the articles of Oseledec [16] and Benettin et al. [17, 18], whose theoretical results have been clearly proven, are followed.

A dynamical system can be characterized by a differential equation as

$$\dot{x}=\frac{dx}{dt}=f(x), \quad t>0 \tag{6}$$

here f is a continuous function. The LEs can be numerically obtained as time limits of appropriately computed quantities Λ_i, which are usually referred to as the finite-time LEs, i.e.

$$\lambda_l=\lim_{t\to\infty}\Lambda_l, \qquad l=1,2,\dots,N \tag{7}$$

which can for example be evaluated by the so-called 'standard method' [12, 18]. In particular, the largest Lyapunov exponent LLE (λ_1) is estimated as the limit for $t\to\infty$ of the finite-time LLE

$$\Lambda_1(t)=\frac{1}{t}\ln\frac{\|\omega(t)\|}{\|\omega(0)\|} \tag{8}$$

Where $\omega(t)=\delta x(t)=\left(\delta q(t),\delta p(t)\right)=\left(\delta q_1(t),\dots,\delta q_N(t)\right), \delta p_1(t),\dots,\delta p_N(t)$ denotes the phase space perturbation vector from the orbit $x(t)=\left(q(t),p(t)\right)$ at time *t*. In the case of regular orbits, $\lambda_1(t)$ tends to zero following the power law [12, 18]

$$\lambda_1(t)\propto t^{-1}, \tag{9}$$

while for chaotic or weakly chaotic orbits, it tends to a non-zero positive value.

The Lyapunov exponents describe the behavior of vectors in the tangent space of the phase space and are defined from the Jacobian matrix

$$J_{i,j}(t) = \frac{df_i(x)}{dx_j}\bigg|_{x(t)} \tag{10}$$

The evolution of an initial deviation vector $\omega(0)$ is governed by the so-called variational equations [12]

$$\dot{\omega}(t) = \begin{bmatrix} \delta\dot{q}_l(t) \\ \delta\dot{p}_l(t) \end{bmatrix} = J.\omega(t), l = 1,2,\dots,N, \tag{11}$$

this Jacobian defines the evolution of the tangent vectors via the equation.

Jacobian matrix evaluated at the position $x(t)$ of the orbit in the system's phase space for all $i, j = 1, 2, \dots, 2N$. The elements of J matrix in (11) depend on the evolution of the orbit $x(t)$ but are independent of $\omega(t)$.

The Lyapunov exponents are calculated for a long observation time of 100000 t units and with a step size 1.0 t unit. The time evolution of the Lyapunov exponents of system (5) is shown in Fig. 6. The $(+,-)$ nature of the Lyapunov exponents confirms the chaotic nature [19]. Numerical calculations of the Lyapunov exponents are integrated together using the tangent map of the model corresponding to the spinor type instanton solutions (5) that describe the evolution of the initial excitation of the system along with the equations of variation (11) for one or more initial perturbations (deviation vectors). method and Verner's "Most Efficient" 9/8 Runge–Kutta solver with tight fault tolerances [20]. The algorithm was implemented on top of the DynamicalSystems.jl software library, which was written entirely in the Julia programming language [21, 22].

3. Results and Discussion

The phase diagram corresponding to the spinor type instanton solutions found in the previous study [5] is given in Figure 1. As can be seen from the phase diagram, the stable points are (-0.5, -0.5) and (0.5,0.5). For β =α(AB) =1, the solution of this system of equations is the Thirring instantons given in Ref. [5]. Based on these stable points of the Thirring instantons, the Lyapunov spectrum of the spinor type instanton solutions (5) with varying parameter β =α(AB) for (q, p) = (0.5,0.5) fixed are examined. Figures 3 and 4 shows LLE spectrum of the spinor type instanton solutions (5), plotted in the interval (0, 100] and (0, 10) respectively by increasing the parameter β by 0.01 for initial condition (q, p) = (0.5, 0.5) fixed. t = 10^5 time units are used for numerical solution.

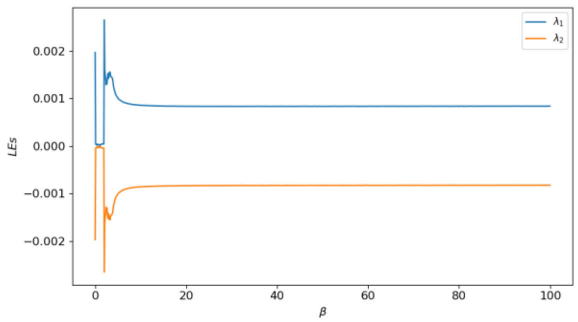

Figure 3. Lyapunov spectrum of spinor type instanton solutions (5) versus parameter β varying in the range of (0,100] for the initial points (0.5,0.5).

Looking at the trajectories in the phase diagram in Figure 1, it is not possible to reach a definite conclusion about the chaoticity. When we look at the Lyapunov spectrum in Figure 3, it is seen that the Lyapunov exponents become to 0 after a while from the beginning. Then a sharp increase occurs. Afterwards, the largest Lyapunov exponents (λ_1) proceed in a positive constant number. The largest Lyapunov exponent is $\lambda_1 \approx 0.003$. Also, since the largest Lyapunov exponents are 0 at the value close to the beginning, it is seen to be periodic. Afterwards, the trajectories can be interpreted as chaotic since they are always positive.

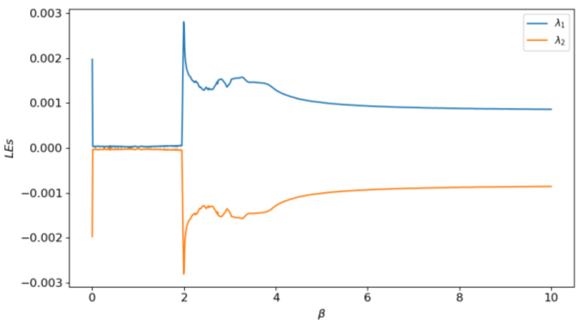

Figure 4. Lyapunov spectrum of spinor type instanton solutions (5) versus parameter β varying in the range of (0,10] for the initial points (0.5,0.5).

In order to observe the changes in the Lyapunov spectrum in more detail, a narrower area was chosen and the Lyapunov spectrum in Figure 4 is plotted. As seen in this spectrum, since largest Lyapunov exponents are 0 around β = 1 and β = 2, they are observed as periodic orbits. It increases sharply around β = 2 and then progresses positively.

Compared with the results obtained by the scale index method in Figure 2 [6], the chaotic situation in the Lyapunov spectrum for the value of β =2 has been determined. In the scale index method, values other than around β =1 are slightly above 0 [6]. When this situation is interpreted together with the Lyapunov spectrum, values slightly above 0 can be interpreted as a weak chaotic state or a quasi-periodic state.

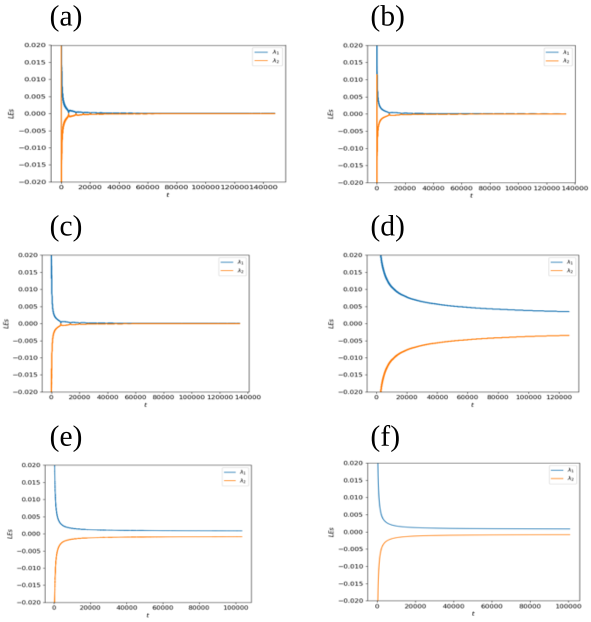

Figure 5. The time evolution of Lyapunov exponents (LEs) of the spinor type instanton solutions (5) for t = 10^5 time units with (a) β =0.1, (b) β =0.5, (c) β =1.5, (d) β =2.0, (e) β =10.0 (f) β =50.0 for initial conditions (q, p) =(0.5, 0.5) fixed.

The time evolution of Lyapunov exponents of the spinor type instanton solutions at t=10^5 time units is investigated in Figure 5. With the iteration number t=10^5, 1000 points are sampled from each trajectory studied in Figure 5. to evaluate the time evolution of the Lyapunov exponents with respect to the initial conditions at different values for the parameter value with (a) β =0.1, (b) β =0.5, (c) β =1.5, (d) β =2.0, (e) β =10.0 (f) β =50.0 for initial conditions (q, p) =(0.5, 0.5) fixed.

In Figure 5, it is seen that the Lyapunov exponents of the system (5) for (a) β =0.1, (b) β =0.5, (c) β =1.5 converge to 0. However, in Figure 5, it is seen that the Lyapunov exponents of the system (5) for (a) β =2, (b) β =10.0, (c) β =50.0 converge towards λ = ± 0.005 (β =2) and λ = ± 0, 002 (β =10.0 and 50.0). According to Figure 5, the time evolution of LEs are very small and zero indicates that the system has periodic and semi-periodic orbits in β< 2. Moreover, positive Lyapunov exponents do not have large values at β>2 values, it can be concluded that they have weak chaotic orbits.

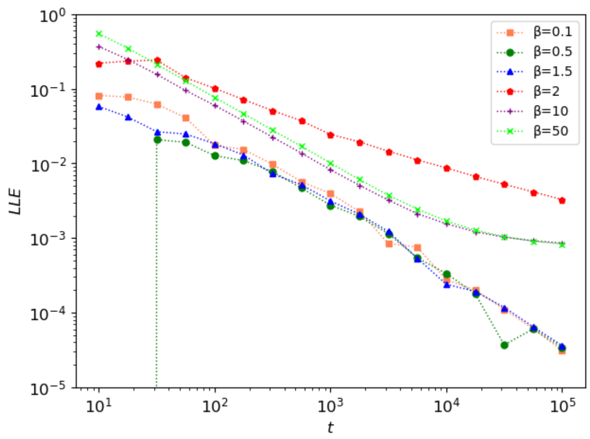

Figure 6. The time evolution of the largest Lyapunov exponents (LLEs) of the spinor type instanton solutions (5) with respect to different 6 different parameters β for t = 10^5 time units and initial conditional (q, p)=(0.5, 0.5) fixed. Parameter values: β =0.1(coral, square), β =0.5(green, circle), β =1.5 (blue, triangle), β =2.0 (red, pentagon), β =10.0 (purple, +), β =50.0 (lime, X). The axes are chosen according to the logarithmic base.

Here, only the largest Lyapunov exponent (LLE) is taken into account, as it determines the predictability of the system [12]. A positive LLE is generally accepted as an indication that the system is chaotic. The time evolution of the largest Lyapunov exponents (LLEs) of the spinor type instanton solutions Eq. (5) is investigated up to t = 10^5 time units. To evaluate the time evolution of the LLE, 1000 points are sampled from each trajectory. The time evolution of the LLE with 1000 test points; dots and lines represent linear regression with respect to the median in Figure 6. $t = 10^5$ is the final integration time of the simulations.

The time evolution of the LLEs of the spinor type instanton solutions with different the parameter value β and keeping the initial condition (q, p)=(0.5, 0.5) is ploted in Figure 6 (LLEs for β values; β =0.1(coral, square), β =0.5(green, circle), β =1.5 (blue, triangle), β =2.0 (red, pentagon), β =10.0 (purple, +), β =50.0 (lime, X)). In Figure 6 . In the case of β =0.1(coral, square), β =0.5(green, circle), β =1.5 (blue, triangle), LLEs tend to zero following the power law, i.e. orbits are in regular state. The other cases β =10.0 (purple, +), β =50.0 (lime, X), the LEEs begin to deviate from the $\lambda_1 \propto t^{-1}$ decrease denoting regular behavior when $t = 10^3$ time and the chaotic nature of the orbits. In Figure 6 , it is clearly seen that there is a convergence of around λ_1=10^{-3} at β =2 and values greater than β =2.

Considering the time evolution of the LLEs according to varying β values in the initial conditions (q, p)=(0.5, 0.5) fixed in Figure 6, it is seen that the fermion-like instanton solutions exhibit weak chaotic behavior.

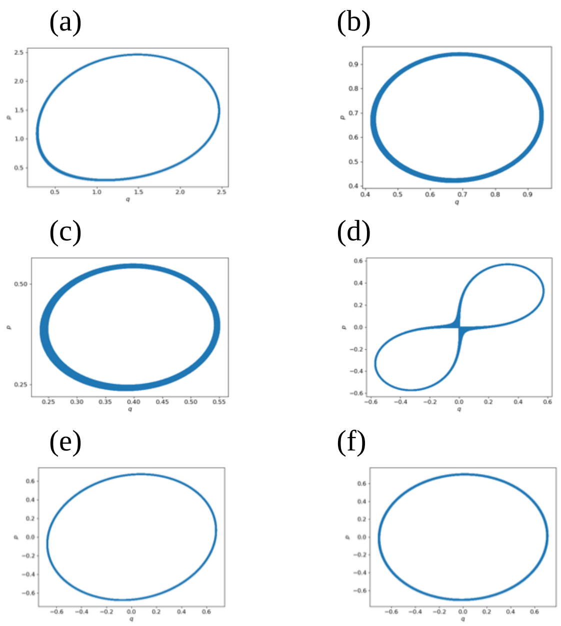

Figure 7. The phase space of the spinor type instanton solutions (5) for t = 10^5 time units with (a) β =0.1, (b) β =0.5, (c) β =1.5, (d) β =2.0, (e) β =10.0 (f) β =50.0 for initial conditions (q, p) =(0.5, 0.5) fixed.

The phase spaces of the system (5) were examined for random selected six points. In Figure 7 d), it is seen that the point (0,0) is an unstable point for β =2, which is the chaotic state, and bifurcation occurs in the phase diagram. In Figure 7. (a) β =0.1, (b) β =0.5, (c) β =1.5, values β<2, regular states appears around a single attractor. In Figure 7. (e) β =10.0 (f) β =50.0 values β>2, the system is periodic around the two attractors.

Although the phase spaces give information about the general states of the orbits, they do not give enough information about the chaotic states. For parameter value β = 2 in the system (6), the phase space and Lyapunov spectrum show a chaotic state. However, there is no chaotic situation in phase space for other parameter values. At β <2 values, the Lyapunov exponents approach zero and decrease proportionally with $\lambda_1(t) \propto t^{-1}$. The fact that the

Lyapunov exponents are very small and zero indicates that the system has periodic or semi-periodic orbits in these intervals. Since positive Lyapunov exponents do not have large values at β>2 values and according to power law (9) LLEs tends to a non-zero positive value, the orbits can be concluded that they have weakly chaotic states.

4. Conclusion and Suggestions

In quantum gauge theory, instantons have an important place in response to tunneling events between vacuums. In this study, the existence of chaos in spinor type instanton solutions found by the Heisenberg approach to the 2D conformal invariant Thirring model was investigated. The phase space and stable equilibrium points, which were found before and corresponding to the Thirring instantons, were taken as Ref. [5]. According to these values, the time evolution of the Lyapunov exponents, the time evolution of largest Lyapunov exponents of the spinor type instanton solutions (5) were plotted. Moreover Lyapunov spectrum of the system (5) was plotted. In addition, phase diagrams were examined for some critical situations.

The Lyapunov exponent is one of the most effective methods of detecting chaos [23]. According to the results found in this study, the fact that the largest Lyapunov exponent is greater than zero shows that the spinor type instanton solutions exhibit chaotic behavior. When the Lyapunov spectrum is also examined, it shows that spinor type instantons have regular orbit when the parameter value is β =α(AB)=1. In general, values less than β<2 indicate that it has regular or quasi-periodic orbits. Lyapunov spectrum, time evolution of LEs, phase space and scale index method [6] show that spinor type instanton solutions exhibit chaotic state at β =2. It is seen that there is a weakly chaotic states when the parameter values are from β>2. These results are consistent with the results found by the GALI method [7] that the orbits of spinor-type instanton solutions are regular around the fixed point, and the orbits become chaotic as they move away from the fixed point. The fact that spinor-type instanton solutions have both periodic orbits and chaotic orbits is an important sign of the existence of chaos in spinor-type instanton solutions. In the Gursey model, which is also a model with spinor type instanton solutions, research on Lyapunov exponents can be done and more information can be learned about the behavior of spinor type instanton solutions[24, 25].

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Statement of Research and Publication Ethics

The study is complied with research and publication ethics

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