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On the weakly completely continuous operators and factorization

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Abstract

In this paper, we establish some relationships between Left and right weakly completely continuous operators and topological centers of module actions and relationships between the factorization and the kinds of amenability. We define the locally topological center of the left and right module actions and investigate some of its properties. Also, we want to examine some conditions that under those the duality of a Banach algebra is strongly Connes-amenable. Finally, we generalize the concept of the weakly strongly connes amenable to even dual in higher orders.

Keywords: Arens regularity, topological centers, module actions, n-th dual,weakly completely continuous, weakly strongly Connes-amenable.
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1. Introduction

Suppose $\digamma^{**}$ is the second dual of a Banach algebra $\digamma$ endowed with the Arens multiplication. The constructions of the two Arens multiplications in $\digamma^{**}$ lead us to definition of topological centers for $\digamma^{**}$ with respect to both Arens multiplications. The topological centers of Banach algebras, module actions and applications of them have been studied in [1, 14, 15].

Baker, Lau and Pym in [19] proved that for Banach algebra A with bounded right approximate identity, $(A^*A)^\perp$ is an ideal of right annihilators in A^{**} and $A^{**} \cong (A^*A)^* \oplus (A^*A)^\perp$. Also in [9, 5, 11], the authors established the Arens regularity of module actions of Banach left or right modules over Banach algebras. They proved that if A has a bounded left approximate identity, then the right (left) module action of A on A^* is Arens regular if and only if A is reflexive.

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Let X, Y, Z be normed spaces and $\iota : X \times Y \to Z$ is a bounded bilinear mapping. Arens in [1] introduced two natural extensions ι^{***} and ι^{t***t} of ι, from $X^{**} \times Y^{**}$ into Z^{**} as follows:

1. $\iota^* : Z^* \times X \to Y^*$, given by $\langle \iota^*(\zeta', \mu), \nu\rangle = \langle \zeta', \iota(\mu, \nu)\rangle$ where, $\mu \in X$, $\nu \in Y$, $\zeta' \in Z^*$,

2. $\iota^{**} : Y^{**} \times Z^* \to X^*$, given by $\langle \iota^{**}(\nu'', \zeta'), \mu\rangle = \langle \nu'', \iota^*(\zeta', \mu)\rangle$ where, $\mu \in X$, $\nu'' \in Y^{**}$, $\zeta' \in Z^*$,

3. $\iota^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by $\langle \iota^{***}(\mu'', \nu''), \zeta'\rangle = \langle \mu'', \iota^{**}(\nu'', \zeta')\rangle$ where, $\mu'' \in X^{**}$, $\nu'' \in Y^{**}$, $\zeta' \in Z^*$.

A mapping ι^{***} is the unique extension of ι with $\mu'' \to \iota^{***}(\mu'', \nu'')$ from X^{**} into Z^{**} is $\mathrm{w}^* - to - \mathrm{w}^*$ continuous for $\nu'' \in Y^{**}$, but $\nu'' \to \iota^{***}(\mu'', \nu'')$ is not in general $\mathrm{w}^* - to - \mathrm{w}^*$ continuous from Y^{**} into Z^{**} unless $\mu'' \in X$. Hence the first topological center of ι is specified by:

$$Z_1(\iota) = \{\mu'' \in X^{**} : \quad \nu'' \to \iota^{***}(\mu'', \nu'') \ \ is \ \ \mathrm{w}^* - to - \mathrm{w}^* - continuous\}.$$

Let $\iota^t : Y \times X \to Z$ be the transpose of ι with $\iota^t(\nu, \mu) = \iota(\mu, \nu)$ for $\mu \in X$ and $\nu \in Y$. ι^t is a continuous bilinear from $Y \times X$ to Z, hence it extended to $\iota^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $\iota^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to ι^{***}, (see [1]), if $\iota^{***} = \iota^{t***t}$, then ι is called Arens regular. $\nu'' \to \iota^{t***t}(\mu'', \nu'')$ is $\mathrm{w}^* - to - \mathrm{w}^*$ continuous for $\mu'' \in X^{**}$, but $\mu'' \to m^{t***t}(\mu'', \nu'')$ from X^{**} into Z^{**} is not in general $\mathrm{w}^* - to - \mathrm{w}^*$ continuous for $\nu'' \in Y^{**}$. So we define the second topological center of ι as

$$Z_2(\iota) = \{\nu'' \in Y^{**} : \quad \mu'' \to \iota^{t***t}(\mu'', \nu'') \ \ is \ \ \mathrm{w}^* - to - \mathrm{w}^* - continuous\}.$$

Obviously ι is Arens regular if and only if $Z_1(\iota) = X^{**}$ or $Z_2(\iota) = Y^{**}$. Arens regularity of ι is characterized also by

$$\lim_i \lim_j \langle \zeta', \iota(\mu_i, \nu_j)\rangle = \lim_j \lim_i \langle \zeta', \iota(\mu_i, \nu_j)\rangle,$$

for bounded sequences $(\mu_i)_i \subseteq X$, $(\nu_i)_i \subseteq Y$ and $\zeta' \in Z^*$, (see [7]).
The constructions of the two Arens products are as below:
The first Arens product is stated in three steps. For α, β in $\mathcal{F}$, f in $\mathcal{F}^*$ and m, n in $\mathcal{F}^{**}$, the phrases $f.\alpha$ and $m.f$ of $\mathcal{F}^*$ and $m.n$ of $\mathcal{F}^{**}$ are specified by:

$$\langle f.\alpha, \beta\rangle = \langle f, \alpha\beta\rangle$$

$$\langle m.f, \alpha\rangle = \langle m, f.\alpha\rangle$$

$$\langle m.n, f\rangle = \langle m, n.f\rangle$$

The second Arens: For α, β in $\mathcal{F}$, f in $\mathcal{F}^*$ and m, n in $\mathcal{F}^{**}$, the phrases $\alpha\Box f$, $f\Box m$ of $\mathcal{F}^*$ and $m\Box n$ of $\mathcal{F}^{**}$ are specified by:

$$\langle \alpha\Box f, \beta\rangle = \langle f, \beta\alpha\rangle$$

$$\langle f\Box m, \alpha\rangle = \langle m, \alpha\Box f\rangle$$

$$\langle m\Box n, f\rangle = \langle n, f\Box m\rangle$$

Let α'' and β'' be elements of $\mathcal{F}^{**}$, the second dual of $\mathcal{F}$. By Goldstin's Theorem, there exist nets $(\alpha_\varrho)_\varrho$ and $(\beta_\upsilon)_\upsilon$ in $\mathcal{F}$ with $\alpha_\varrho \overset{\mathrm{w}^*}{\to} \alpha''$ and $\beta_\upsilon \overset{\mathrm{w}^*}{\to} \beta''$. So for $\alpha' \in \mathcal{F}^*$,

$$\lim_\varrho \lim_\upsilon \langle \alpha', \iota(\alpha_\varrho, \beta_\upsilon)\rangle = \langle \alpha''\beta'', \alpha'\rangle$$

and

$$\lim_{\upsilon}\lim_{\varrho}\langle \alpha', \iota(\alpha_\varrho, \beta_\upsilon)\rangle = \langle \alpha''\Box\beta'', \alpha'\rangle,$$

where $\alpha''\beta''$ and $\alpha''\Box\beta''$ are the first and second Ares products of $\mathcal{F}^{**}$, respectively, (see [6]).

Let $\mathcal{B}$ be a Banach $\mathcal{F}$-bimodule, and let

$$\pi_\ell : \ \mathcal{F} \times \mathcal{B} \longrightarrow \mathcal{B} \quad \text{and} \quad \pi_r : \ \mathcal{B} \times \mathcal{F} \longrightarrow \mathcal{B},$$

be the right and left module actions of $\mathcal{F}$ on $\mathcal{B}$. By previous description, the transpose of π_r denoted by $\pi_r^t : \mathcal{F} \times \mathcal{B} \to \mathcal{B}$. Then

$$\pi_\ell^* : \mathcal{B}^* \times \ \mathcal{F} \longrightarrow \mathcal{B}^* \quad \text{and} \quad \pi_r^{t*t} : \mathcal{F} \times \mathcal{B}^* \longrightarrow \mathcal{B}^*.$$

Thus $\mathcal{B}^*$ is a left Banach $\mathcal{F}$-module and a right Banach $\mathcal{F}$-module with respect to the module actions π_r^{t*t} and π_ℓ^*, respectively. The second dual $\mathcal{B}^{**}$ is a Banach $\mathcal{F}^{**}$-bimodule with the following module actions

$$\pi_\ell^{***} : \ \mathcal{F}^{**} \times \mathcal{B}^{**} \longrightarrow \mathcal{B}^{**} \quad \text{and} \quad \pi_r^{***} : \ \mathcal{B}^{**} \times \mathcal{F}^{**} \longrightarrow \mathcal{B}^{**},$$

where $\mathcal{F}^{**}$ is considered as a Banach algebra with respect to the first Arens product. Similarly, $\mathcal{B}^{**}$ is a Banach $\mathcal{F}^{**}$-bimodule with the module actions

$$\pi_\ell^{t***t} : \ \mathcal{F}^{**} \times \mathcal{B}^{**} \longrightarrow \mathcal{B}^{**} \quad \text{and} \quad \pi_r^{t***t} : \ \mathcal{B}^{**} \times \mathcal{F}^{**} \longrightarrow \mathcal{B}^{**},$$

where $\mathcal{F}^{**}$ is considered as a Banach algebra with respect to the second Arens product.

Let $\mathcal{B}$ be a Banach $\mathcal{F}$-bimodule. Then $\mathcal{B}$ is called factors on the left (right) with respect to $\mathcal{F}$, if $\mathcal{B} = \mathcal{B}\mathcal{F}$ ($\mathcal{B} = \mathcal{F}\mathcal{B}$). Thus $\mathcal{B}$ factors on both sides, if $\mathcal{B} = \mathcal{B}\mathcal{F} = \mathcal{F}\mathcal{B}$.

The member e'' of $\mathcal{F}^{**}$ is said to be a bilateral unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. Indeed, e'' is a bilateral unit if and only if, for each $\alpha'' \in \mathcal{F}^{**}$, $\alpha''e'' = e''\Box\alpha'' = \alpha''$. By [[2], p.146], an element e'' of $\mathcal{F}^{**}$ is bilateral unit if and only if it is a w* cluster point of some BAI (bounded approximate identity) $(e_\varrho)_{\varrho\in I}$ in $\mathcal{F}$.

2. Weakly completely continuous operators and its relationships with the topological centers of module actions

In throughout of this section, we denote weakly sequentially complete Banach algebra $\mathcal{F}$ by WSC, that is $\mathcal{F}$ is said to be weakly sequentially complete, if every weakly Cauchy sequence in $\mathcal{F}$ has a weak limit in $\mathcal{F}$. Here we investigate some conditions that under those conditions a bilinear mapping on WSC is weakly compact.

Suppose that $\mathcal{F}$ is a Banach algebra and $\mathcal{B}$ is a Banach $\mathcal{F} - bimodule$. According to [7], $\mathcal{B}^{**}$ is a Banach $\mathcal{F}^{**} - bimodule$, where $\mathcal{F}^{**}$ is equipped with the first Arens product. We define $\mathcal{B}^*\mathcal{B}$ as a subspace of $\mathcal{F}$, that is, for all $\beta' \in \mathcal{B}^*$ and $\beta \in \mathcal{B}$, we define

$$\langle \beta'\beta, \alpha\rangle = \langle \beta', \beta\alpha\rangle;$$

We similarly define $\mathcal{B}^{***}\mathcal{B}^{**}$ as a subspace of $\mathcal{F}^{**}$ and we take $\mathcal{F}^{(0)} = \mathcal{F}$ and $\mathcal{B}^{(0)} = \mathcal{B}$.

In the following, we will study some properties of topological centers of module actions and we will extend some problems from topological centers of Banach algebras into module actions with some relationships of them.

When there is not confused, we set $\pi_\ell(\alpha, \beta) = \alpha\beta$ and $\pi_r(\beta, \alpha) = \beta\alpha$, then we use the notions $Z^\ell_{\mathcal{B}^{**}}(\mathcal{F}^{**})$ and $Z^r_{\mathcal{F}^{**}}(\mathcal{B}^{**})$ for topological centers of module actions as follows.

$$Z^\ell_{\mathcal{B}^{**}}(\mathcal{F}^{**}) = \{\alpha'' \in \mathcal{F}^{**} : \ \textit{the map} \ \ \beta'' \to \alpha''\beta'' \ \ : \ \mathcal{B}^{**} \to \mathcal{B}^{**}$$

$$is \quad \mathrm{w}^* - to - \mathrm{w}^* \ continuous\}.$$

$$Z^r_{\mathcal{F}^{**}}(\mathcal{B}^{**}) = \{\beta'' \in \mathcal{B}^{**} : \ the \ map \ \ \alpha'' \to \beta''\alpha'' \ : \ \mathcal{F}^{**} \to \mathcal{B}^{**}$$
$$is \quad \mathrm{w}^* - to - \mathrm{w}^* \ continuous\}$$

Definition 2.1. *If $\mathcal{F}$ is a Banach algebra and $\varsigma_\alpha : \mathcal{F} \longrightarrow \mathcal{F}$ is defined by $\beta \mapsto \alpha\beta$ for all $\alpha \in \mathcal{F}$, then $\alpha \in \mathcal{F}$ is called left weakly completely continuous, if the mapping ς_α be weakly compact on $\mathcal{F}$. Banach algebra $\mathcal{F}$ is left weakly completely continuous if every element α of $\mathcal{F}$ be left weakly completely continuous. The right weakly completely continuous is defined analogously. We say that $\mathcal{F}$ is weakly completely continuous(wcc) if $\mathcal{F}$ is left and right weakly completely continuous.*

Theorem 2.2. *Let $\mathcal{F}$ be a WSC and wcc Arens regular Banach algebra with a BAI and $\mathcal{B}$ be a Banach algebra. Then every bounded linear operator $T \in \mathcal{B}(\mathcal{F}, \mathcal{B})$ is weakly compact.*

Proof. Assume that $\mathcal{F}$ is Arens regular. We prove that $\mathcal{F}$ is reflexive. In this state if $\mathcal{F}$ is reflexive, then ball $\mathcal{F}$ is $\sigma(\mathcal{F}, \mathcal{F}^*)$-compact. Thus, since $T : (\mathcal{F}, \sigma(\mathcal{F}, \mathcal{F}^*)) \to (\mathcal{B}, \sigma(\mathcal{B}, \mathcal{B}^*))$ is continuous, then $\overline{T(ball\mathcal{F})}^w$ is weakly compact. Then T is weakly compact.
Now we prove that $\mathcal{F}$ is reflexive. Let D_0 be a separable subalgebra of $\mathcal{F}$. Then by Lemma 3.4 of [18] D_0 is contained in a separable subalgebra D of $\mathcal{F}$ that has a sequential $\mathcal{BF}I$ $(e_n)_{n\in\mathbb{N}}$. Hence by assumptions, D will be Arens regular and WSC and by Theorem 3.1 of [18] D^* factors($D^*D = D^* = DD^*$). Now, for any $\alpha' \in D^*$ there exist $\mu \in D$ and $\beta' \in D^*$ with$\alpha' = \beta'\mu$. Since $\mu e_n \to \mu$, then

$$\langle \alpha', e_n \rangle = \langle \beta'\mu, e_n \rangle = \langle \beta', \mu e_n \rangle \to \langle \beta', \mu \rangle.$$

This shows that the sequence $(e_n)_{n\in\mathbb{N}}$ is weakly Cauchy. since D is WSC, it converges weakly to some element e of D. Then e is the unit element of D. Thus D is unital . Now we suppose $\mathcal{F}$ is *wcc* and let $(\alpha_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{F}$. Then $(\alpha_n)_{n\in\mathbb{N}}$ is contained in a unital subalgebra D. Since D is also *wcc* and unital, it is reflexive. Hence $(\alpha_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence, and $\mathcal{F}$ is reflexive. □

Example 2.3. *Let G be a locally not compact group but no compact. Then the only element of $L^1(G)$ with left weakly completely continuous, is zero. But if G is compact, then each $\phi \in L^1(G)$ is left weakly completely continuous.*

Definition 2.4. *Let $\mathcal{B}$ be a left Banach $\mathcal{F}-$module. Then $\mathcal{B}^*$ is said to be left weakly completely continuous $= (\widetilde{Lwcc})$, if for each $\beta' \in \mathcal{B}^*$, the mapping $\alpha \to \pi^*_\ell(\beta', \alpha)$ from $\mathcal{F}$ into $\mathcal{B}^*$, maps weakly Cauchy sequence into weakly convergence ones. If every $\beta' \in \mathcal{B}^*$ is $\widetilde{Lwcc}$, then we say that $\mathcal{B}^*$ is $\widetilde{Lwcc}$.*
The definition of right weakly completely continuous $(= \widetilde{Rwcc})$ is similar. We say that $\beta' \in \mathcal{B}^*$ is weakly completely continuous $(= \widetilde{wcc})$, if β' is $\widetilde{Lwcc}$ and $\widetilde{Rwcc}$.

Theorem 2.5. *Let $\mathcal{B}$ be a left Banach $\mathcal{F}-$module. Then by one of the following conditions, $\mathcal{B}^*$ is $\widetilde{Lwcc}$.*

1. *$\mathcal{F}$ is WSC.*
2. *$\mathcal{B}^*$ is WSC.*
3. *$Z_{\mathcal{B}^{**}}(\mathcal{F}^{**}) = \mathcal{F}^{**}$.*

Proof. 1. Let $(\alpha_n)_n \subseteq \mathcal{F}$ be weakly Cauchy sequence. Since $\mathcal{F}$ is WSC, there is $\alpha \in \mathcal{F}$ with $\alpha_n \xrightarrow{w} \alpha$. Now, let $\beta' \in \mathcal{B}^*$ and $\beta'' \in \mathcal{B}^{**}$. Then we have

$$\langle \beta'', \pi^*_\ell(\beta', \alpha_n) \rangle = \langle \pi^{**}_\ell(\beta'', \beta'), \alpha_n \rangle \to \langle \pi^{**}_\ell(\beta'', \beta'), \alpha \rangle = \langle \beta'', \pi^*_\ell(\beta', \alpha) \rangle.$$

Thus $\pi^*_\ell(\beta', \alpha_n) \xrightarrow{w} \pi^*_\ell(\beta', \alpha)$.

2. Proof is similar to (1).

3. Let $(\alpha_n)_n \subseteq \digamma$ be weakly Cauchy sequence. Since the sequence $(\alpha_n)_n \subseteq \digamma$ is weakly bounded in $\digamma$, it has subsequence such as $(\alpha_{n_k})_k \subseteq \digamma$ with $\alpha_{n_k} \overset{w^*}{\to} \alpha''$ for some $\alpha'' \in \digamma^{**}$. Then for each $\beta' \in \mathcal{B}^*$ and $\beta'' \in \mathcal{B}^{**}$, we have

$$\begin{aligned}\langle \beta'', \pi_\ell^*(\beta', \alpha_{n_k})\rangle &= \langle \pi_\ell^{***}(\alpha_{n_k}, \beta''), \beta'\rangle \to \langle \pi_\ell^{***}(\alpha'', \beta''), \beta'\rangle \\ &= \langle \alpha'', \pi_\ell^{**}(\beta'', \beta')\rangle = \langle \pi_\ell^{*****}(\beta'', \beta'), \alpha''\rangle \\ &= \langle \beta'', \pi_\ell^{****}(\beta', \alpha'')\rangle.\end{aligned}$$

It is enough, we show that $\pi_\ell^{****}(\beta', \alpha'') \in \mathcal{B}^*$. Suppose that $(\beta''_\varrho)_\varrho \subseteq \mathcal{B}^{**}$ with $\beta''_\varrho \overset{w^*}{\to} \beta''$. Then since $Z_{\mathcal{B}^{**}}(\digamma^{**}) = \digamma^{**}$, we have

$$\begin{aligned}&\langle \pi_\ell^{****}(\beta', \alpha''), \beta''_\varrho\rangle = \langle \beta', \pi_\ell^{***}(\alpha'', \beta''_\varrho)\rangle = \langle \pi_\ell^{***}(\alpha'', \beta''_\varrho), \beta'\rangle \\ &\to \langle \pi_\ell^{***}(\alpha'', \beta''), \beta'\rangle = \langle \pi_\ell^{****}(\beta', \alpha''), \beta''\rangle.\end{aligned}$$

It follows that $\pi_\ell^{****}(\beta', \alpha'') \in (\mathcal{B}^{**}, \mathrm{w}^*)^* = \mathcal{B}^*$.

□

Example 2.6. *i) Suppose that G is a compact group and $1 \leq p \leq \infty$. Take $L^p(G)$ and $M(G)$ as $L^1(G)-$bimodule. Since $L^1(G)$ is WSC, by using Theorem 2.5, $L^p(G)$ and $M(G)^*$ are $\widetilde{Lwcc}$. We know that $c_0^* = \ell^1$ and c_0 is a ℓ^1-bimodule. Then since ℓ^1 is WSC, by using Theorem 2.5, ℓ^1 is $\widetilde{Lwcc}$.*

ii) Let $\mathcal{B}$ be a reflexive Banach space. $\mathcal{B}\widehat{\otimes}\mathcal{B}^*$, $N(\mathcal{B})$, the space of nuclear operator on $\mathcal{B}$, $K(\mathcal{B})$, the space of compact operators on $\mathcal{B}$ and $W(\mathcal{B})$, the space of weakly compact operators on $\mathcal{B}$ are Arens regular, so, they are $\widetilde{Lwcc}$ (see[6]).

Definition 2.7. *Let $\digamma$ be a Banach space. The member α'' of $\digamma^{**}$ is said to be Baire -1, if there exists a sequence $(\alpha_n)_n$ in $\digamma$ that converges to α'' in the w^* topology of $\digamma^{**}$. The collection of Baire -1 elements of $\digamma^{**}$ is denoted by $\mathcal{B}_1(\digamma)$.*

Theorem 2.8. *Let $\mathcal{B}$ be a left Banach $\digamma-$module. Then $\mathcal{B}_1(\digamma) \subseteq Z_{\mathcal{B}^{**}}(\digamma^{**})$ if and only if $\mathcal{B}^*$ is $\widetilde{Lwcc}$.*

Proof. Let $\mathcal{B}^*$ is $\widetilde{Lwcc}$ and suppose that $\alpha'' \in \mathcal{B}_1(\digamma)$. Then there is a sequence $(\alpha_n)_n \subseteq \digamma$ with $\alpha_n \overset{w^*}{\to} \alpha''$. It follows that $(\alpha_n)_n$ is weakly Cauchy sequence in $\digamma$. Then there is $\nu' \in \mathcal{B}^*$ with $\pi_\ell^*(\beta', \alpha_n) \overset{w}{\to} \nu'$. Moreover, for $\beta'' \in \mathcal{B}^{**}$, we have:

$$\begin{aligned}\langle \beta'', \nu'\rangle &= \lim_n \langle \beta'', \pi_\ell^*(\beta', \alpha_n)\rangle \\ &= \lim_n \langle \pi_\ell^{**}(\beta'', \beta'), \alpha_n\rangle \\ &= \lim_n \langle \alpha_n, \pi_\ell^{**}(\beta'', \beta')\rangle \\ &= \langle \alpha'', \pi_\ell^{**}(\beta'', \beta')\rangle \\ &= \langle \pi_\ell^{*****}(\beta'', \beta'), \alpha''\rangle \\ &= \langle \beta'', \pi_\ell^{****}(\beta', \alpha'')\rangle.\end{aligned}$$

It follows that $\pi_\ell^{****}(\beta', \alpha'') \in \mathcal{B}^*$. Suppose that $(\beta''_\varrho)_\varrho \subseteq \mathcal{B}^{**}$ with $\beta''_\varrho \overset{w^*}{\to} \beta''$. Then for each $\beta' \in \mathcal{B}^*$, we have

$$\begin{aligned}\langle \pi_\ell^{***}(\alpha'', \beta''_\varrho), \beta'\rangle &= \langle \pi_\ell^{****}(\beta', \alpha''), \beta''_\varrho\rangle \\ &= \langle \beta''_\varrho, \pi_\ell^{****}(\beta', \alpha'')\rangle \to \langle \beta'', \pi_\ell^{****}(\beta', \alpha'')\rangle \\ &= \langle \pi_\ell^{***}(\alpha'', \beta''), \beta'\rangle.\end{aligned}$$

Therefore we have $\alpha'' \in Z_{\mathcal{B}^{**}}(\digamma^{**})$.
Conversely, Let $\mathcal{B}_1(\digamma) \subseteq Z_{\mathcal{B}^{**}}(\digamma^{**})$ and suppose that $(\alpha_n)_n \subseteq \digamma$ is weakly Cauchy sequence in $\digamma$. Then it has subsequence such as $(\alpha_{n_k})_k \subseteq \digamma$ with $\alpha_{n_k} \xrightarrow{w^*} \alpha''$ for some $\alpha'' \in \digamma^{**}$. It follows that $\alpha'' \in \mathcal{B}_1(\digamma)$, and so $\alpha'' \in Z_{\mathcal{B}^{**}}(\digamma^{**})$. It is similar to Theorem 2.5, we have $\pi_\ell^{****}(\beta', \alpha'') \in \mathcal{B}^*$ for some $\beta' \in \mathcal{B}^*$. Then for $\beta'' \in \mathcal{B}^{**}$, we have

$$\begin{aligned}
\lim_n \langle \beta'', \pi_\ell^*(\beta', \alpha_n) \rangle &= \lim_n \langle \pi_\ell^{**}(\beta'', \beta'), \alpha_n \rangle \\
&= \lim_k \langle \pi_\ell^{**}(\beta'', \beta'), \alpha_{n_k} \rangle \\
&= \lim_k \langle \pi_\ell^{***}(\alpha_{n_k}, \beta''), \beta' \rangle \\
&= \langle \pi_\ell^{***}(\alpha'', \beta''), \beta' \rangle \\
&= \langle \pi_\ell^{*****}(\beta', \beta''), \alpha'' \rangle \\
&= \langle \beta'', \pi_\ell^{****}(\beta', \alpha'') \rangle.
\end{aligned}$$

It follows that $\pi_\ell^*(\beta', \alpha_n) \xrightarrow{w} \pi_\ell^{****}(\beta', \beta'')$ in $\mathcal{B}^*$. Thus $\mathcal{B}^*$ is $\widetilde{Lwcc}$. □

Definition 2.9. *Let $\mathcal{B}$ be a Banach $\digamma -$bimodule and $\alpha'' \in \digamma^{**}$. We define the locally topological center of the left and right module actions of α'' on $\mathcal{B}^{**}$, respectively, as follows*

$$\begin{aligned}
Z^t_{\alpha''}(\mathcal{B}^{**}) = Z^t_{\alpha''}(\pi^t_\ell) = \{\beta'' \in \mathcal{B}^{**} : \quad \pi_\ell^{t***t}(\alpha'', \beta'') = \pi_\ell^{***}(\alpha'', \beta'')\}, \\
Z_{\alpha''}(\mathcal{B}^{**}) = Z_{\alpha''}(\pi^t_r) = \{\beta'' \in \mathcal{B}^{**} : \quad \pi_r^{t***t}(\beta'', \alpha'') = \pi_r^{***}(\beta'', \alpha'')\}.
\end{aligned}$$

It is clear that

$$\bigcap_{\alpha'' \in \digamma^{**}} Z^t_{\alpha''}(\mathcal{B}^{**}) = Z^t_{\digamma^{**}}(\mathcal{B}^{**}) = Z(\pi^t_\ell),$$

$$\bigcap_{\alpha'' \in \digamma^{**}} Z_{\alpha''}(\mathcal{B}^{**}) = Z_{\digamma^{**}}(\mathcal{B}^{**}) = Z(\pi_r).$$

Theorem 2.10. *Let $\mathcal{B}$ be a left Banach $\digamma -$module. Then we have:*

1. *Suppose that $\mathcal{B}$ has a sequential $\mathcal{B}\digamma I$ $(e_n)_n \subseteq \digamma$ with$Z_{e''}(\mathcal{B}^{**})\digamma \subseteq \mathcal{B}$ where e'' is a bilateral unit for $\digamma^{**}$ and $e_n \xrightarrow{w^*} e''$. If $\mathcal{B}$ is WSC, then $Z_{e''}(\mathcal{B}^{**}) = \mathcal{B}$.*
2. *If $\mathcal{B}^*$ is WSC, then $\mathcal{B}_1(\digamma) \subseteq Z_{\mathcal{B}^{**}}(\digamma^{**})$.*
3. *Assume that $\mathcal{B}$ has a sequential $LBAI$ $(e_n)_n \subseteq \digamma$ and $\mathcal{B}^*$ is WSC. If $\digamma$ is a left ideal in $\digamma^{**}$, then $Z_{\mathcal{B}^{**}}(\digamma^{**}) = \digamma^{**}$.*
4. *Assume that $\mathcal{B}^*$ is WSC and $\digamma$ is a right ideal in $\digamma^{**}$. If $Z_{e''}(\digamma^{**}) = \digamma^{**}$, then $Z_{\mathcal{B}^{**}}(\digamma^{**}) = \digamma^{**}$.*

Proof. 1. Since $\mathcal{B} \subseteq Z_{e''}(\mathcal{B}^{**})$ for every $\beta \in \mathcal{B}$, we have $\pi_r^{***}(\beta, e'') = w^* - \lim_n \pi_r(\beta, e_n) = \beta$. Let $\beta'' \in Z_{e''}(\mathcal{B}^{**})$. Suppose that $(\beta_\varrho)_\varrho \subseteq \mathcal{B}$ with$\beta_\varrho \xrightarrow{w^*} \beta''$. Then for every $\beta' \in \mathcal{B}^*$, we have

$$\begin{aligned}
\lim_n \langle \pi_r^{***}(\beta'', e_n), \beta' \rangle &= \langle \pi_r^{***}(\beta'', e''), \beta' \rangle \\
&= \lim_\varrho \langle \pi_r^{***}(\beta_\varrho, e''), \beta' \rangle \\
&= \lim_\varrho \langle \beta', \beta_\varrho \rangle \\
&= \langle \beta'', \beta' \rangle.
\end{aligned}$$

It follows that $w^* - \lim \pi_r^{***}(\beta'', e_n) = \beta''$. Since $\pi_r^{***}(\beta'', e_n) \in \mathcal{B}$ and $\mathcal{B}$ is WSC, $\beta'' \in \mathcal{B}$.

2. Assume that $\mathcal{B}^*$ is WSC. Then by Theorem 2.5, $\mathcal{B}^*$ is $\widetilde{Lwcc}$, by Theorem 2.8, we have $\mathcal{B}_1(\digamma) \subseteq Z_{\mathcal{B}^{**}}(\digamma^{**})$.
3. Assume that $\mathcal{B}^*$ is WSC, by using part (2), we have $\mathcal{B}_1(\digamma) \subseteq Z_{\mathcal{B}^{**}}(\digamma^{**})$. Let $\alpha'' \in \digamma^{**}$ and suppose that $e'' \in \digamma^{**}$ is a left unit for $\digamma^{**}$ with $e_n \xrightarrow{w^*} e''$. Then for every $\alpha'' \in \digamma^{**}$, we have $e_n\alpha'' \xrightarrow{w^*} e''\alpha'' = \alpha''$. Since $\digamma\digamma^{**} \subseteq \digamma$, $\alpha'' \in \mathcal{B}_1(\digamma)$. Consequently we have $\alpha'' \in Z_{\mathcal{B}^{**}}(\digamma^{**})$.
4. Proof is similar to (3).

□

Example 2.11. 1. *Let G be a compact group. We know that $L^1(G)$ is WSC with a sequential $\mathcal{B}\digamma I$. Assume that e'' is a bilateral unit for $L^1(G)^{**}$. Since*

$$Z_{e''}(L^1(G)^{**})L^1(G) \subseteq L^1(G)^{**}L^1(G) \subseteq L^1(G),$$

by using the preceding theorem, we have

$$Z_{e''}(L^1(G)^{**}) = L^1(G).$$

2. *Let G be a locally compact group. In the preceding theorem, if we take $\mathcal{B} = c_0(G)$ and $\digamma = \mathbf{L}^1(G)$, then it is clear that $\mathcal{B}$ is a Banach $\digamma - bimodule$. Since $\mathbf{L}^1(G) = c_0(G)^*$ is a WSC, $\subseteq Z^{\ell}_{(\ell^1(G))^\infty}(\ell^\infty(G)^*)$.*

3. Arens regularity and factorization property and weakly strongly Connes-amenability

A variant of that definition amenability was introduced in [16], but is most commonly associated with A. Connes' paper [3]. For this reason, we refer to this notion of amenability as to Connes-amenability.

A derivation from Banach algebra $\digamma$ into Banach $\digamma$−bimodule $\mathcal{B}$ is a bounded linear mapping $D : \digamma \longrightarrow \mathcal{B}$ with

$$D(\mu\nu) = \mu D(\nu) + D(\mu)\nu \quad for \quad all \quad \mu,\ \nu \in \digamma.$$

The space of continuous derivations from $\digamma$ into $\mathcal{B}$ is denoted by $Z^1(\digamma, \mathcal{B})$.
Easy examples of derivations are the inner derivations, which are given for each $b \in \mathcal{B}$ by

$$\delta_\beta(\alpha) = \alpha\beta - \beta\alpha \quad for \quad all \quad \alpha \in \digamma.$$

The space of inner derivations from $\digamma$ into $\mathcal{B}$ is denoted by $\mathcal{B}^1(\digamma, \mathcal{B})$. The Banach algebra $\digamma$ is said to be amenable, if for every Banach $\digamma - bimodule$ $\mathcal{B}$, the inner derivations are only derivations existing from $\digamma$ into $\mathcal{B}^*$. It is clear that $\digamma$ is amenable if and only if $H^1(\digamma, \mathcal{B}^*) = Z^1(\digamma, \mathcal{B}^*)/\mathcal{B}^1(\digamma, \mathcal{B}^*) = \{0\}$ and $\digamma$ is weakly amenable if $H^1(\digamma, \digamma^*) = \{0\}$. The concept of amenability for a Banach algebra $\digamma$, was introduced by Johnson in 1972, (see [13]). For a Banach $\digamma - bimodule$ $\mathcal{B}$, the quotient space $H^1(\digamma, \mathcal{B})$ of all continuous derivations from $\digamma$ into $\mathcal{B}$ modulo the subspace of inner derivations is called the first cohomology group of $\digamma$ with coefficients in $\mathcal{B}$.

A Banach algebra $\mathcal{B}$ is said to be dual if there is a closed submodule $\mathcal{B}_*$ of $\mathcal{B}^*$ with$\mathcal{B} = (\mathcal{B}_*)^*$ and we know that $\mathcal{B}_*$ need not be unique. For example, if $\mathcal{B}$ is Arens regular, then $\mathcal{B}^{**}$ is dual, or if G is locally compact group, then $M(G)$ is dual (with $M(G)_* = C_0(G)$).
Let $\digamma$ be a Banach Algebra. A dual Banach $\digamma - bimodule$ $\mathcal{B}$ is called normal if, for $\mu \in \mathcal{B}$ the map $\alpha \mapsto \alpha\mu$ and $\alpha \mapsto \mu\alpha$ from $\digamma$ into $\mathcal{B}$ is weak*-to- weak* continuous.

A dual Banach algebra $\digamma$ is strongly Connes-amenable if, for every normal, dual Banach $\digamma - bimodule$ $\mathcal{B}$, every weak*-to- weak* continuous derivation $D \in Z^1(\digamma, \mathcal{B}^*)$ is inner. Then we write $H^1_{w^*}(\digamma, \mathcal{B}^*) = \{0\}$.

Theorem 3.1. *Suppose $\mathcal{F}$ is a Banach algebra that $(\mathcal{F}^{**}, \Diamond)$ is amenable($\Diamond$ is the first Arens product) with$\mathcal{F}$ is ideal in $\mathcal{F}^{**}$ and if $\widehat{\mathcal{F}}$ is the image of $\mathcal{F}$ in $\mathcal{F}^{**}$ under the canonical mapping, with$\widehat{\mathcal{F}}\Diamond\mathcal{F}^{**} \subset Z_1$. Then $\mathcal{F}^{**}$ is strongly Connes-amenable.*

Proof. Since $(\mathcal{F}^{**}, \Diamond)$ is amenable, then by Theorem 1.8 of [12], $\mathcal{F}$ will be amenable. So $\mathcal{F}$ has a bounded approximate identity. Hence $\mathcal{F}^*.\mathcal{F} = \mathcal{F}^*$(namely $\mathcal{F}^*$ factors on the left). Now, if $f \in \mathcal{F}^*$, then there exist $g \in \mathcal{F}^*$ and $\alpha \in \mathcal{F}$ with$f = g.\alpha$. We prove that $\mathcal{F}$ is Arens regular. Let $m, n \in \mathcal{F}^{**}$. Then $\widehat{\alpha}\Diamond m \in Z_1$ and $\widehat{\alpha}\Diamond m = \widehat{\alpha}\Box m$, and we have

$$\begin{aligned}\langle m\Diamond n, f\rangle &= \langle m\Diamond n, g.\alpha\rangle \\ &= \langle \widehat{\alpha}\Diamond(m\Diamond n), g\rangle \\ &= \langle (\widehat{\alpha}\Diamond m)\Diamond n), g\rangle \\ &= \langle (\widehat{\alpha}\Diamond m)\Box n, g\rangle \\ &= \langle (\widehat{\alpha}\Box m)\Box n, g\rangle \\ &= \langle \widehat{\alpha}\Box(m\Box n), g\rangle \\ &= \langle m\Box n, g.\alpha\rangle \\ &= \langle m\Box n, f\rangle,\end{aligned}$$

for $f \in \mathcal{F}^*$, $m\Diamond n = m\Box n$, then $\mathcal{F}$ is Arens regular and we get $\mathcal{F}^{**}$ is dual. Since $\mathcal{F}$ is amenable and Arens regular and $\mathcal{F}$ is ideal in $\mathcal{F}^{**}$ and $\mathcal{F}^{**}$ is dual, by Theorem 4.4 of [17], $\mathcal{F}^{**}$ is strongly Connes-amenable. □

Theorem 3.2. *Let $\mathcal{F}$ be an Arens regular dual Banach algebra with $\mathcal{F}^*$ is WSC . If $\mathcal{F}^{**}$ is weakly strongly Connes-amenable. Then $\mathcal{F}$ is weakly strongly Connes-amenable*

Proof. Let $D : \mathcal{F} \to \mathcal{F}^*$ be a w^*-continuous derivation. Since $\mathcal{F}$ is Arens regular, $\mathcal{F}^{**}$ is dual and since $\mathcal{F}^*$ is WSC, then every derivation, $D : \mathcal{F} \to \mathcal{F}^*$ is weakly compact. Then $D^{**}(\mathcal{F}^{**}) \subseteq \mathcal{F}^*$ by (Theorem 5.5 of [4]), hence, by Arens regularity of $\mathcal{F}$, $\mathcal{F}^*$ is an $\mathcal{F}^{**}-$submodule of $(\mathcal{F}^{**})^*$ and $D^{**}(\mathcal{F}^{**}).\mathcal{F}^{**} \subseteq \mathcal{F}^*.\mathcal{F}^{**} \subseteq \mathcal{F}^*$. Thus, according to Theorem 7.1 of [7], $D^{**} : \mathcal{F}^{**} \to \mathcal{F}^{***}$ is a w^*-continuous derivation. Since $\mathcal{F}^{**}$ is weakly strongly Connes-amenable, there exists $\alpha''' \in \mathcal{F}^{***}$ with $D^{**}(F) = F.\alpha''' - \alpha'''.F$ for each $F \in \mathcal{F}^{**}$. Now, if $E : \mathcal{F} \to \mathcal{F}^{**}$ is the canonical map, setting $f = E^*(\alpha''')$, then $D(\alpha) = \alpha.f - f.\alpha$ for all $\alpha \in \mathcal{F}$ and so D is inner.Thus $\mathcal{F}$ is weakly strongly Connes-amenable. □

Theorem 3.3. *If $\mathcal{F}$ is a Banach algebra with a left ideal in $\mathcal{F}^{**}$ and $\mathcal{F}^*$ factors on the right. Then $\mathcal{F}$ is Arens regular.*

Proof. Consider $\mathcal{F}^*$ as a Banach $\mathcal{F}$-bimodule. Let $\beta'' \in \mathcal{F}^{**}$. Also, let $(\alpha_\varrho)_\varrho$ be a net in $\mathcal{F}$ with $\alpha''_\varrho \xrightarrow{w^*} \alpha''$ in $\mathcal{F}^{**}$. We show that $\beta''\alpha''_\varrho \xrightarrow{w^*} \beta''\alpha''$. Let $\alpha' \in \mathcal{F}^*$, Then, since $\mathcal{F}^*$ factors on the right, there are $\beta \in \mathcal{F}, \beta' \in \mathcal{F}^*$ with $\alpha' = \beta\beta'$. Since $\mathcal{F}$ is the left ideal in $\mathcal{F}^{**}$, thus we have $\alpha''_\varrho\beta \longrightarrow \alpha''\beta$ in $\mathcal{F}^{**}$ if and only if $\alpha''_\varrho\beta \longrightarrow \alpha''\beta$ in $\mathcal{F}$. Also, since $\mathcal{F} \longrightarrow \mathcal{F}^*$ with $\beta \mapsto \beta\beta'$ is continuous, it follows that $\alpha''_\varrho\beta\beta' \xrightarrow{w} \alpha''\beta\beta'$ in $\mathcal{F}^*$. Thus

$$\begin{aligned}\lim_\varrho\langle \beta''\alpha''_\varrho, \alpha'\rangle &= \lim_\varrho\langle \beta''\alpha''_\varrho, \beta\beta'\rangle \\ &= \lim_\varrho\langle \beta'', \alpha''_\varrho(\beta\beta')\rangle \\ &= \langle \beta'', \alpha''(\beta\beta')\rangle \\ &= \langle \beta''\alpha'', \beta\beta'\rangle \\ &= \langle \beta''\alpha'', \alpha'\rangle.\end{aligned}$$

This shows that $\beta'' \in Z_{\mathcal{F}}(\mathcal{F}^{**})$. Hence $\mathcal{F}$ is Arens regular. □

Theorem 3.4. *If $\mathcal{F}$ is a Banach algebra and is a right ideal in $\mathcal{F}^{**}$ with $\mathcal{F}^*$ factors on the left. Then $\mathcal{F}$ is Arens regular.*

Proof. similar to Theorem 3.3 □

Corollary 3.5. *If $\digamma$ is a two-sided ideal in $\digamma^{**}$ and $\digamma^{*}$ factors. Then $\digamma$ is Arens regular.*

Theorem 3.6. *For a Banach algebra $\digamma$, suppose that $\digamma$ is weakly completely continuous and $\digamma^{*}$ factors. Then $\digamma$ is Arens regular.*

Proof. By Lemma (3) of [8], since $\digamma$ is weakly completely continuous then $\digamma$ is an ideal in $\digamma^{**}$ and if we apply Theorem 3.4, then $\digamma$ is Arens regular. □

Now we can extend the Theorems 3.3 and 3.4 and Corollary 3.5 to even dual of $\digamma$:

Theorem 3.7. *[10] If $\digamma^{(2n)}$ is a two-sided ideal in $\digamma^{(2n+2)}$ and $\digamma^{(2n+1)}$ factors, then $\digamma^{(2n)}$ is Arens regular.$(n \geq 0, n \in \mathbb{N})$.*

Theorem 3.8. *For a Banach algebra $\digamma$, suppose that $\digamma^{**}$ is an ideal in $\digamma^{****}$ and $\digamma^{***}$ factors. Then if $\digamma$ is weakly amenable, then $\digamma^{**}$ is weakly strongly Connes-amenable.*

Proof. If we apply Corollary 3.5, then we conclude that $\digamma^{**}$ is Arens regular. Now let $D : \digamma^{**} \longrightarrow \digamma^{***}$ be a $w^{*} - w^{*}$-continuous derivation. First, we prove that $\digamma^{***}$ is a normal Banach $\digamma^{**}$-bimodule. Let $(\alpha''_{\varrho})_{\varrho}$ is a net in $\digamma^{**}$ and $\alpha''' \in \digamma^{***}$. Then, by Arens regularity of $\digamma^{**}$, for every $\beta'' \in \digamma^{**}$ we have

$$\begin{aligned}
\langle (w^{*} - \lim_{\varrho} \alpha''_{\varrho}).\alpha''', \beta'' \rangle &= \langle \alpha''', \beta''.(w^{*} - \lim_{\varrho} \alpha''_{\varrho}) \rangle \\
&= \lim_{\varrho} \langle \alpha''', \beta''.\alpha''_{\varrho} \rangle \\
&= \lim_{\varrho} \langle \alpha''_{\varrho}.\alpha''', \beta'' \rangle \\
&= \langle w^{*} - \lim_{\varrho} (\alpha''_{\varrho}.\alpha'''), \beta'' \rangle.
\end{aligned}$$

Also

$$\begin{aligned}
\langle \alpha'''.(w^{*} - \lim_{\varrho} \alpha''_{\varrho}), \beta'' \rangle &= \langle \alpha''', w^{*} - \lim_{\varrho} \alpha''_{\varrho}.\beta'' \rangle \\
&= \lim_{\varrho} \langle \alpha''', \alpha''_{\varrho}, \beta'' \rangle \\
&= \lim_{\varrho} \langle \alpha'''.\alpha''_{\varrho}, \beta'' \rangle \\
&= \langle w^{*} - \lim_{\varrho} (\alpha'''.\alpha''_{\varrho}), \beta'' \rangle.
\end{aligned}$$

Hence, the mapping $\alpha'' \mapsto \alpha''.\alpha'''$ and $\alpha'' \mapsto \alpha'''.\alpha''$ are weak*-weak*-contnuous from $\digamma^{**} \to \digamma^{***}$. Consequently, $\digamma^{***}$ is a normal Banach $\digamma^{**}-$bimodule.

For each $\alpha \in \digamma$, we define $\bar{D} : \digamma \longrightarrow \digamma^{*}$ by

$$\bar{D}(\alpha) = D(\widehat{\alpha}) \mid_{\digamma} .$$

Then $\bar{D}$ is a continuous derivation from $\digamma$ into $\digamma^{*}$. Because for every $\alpha, \beta \in \digamma$, we have

$$\bar{D}(\alpha\beta) = D(\widehat{\alpha\beta}) = D(\widehat{\alpha} \Diamond \widehat{\beta}) = \alpha.D(\widehat{\beta}) + D(\widehat{\alpha}).\beta = \alpha.\bar{D}(\beta) + \bar{D}(\alpha).\beta.$$

Since $\digamma$ is weakly amenable, thus $\bar{D}$ is inner.Then there exist $\alpha''' \in \digamma^{***}$ with

$$D(\widehat{\alpha}) = \bar{D}(\alpha) = \alpha.\alpha''' \mid_{\digamma} - \alpha''' \mid_{\digamma} .\alpha = \widehat{\alpha}.\alpha''' \mid_{\digamma} - \alpha''' \mid_{\digamma} .\widehat{\alpha}.$$

We consider canonical mapping $E : \digamma^{*} \longrightarrow \digamma^{***}$. Then there is $b''' \in \digamma^{***}$ with $E(\alpha''' \mid_{\digamma}) = \beta'''$. So

$$D(\widehat{\alpha}) = \widehat{\alpha}.\beta''' - \beta'''.\widehat{\alpha}.$$

There for D is inner. Thus $\digamma^{**}$ is weakly strongly Connes-amenable. □

Corollary 3.9. *Let $\digamma^{(2n+2)}$ be a two sided ideal in $\digamma^{(2n+4)}$ and $\digamma^{(2n+3)}$ factors. If $\digamma^{(2n)}$ be weakly amenable, then $\digamma^{(2n+2)}$ is weakly strongly connes amenable.*

Proof. It is easily followed from Theorems 3.7 and 3.8. □

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