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Suggestion of a Perimeter Formula for Super Ellipses and their Use in Rectangular Boundary Value Problems in Physics

Süper Elipsler için bir Çevre Formülü Önerisi ve Fizikte Dikdörtgensel Sınır Değer Problemlerinde Kullanımı

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Abstract

Super ellipses or Lame curves introduced by Gabriel Lame have been frequently studied subjects in physics and engineering in recent years. While the calculation of the area of super-ellipses is analytically possible, the lack of formulations for circumference calculations is noteworthy. To overcome this deficiency, this study aimed to write a code that computes the circumferences of super ellipses numerically and to find an approximate circumference formulation compatible with the numerical results. Additionally, the perimeter formulation obtained shows that in a rectangular boundary condition, Laplace's equation can be reduced to approximately one dimension with super ellipses and a practical approximate solution to a difficult physical problem can be found.

Keywords: Super ellipse, Perimeter, Circumference, Laplace equation, Rectangular boundary

Öz

Gabriel Lame tarafından tanıtılan süper elipsler veya Lame eğrileri, son yıllarda fizik ve mühendislikte sıkça çalışılan konular olmuştur. Süper elips alanlarının hesaplanması analitik olarak mümkün olsa da, çevre hesaplamaları için formüllerin eksikliği dikkat çekicidir. Bu eksikliği gidermek için bu çalışma, süper elipslerin çevrelerini sayısal olarak hesaplayan bir kod yazmayı ve sayısal sonuçlarla uyumlu yaklaşık bir çevre formülasyonu bulmayı amaçlamıştır. Ek olarak, elde edilen çevre formülasyonu, bir dikdörtgen sınır koşulunda, Laplace denkleminin süper elipslerle yaklaşık bir boyuta indirgenebileceğini ve zor bir fiziksel probleme pratik bir yaklaşık çözüm bulunabileceğini göstermektedir.

Anahtar Kelimeler: Süper elips, Çevre, Laplace denklemi, Dikdörtgensel sınırlar

1. Introduction

Super ellipses are shapes that can transform into ellipses, rectangles and asteroids, first proposed by Gabriel Lame (Gridgeman 1970). In Cartesian coordinates, it is represented by using Equation 1.

$$\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1 \quad (1)$$

where a, b and n are positive real numbers. For different values of n, the shape of the super ellipse changes in angularity, while variations in parameters a and b lead to changes in the aspect ratio and size of the shape (Figure 1). For similar figures, readers can see the paper of Erbaş (Erbaş 2020) and web page of Wikipedia (Superellipse 2021). When the parameters a and b are equal, equilateral shapes such as square ($n \rightarrow \infty$) and circle ($n=2$) appear. Some authors prefer to call $n=4$ of these shapes as "*squircle*" and some as "*super circle*" (Li and Boyd 2015) When $n=4$ and $a \neq b$, it is called "*rectellipse*" (Weisstein 2021a, 2021b).

After Piet Hein's works on design (Hein 2021), super-ellipses have become an interesting topic for many disciplines such as architecture, design, physics, engineering and computer vision (Gardner 2020). Recently, studies have shown that products designed as round or curved polygons are more preferred than angular or pointed polygons (Bar and Neta 2006, 2007, Silvia and Barona 2009, Westerman et al. 2012). Based on this, the popularity of super ellipses has also manifested itself in the field of design and architecture.

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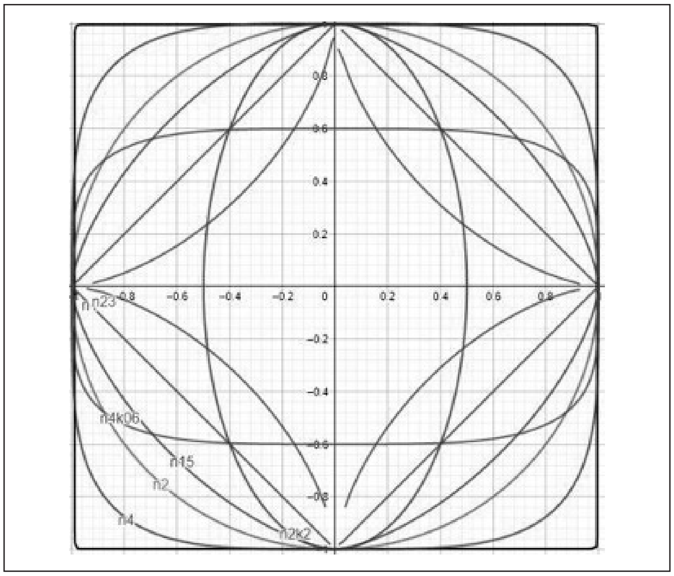

Figure 1. Super ellipses for different parameters. **n100** (approximately square): *n=100, a=b=1*; **n4** (squircle): *n=4, a=b=1*; **n2** (circle): *n=2, a=b=1*; **n15** (curved rhombus): *n=1.5, a=b=1*; **n1** (rhombus): *n=1, a=b=1*; **n23** (asteroid): *n=2/3, a=b=1*; **n4k06** (super ellipse): *n=4, a=1, b=0.6*; **n2k2** (ellipse): *n=2, a=0.5, b=1.*

Not only in design and architecture, but also in physics and engineering, super-ellipses have reached widespread use. Studies in this subject are summarized as diffraction optics (Manenkov 2014, Kleev and Kyurkchan 2015, Manenkov 2019, Kyurkchan et al. 2019), superelliptical boundary conditions in the Helmholtz equation (Panda and Hazra 2014), tire-ground contact area (Hallonborg 1996), contact-force for packing (Delaney and Cleary 2010, Arifuzzaman et al. 2020), radiotherapy (Méndez and Casar 2021), electromagnetic source transformations (Allen et al. 2009), shape fitting (Rosin 2000, Osian et al. 2004), computer vision (Zhang and Rosin 2003), skull prosthesis modeling (Lin et al. 2016) and superplasticity (Anishchenko et al. 2019, Anishchenko et al. 2020).

When studies in physics and engineering are examined in more detail, it is seen that super ellipses are used in calculations in mathematical physics. Nagornov et al. report that equipotential lines of the capture potential of the 2x NADEL ICR cell are formed as super-ellipses in the plane perpendicular to the magnetic field (Nagornov et al. 2021). Huang et al. 2021 reported that the interface between solid and fluid domains in a Stefan problem changed from the initial circular configuration to an up-down asymmetric egg-shaped geometry. Also, square cylinders have become a hot topic in engineering (Sert 2021) and therefore super-ellipses can be used to model the temperature contour lines that will form around the square cylinder.

Some researchers report that solutions of differential or integral equations in near corner regions are difficult compared to smooth bounds. (Bremer et al. 2010, Dhia et al. 2021). Applying the finite element method in a boundary condition with rounded corners is a time-consuming and costly method because it is necessary to create a mesh with many nodes. Even when the rounded corners are sharpened, the calculations are correct, but locally incorrect (Krähenbühl et al. 2011).

To overcome the difficulties in the corner regions, Erbaş showed that he could model the contour lines (equipotential curves) of super ellipses with a good approximation in his studies, in which he examined the approximate solution of Laplace's equation in rectangular boundary conditions (Erbaş 2019, Erbaş 2020). Applying Gauss's law for a Gaussian surface defined on equipotential curves around a square conductor, he could calculate the characteristic impedance of a rectangular transmission line with a reasonable approximation.

Assuming that the contours that will form around the angular boundaries in two dimensions can be modeled with convex super ellipses, it is important to know the circumferences of the convex super ellipses in accordance with the 2D divergence theorem. To the best of our knowledge, there is no large-scale study in the literature that covers the circumferences of all degrees and edge ratios of convex superellipses. To address this important shortcoming, a numerical method that can calculate the circumferences of super-ellipses much more precisely and an analytical formulation that overlaps with the numerical results with a good approximation is proposed in this study. With the proposed contour formulation, one-dimensional approximate solution of Laplace's equation in rectangular boundary conditions is also obtained.

With the results of the study, perimeter calculations of super-ellipses, which are increasingly used, will be facilitated and practical solutions to the boundary value problems used in physics will be offered. In addition, it will be able to offer a solution to the computational problems seen in the regions with angular borders mentioned in the literature.

2. Material and Method

In Cartesian coordinates, arc length of a smooth function $y=f(x)$ between $x=x_1$ and $x=x_2$ is calculated using Equation 2 (Weir et al. 2005).

$$L=\int_{x1}^{x2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{2}$$

In most cases, this integral does not come in terms of familiar conventional functions and is expressed in functions of complex structure, such as hypergeometric functions. Although the difficulty of integrating due to the square root is tried to be overcome with numerical calculations, it costs a lot of effort to write the computer codes for numerical calculations. Moreover, in regions where the derivative approaches infinity, the numerical sum may produce erroneous results.

A familiar solution to overcome this problem is to calculate the perimeter in a different coordinate system. The first coordinate system that comes to mind is polar coordinates, the formulation of which is given in Equation 3. For a closed curve, θ_1 and θ_2 limits can be selected as *0* and *2π*. Equation 4 is a parametric arc length formula for an ellipse (Weir et al. 2005). To compute the quarter perimeter of ellipse, t_1 and t_2 can be chosen as *0* and *π/2* (Abbot 2011).

$$L=\int_{\theta 1}^{\theta 2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \tag{3}$$

$$L=\int_{t1}^{t2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \tag{4}$$

2.1. Numerical Computation of the Perimeter

To find a formulation that predicts the circumference of the super ellipse expressed by Equation 1, the notation and basic concepts are explained in Equation 5 (Superellipse 2011). Perimeter of a super ellipse of degree *n* is given by

$$P_n(a,b)=aP_n(1,k)=4a\left(L_n(k)+S_n(k)\right) \tag{5}$$

where *k=b/a* is the ratio of the minor axis to the major axis and the subscript *n* denotes the degree of the super ellipse. $L_n(k)$ and $S_n(k)$ show the arc lengths above and below the diagonal respectively. The numerical calculation of the L_n value is easier than for the S_n value because while the *dy/dx* derivative takes values close to zero in L_n, it approaches infinity for S_n (Figure 2A). Large derivatives can cause some numerical difficulties in the perimeter integral. Therefore, the length S_n can be calculated for a super-ellipse where the arc is positioned more horizontally (Figure 2B). The integral expressions for L_n and S_n are as follows:

$$y=k\left(1-x^2\right)^{\frac{1}{n}} \tag{6}$$

$$\frac{dy}{dx}=-kx^{n-1}\left(1-x^n\right)^{\frac{1}{n}-1}=\frac{-kx^{n-1}}{\left(1-x^n\right)^{\frac{n-1}{n}}}=-k\left(\frac{x^n}{1-x^n}\right)^{\frac{n-1}{n}} \tag{7}$$

Using Equation 2 leads

$$L_n(k)=\int_{x=0}^{x=2^{-1/n}}\sqrt{1+k^2\left(\frac{x^n}{1-x^n}\right)^{\frac{2n-2}{n}}}\,dx \tag{8}$$

$S_n(k)$ can be calculated from both Figure 2A and B as in Equation 9.

$$\begin{aligned}S_n(k)&=\int_{x=2^{-1/n}}^{x=1}\sqrt{1+k^2\left(\frac{x^n}{1-x^n}\right)^{\frac{2n-2}{n}}}\,dx\\&=\int_{x=0}^{x=k.2^{-1/n}}\sqrt{1+\frac{1}{k^2}\left(\frac{(x/k)^n}{1-(x/k)^n}\right)^{\frac{2n-2}{n}}}\,dx\end{aligned} \tag{9}$$

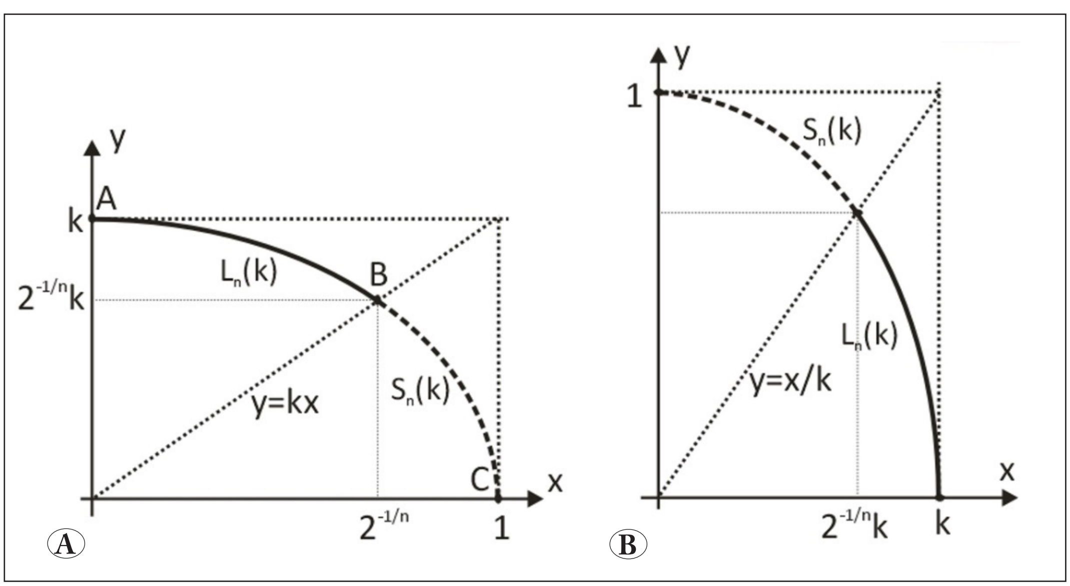

Figure 2. A) The first quarter of a super ellipse with *a=1* and *b=k*. Arc lengths of the arcs AB and BC are denoted by $L_n(k)$ and $S_n(k)$ respectively.
B) Rotated and mirrored version with *a=k* and *b=1*.

The curve in Figure 2B is obtained by writing 1/k instead of k in the curve in Figure 2A and enlarging it k times. Applying the *t=x/k* transformation to the second integral of Equation 9, Equation 10 and 11 are obtained.

$$S_n(k) = k\int_{x=0}^{t=2^{-1/n}} \sqrt{1+\frac{1}{k^2}\left(\frac{t^n}{1-t^n}\right)^{\frac{2n-2}{n}}}\,dt = k\ L_n\left(\frac{1}{k}\right) \quad (10)$$

$$P_n(1,k) = 4\left(L_n(k) + k\ L_n\left(\frac{1}{k}\right)\right) \quad (11)$$

In Equation 9, the first and second integrals come from Figure 2A and B respectively. Equation 10 gives the relationship between L_n and S_n. Finally, Equation 11 expresses the perimeter of a super ellipse with unit semi-major axis in terms ok $L_n(k)$ defined in Equation 8. $P_n(1,k)$ in Equation 11 (Superellipse 2011) is the key factor for the calculation of all super ellipses with different ratios and degrees by the help of Equation 5.

Using Matlab's numerical integral tool (R2021b), $L_n(k)$ values in Equations 8 were computed with 0.10 step size in the range $0<k<1$ and $1<n<20$. Perimeters $P_n(1,k)$ in Equation 11 were finally computed and recorded in Table 1. The Matlab code for this numerical computation is given in the Appendix.

2.2. Suggestion of a General Perimeter Formula

The two special cases of super ellipses, $n=1$ and $n=\infty$, correspond to rhombus and rectangular shapes, respectively (*n1* and *n100* in Figure 1). Their perimeters with semi-axis *1* and *k* are given by Equations 12 and 13 (Mathisfun 2020).

$$P_1(k) = 4\sqrt{1+k^2} = 4(1+k^2)^{\frac{1}{2}} \quad (12)$$

$$P_\infty(k) = 4(1+k) = 4(1+k^1)^{\frac{1}{1}} \quad (13)$$

One can assert that the power in Equation 9 and 10 change with the degree of super ellipse (*n*). Therefore, a general perimeter formula for a super ellipse of degree *n* and axis ratio *k* can be assigned as

$$P_n(k) = 4\left(1+k^{s(n)}\right)^{\frac{1}{s(n)}} \quad (14)$$

where *s(n)* represents the power in terms of the degree with specific values $s(1)=2$ and $s(\infty)=1$. However, this is not the only suggestion without any alternatives. In equation 15, by writing $\alpha(1)=0$ and $\alpha(\infty)=2$, equations 12 and 13 can be obtained, respectively. Moreover, by combining Equations 14 and 15, Equation 16 can also be assigned as an alternative formula for super ellipse perimeter formulation.

$$P_n(1,k) = 4\left(1+\alpha(n).k+k^2\right)^{\frac{1}{2}} \quad (15)$$

$$P_n(1,k) = 4\left(1+\alpha k^{s(n)}+k^{2s(n)}\right)^{\frac{1}{2s(n)}} \quad (16)$$

As a third alternative, Ramanujan's (Ramanujan and Hardy 1962) ellipse approximation in Equation 17 can be used as a basis (Villarino 2005). Equation 18 is Ramanujan's approximation expressed with four parameters (*A, B, C and D*). The values $A_2=\pi$, $B_2=3\pi$, $C_2=10$ and $D_2=7$ are valid for the ellipse.

$$P_{n=2}(1,k) = \pi\left[1+k+\frac{3(1-k)^2}{10(1+k)+\sqrt{1+14k+k^2}}\right] \quad (17)$$

$$P_n(1,k) = A_n(1+k)+\frac{B_n(1-k)^2}{C_n(1+k)+\sqrt{1+2D_nk+k^2}} \quad (18)$$

It is clear that $2A_n$ will give the circumference of the k=1 *super circle*, since in the case of k=1 there will be $2A_n$ remaining from Equation 18. Of the other special super ellipses, the relevant parameters for n=1 should be $A_1=2\sqrt{2}$, $B_1=2$, $C_1=\sqrt{2}/2$ and $D_1=0$ because under these conditions

Table 1. Perimeters of super ellipses calculated by numerical integration 2 in the Appendix.

n\k	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
1.0	4.020	4.079	4.176	4.308	4.472	4.665	4.883	5.122	5.381	5.657
1.5	4.033	4.122	4.257	4.429	4.631	4.857	5.105	5.369	5.648	5.939
2.0	4.064	4.202	4.386	4.603	4.844	5.105	5.382	5.672	5.973	6.283
2.5	4.099	4.277	4.497	4.746	5.015	5.300	5.598	5.906	6.223	6.547
3.0	4.130	4.339	4.586	4.857	5.146	5.448	5.761	6.082	6.410	6.745
4.0	4.179	4.430	4.712	5.013	5.328	5.653	5.985	6.324	6.669	7.018
6.0	4.240	4.538	4.857	5.190	5.531	5.880	6.234	6.592	6.953	7.318
9.0	4.288	4.618	4.963	5.318	5.678	6.043	6.412	6.783	7.157	7.533
20.0	4.347	4.715	5.090	5.469	5.851	6.235	6.620	7.007	7.395	7.784
inf	4.400	4.800	5.200	5.600	6.000	6.400	6.800	7.200	7.600	8.000

the Rhombus circumference in equation 12 is obtained. In the case of n=∞, B_∞=0 is sufficient to obtain equation 13 with a rectangular perimeter.

Since at *k=0* all $P_n(1,k)$ values will approach 4, parameter B_n can be expressed in terms of others. Equation 19 guarantees to give the perimeter value correctly for the extreme values of k=0 and k=1.

$$P_n(1,k) = \frac{P_n(1,1)}{2}(1+k) + \frac{(4-P_n(1,1)/(1+C_n)(1-k)^2}{C_n(1+k)+\sqrt{1+2D_nk+k^2}} \tag{19}$$

In short, there are 4 different perimeter formulation alternatives so far (Equations 14-16, 19). As a result of many curve fitting methods tried with these functions, it was seen that Equation 19 gave the best values. As a result, it was aimed to find the parameters in Equation 19 for each *n* value.

2.3. Computation of the Parameters for the Perimeter Formula

In Equation 19, the two unknown parameters to be calculated by curve fitting are parameters C and D. The desired parameters were calculated for certain n values with the Matlab Curve fit tool (Matlab 2021). When choosing n values, those giving approximately equally distributed perimeter values for the super circle were chosen. For the k ratio of the super ellipse, steps with 0.05 were preferred. Selected *n* and *k* values for Curve fit are

n = {1.0, 1.3, 1.5, 1.7, 1.8, 2.0, 2.2, 2.5, 3.0, 4.0, 6.0, 9.0, 20.0};
k={0.0, 0.05, 0.10, 0.15, ….. 1}. (20)

First, the parameters for the n=2 ellipse were found with curve fit, unlike the values found by Ramanujan (Villarino 2005), as in Table 2.

For all n values in Equation 20, parameters C_n and D_n were obtained with errors less than 10^{-4}. The C_n and D_n parameter sets were fitted as a function of *n* and directed to surface fit tool to find the most general $P_n(1, k)$ function. After several trials and intervening curve fits, an appropriate $P_n(1, k)$ function was found with surface fit with RMSE: 0.001598. Surface fit results are presented in Equation 21.

$$P_n(1,k) = \left(1.176\left(1-\frac{1}{n}\right)^{1.89} + 2.825\right)(1+k) + \frac{(-0.03*n^2+25*n-16.6)(1-k)^2}{(10*n^2+0.2*n-7.6)(1+k)+(n^2-1.5*n+5)\sqrt{1+240*\left(1-\frac{1}{n}\right)^{4.74}*k+k^2}} \tag{21}$$

3. An application of Physics: Potential Around a Square Conductor

3.1. Numerical Solution

Can contour lines be modeled with super ellipses and reduced to 1 dimension under ideal conditions? To find the answer to this question, the potential distribution around a conductive square whose potential is held constant was studied as a boundary value problem, whose contour lines resemble super ellipses. The problem in question was defined between a 50 unit radius circle, held at a concentric potential of 0 volts, and a unit radius square with a potential of 1 volt, as shown in Figure 3. The boundaries of the square are the 1-volt equipotential curve. As you move away from the square, the equipotential curves will become more and more rounded because at infinity the square becomes pointlike and the equipotential curve approaches the circle. This means that the first contour line at the square is n=∞, and the one at infinity is n=2 super circles. Contours in between form a family of super ellipses going from n=∞ to n=2.

This problem can be expressed mathematically as in Equation 22-24 and Figure 3A, B. It was solved by MATLAB PDE tool and the potentials *V(x,0)* at some points were recorded as in Table 5.

$$\nabla^2 V(x,y) = 0 \tag{22}$$

$$V(1,y) = V(-1,y) = 1 at -1 \leq y \leq \; and \tag{23}$$

$$\begin{aligned} &V(x,1) = V(x,-1) = 1\, at -1 \leq x \leq 1 \\ &V(x,y) = 0\; at\; x^2+y^2 = 50^2 \end{aligned} \tag{24}$$

3.2. Approximated Solution

Can this boundary value problem be made 1-dimensional with respect to *μ* by transforming $x^n+y^n=\mu^n$ in symmetric

Table 2. Curve fit results for n=2.

General model	$P_2(1,k)$ = pi*(1+k)+(4-pi)*(C+1)*(1-k)^2/(C*(1+k)+sqrt(1+2*D*k+k^2))		
Parameters	C =12.33	D=8.997	
Goodness of fit	SSE: 1.918e-008	R-square: 1	RMSE: 3.177e-005

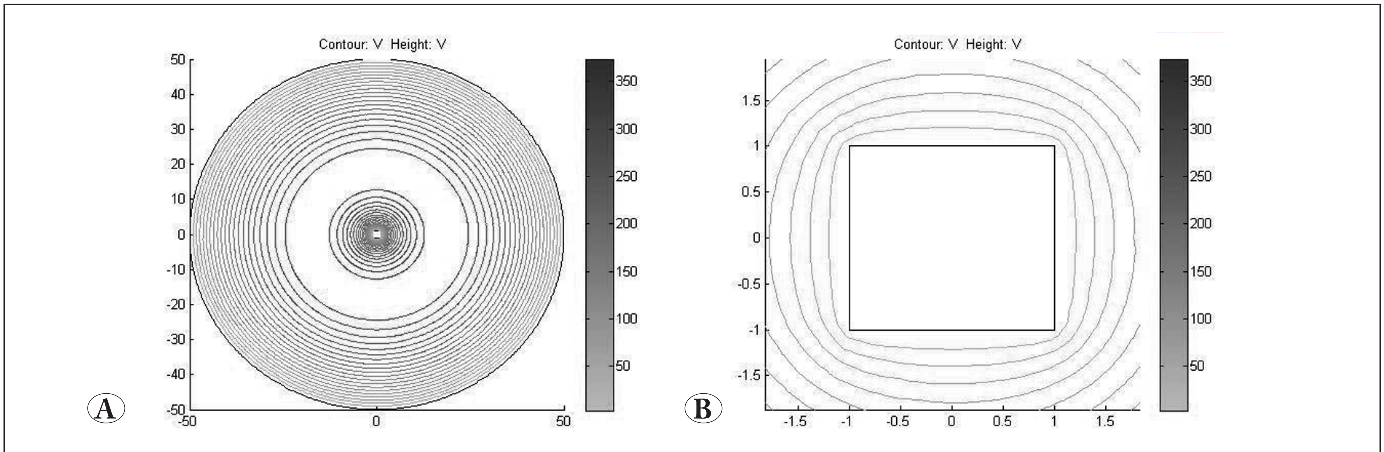

Figure 3. Equipotential lines of Equations 22-24. **A)** Full scale, **B)** zoomed scale. Inner square and outer circle are held at V=1 and V=0 respectively.

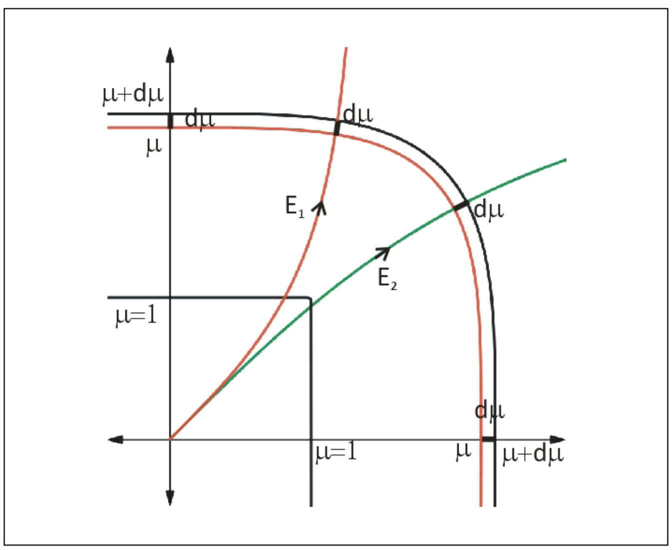

Figure 4. Equipotential lines between two neighboring contours. E_1 and E_2 represent any of two electric field lines (gradient).

conditions, similar to how Laplace's equation is reduced to 1 dimension with respect to r with $x^2+y^2=r^2$ transform in circularly symmetric conditions? Proving this is the subject of a separate study. In this study, it will be assumed that it can be reduced to 1 dimension and the results will be compared. Under this assumption, the superellipse degree "n" should be taken as a function of μ.

3.2.1. Finding n(μ)

In a one-dimensional problem, the electric field must be of constant magnitude on each equipotential curve. As you move away from the squared boundary, μ increases and n decreases. A suitable choice of $n(\mu)$ in Equation 25 (Erbaş 2019, Erbaş 2020) can explain increasingly rounded families of super circles. To find the appropriate $n(\mu)$ function, on the contour of a given μ value, it must depend only on μ in the electric field (Equation 26 and Figure 4).

$$x^{n(\mu)}+y^{n(\mu)}=\mu^{n(\mu)} \tag{25}$$

$$\vec{\mathbf{E}}=-\vec{\nabla}V \Rightarrow E(\mu)=-\frac{dV(\mu)}{d\mu} \tag{26}$$

If the electric field (E) and potential difference (dV) are homogeneous on the contour, then according to equation 26, the perpendicular coordinate length $d\mu$ is also homogeneous on the contour. The area dA of the region of thickness $d\mu$ is expressed as in Equation 27, depending on the contour perimeter P.

$$P(\mu)d\mu=dA(\mu)\Rightarrow P(\mu)=\frac{dA(\mu)}{d\mu} \tag{27}$$

The perimeters $P(\mu)$ and areas $A(\mu)$ of super circles are directly proportional to their radii and their squares, respectively, with coefficients $p(n)$ and $a(n)$ (Erbaş 2020). Therefore, perimeters and areas can be expressed as in Equation 28.

$$P(\mu)=\mu\cdot p(n)\ and\ A(\mu)=\mu^2 a(n) \tag{28}$$

If equation 28 is substituted in equation 27, a differential equation depending on μ and n is obtained as in Equation 29.

$$\begin{aligned}\mu p(n)&=2\mu a(n)+\mu^2\frac{da(n)}{d\mu}\\ &\Rightarrow p(n)=2a(n)+\mu\frac{dn}{d\mu}\frac{da(n)}{dn}\end{aligned} \tag{29}$$

The solution of Equation 29 is Equation30 with a_n denotes derivative of a with respect to n.

$$\frac{\mu_n(n)}{\mu(n)} = \frac{a_n(n)}{p(n)-2a(n)} \to \mu(n)-1 = \exp\left[\int_{\infty}^{n} \frac{a_n(n)\cdot dn}{p(n)-2a(n)}\right] \tag{30}$$

Since the boundaries have equal axis, the perimeter coefficient *p(n)* in Equation 31 can be obtained by writing *k=1* in Equation 21 and area coefficients *a(n)* (Weisstein 2021a) is given by Equation 32.

$$p(n) = 2.352(1-\frac{1}{n})^{1.89} + 5.65 \tag{31}$$

$$a(n) = 4.^{(1-\frac{1}{n})}\sqrt{\pi}\frac{\Gamma\left(1+\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)} \tag{32}$$

From the solution of the equations, a function that gives *n(μ)* is obtained. When these functions are fitted to the appropriate functions, the following results are obtained with RMSE=1.333*10^{-9} which has very high accuracy. Equation 33 gives the degrees of the rounding super circles shown in Figure 4.

$$n(\mu) = 2 + \frac{3.19}{\mu - 1} \tag{33}$$

3.2.2. Finding the Potential V(μ)

The problem given by Equation 22-24 is a Laplace equation in quadratic symmetric boundary conditions. Assuming the contour lines will be modeled with Equation 33, Laplace's equation can be expressed in 1 dimension according to μ with the help of Gauss's Law. The equipotential surface corresponding to a given value μ defines a super circular closed cylinder of circumference *P(μ)* of length *L*. Since the electric field will be constant on this cylinder, the closed surface integral can be written as in Equation 34 and potential *V* can be solved as in Equation 35. Parameters *q* and ε_0 denote charge and permittivity of vacuum respectively.

$$\oiint_S \vec{E}\cdot d\vec{A} = E\cdot A = \frac{q}{\varepsilon_0} \to -\frac{dV}{d\mu}\cdot L\cdot P(\mu) = \frac{q}{\varepsilon_0} \tag{34}$$

$$V(\mu) - V(1) = -\frac{q}{L_{\varepsilon_0}}\int_1^{\mu}\frac{d\mu}{P(\mu)} \tag{35}$$

To obtain the *1/P(μ)* function in the integral in Equation 35, Equations 31 and 33 are combined to find *P(μ)* as in Equation 36. The inverse of this function, *1/P(μ)*, fits very well with the function shown in Equation 37.

$$P(\mu) = \mu p(n) = 2.352\mu\left(1-\frac{1}{n}\right)^{1.89} + 5.65\mu \tag{36}$$

$$\frac{1}{P(\mu)} = \frac{0.1608}{\mu + 0.2595} \tag{37}$$

When the function in Equation 37 is substituted into the integral in 35, it gives a function in Equation 38 form with two unknown parameters (A and C). Here, A and C depend on the boundary conditions.

$$V(\mu) = A\, In(\mu + 0.2595) + C \tag{38}$$

Since the boundary values are *V(1)=1* and *V(50)=0*, the function *V(μ)* that provides these values is represented by Equation 39.

$$V(\mu) = In\left[\frac{\mu+0.2595}{50+0.295}\right] / In\left[\frac{1+0.2595}{50+0.295}\right] \tag{39}$$

Finally, equation 40 is the intended approximate solution to the boundary value problem. Its derivative function is the expression of the electric field and is given in Equation 41.

$$V(\mu) = -0.2712\cdot In\left[\frac{\mu+0.2595}{50.295}\right] \tag{40}$$

$$E(\mu) = \frac{0.2712}{\mu + 0.2595} \tag{41}$$

The formulations shown by Equation 40 and 41 can explain the potential or electric field on the x-axis, y-axis and y=+-x lines with good approximation. To find the value at any (*x,y*) point, it is necessary to find the μ value corresponding to each (*x,y*) point based on Equation 25. Using the modeling of contour lines with super ellipses, Erbaş calculated the capacitance of square-shaped capacitors or the characteristic impedance of transmission lines. (Erbaş 2020).

4. Results and Discussion

Matlab numerical integral tool was used in two different ways to calculate the circumferences of convex superellipses numerically. In the first, the integral is taken directly from zero to one, while in the second, it is taken from two different parts (as in Equation 11). The consistency of the two different numerical methods tried was tested with the elliptic integral function giving the circumference of the ellipse (*n=2*). The codes of these integral methods are given in Appendix as "*Numerical Integration 1 (NI 1)*" and "*Numerical Integration 2 (NI 2)*", respectively. The results for some extreme values of the ellipse are listed in Table 3 with their relative errors.

As can be seen from Table 3, NI2 gives better results for the ellipse circumference. Therefore, when calculating the circumferences of superellipses with different degrees, NI 2 was taken as reference. The approximate perimeter formulation proposed in this study (Equation 21) and Aldaher's approximate perimeter formulation (Aldaher

Table 3. Peripheral arc lengths of some ellipses for different values of k by numerical integration 1 and 2 with their relative errors with respect to elliptic integral of the second kind.

k	Elliptic Int	NI 1	NI 2	Error 1	Error 2
0.05	4.019426	4.019426	4.019426	1.33E-12	0
0.5	4.844224	4.844224	4.844224	3.08E-14	0
1	6.283185	6.283185	6.283185	4.93E-14	0

Table 4. Peripheral arc lengths of some superellipses for different values of n and k by numerical integrations, web calculators and approximated formulas with relative errors with respect to NI 2.

n	k	NI 1	NI 2	Procato	Had2know	Aldaher	Eq.21
1.1	0.05	4.00515	4.00515	4.01000	4.00600	4.00908	4.00920
1.1	0.5	4.48252	4.48252	4.48000	4.48300	4.50899	4.47944
1.1	1	5.67712	5.67712	5.68000	5.67800	5.70411	5.67530
4	0.05	4.07324	4.07326	4.07000	4.07300	4.09252	4.07471
4	0.5	5.32784	5.32803	5.33000	5.32700	5.39194	5.32755
4	1	7.01730	7.01770	7.02000	7.01600	7.04689	7.01554
80	0.05	4.05878	4.19159	4.19000	1.23200	4.21641	4.19015
80	0.5	4.63377	5.96195	5.96000	1.77300	5.94028	5.96280
80	1	5.28863	7.94499	7.94000	2.30600	7.85535	7.94674
		Rel.Err	Rel.Err	Rel.Err	Rel.Err	Rel.Err	Rel.Err
1.1	0.05	2.15E-09	0	1.21E-03	2.13E-04	9.83E-04	1.01E-03
1.1	0.5	2.81E-08	0	-5.62E-04	1.07E-04	5.90E-03	-6.87E-04
1.1	1	1.34E-08	0	5.08E-04	1.55E-04	4.75E-03	-3.19E-04
4	0.05	-4.85E-06	0	-8.01E-04	-6.42E-05	4.73E-03	3.54E-04
4	0.5	-3.71E-05	0	3.69E-04	-1.94E-04	1.20E-02	-9.05E-05
4	1	-5.64E-05	0	3.28E-04	-2.42E-04	4.16E-03	-3.08E-04
80	0.05	-3.17E-02	0	-3.80E-04	-7.06E-01	5.92E-03	-3.44E-04
80	0.5	-2.23E-01	0	-3.27E-04	-7.03E-01	-3.64E-03	1.43E-04
80	1	-3.34E-01	0	-6.27E-04	-7.10E-01	-1.13E-02	2.21E-04

2012) were calculated and compared with numerical studies. Also, two calculator sites related to super ellipse circumference calculations have been found on the internet and the results on these sites are *Procato* (Natural Superellipse 2022) and *had2know* (Lamé Curve Calculator 2022) are included in the comparison in Table 4.

It can be seen from Table 4 that the calculations deviate more from each other as the degree of superellipse gets larger. If the degree is 80, the superellipse will look more like a rectangle, so its circumference will be closer to the value of Equation 13. Similarly, the n=1.1 super ellipse should come out close to the value from equation 12. According to the results in the table, it can be said that the *NI 1* and *had2know* calculator results are largely bankrupt. At small degrees, the *procato* calculator gives a larger error than the others. According to these results, it can be said that the best numerical method is *NI 2* and the best approximate formula is Equation 21. Equation 21, which is the product of this study, proves that the degree of super ellipse n and the edge ratio k is a large-scale approximation with a relative error between *1e-3* and *1e-4* in almost every scale. In more detailed error measurements, it was seen that Equation 21 gave an absolute mean relative error of *2.43e-4* compared to *NI2*.

The potential values computed with the numerical solutions of Laplace's equation with MATLAB PDE tool and the potential values calculated from Equation 40 approximately are shown in Table 5. Additionally, this table also shows the percentage errors of the approximate method compared to the numerical method.

Table 5. Numerical solution (V) and approximate analytical solution (V_a) with percentage errors

μ	V	V_a	%er	μ	V	V_a	%er
1.00	1.0000	1.0000	0.00	8.47	0.4744	0.4748	0.09
1.33	0.9478	0.9364	-1.21	10.11	0.4270	0.4281	0.27
1.65	0.9004	0.8866	-1.54	12.07	0.3797	0.3812	0.39
2.02	0.8531	0.8397	-1.57	14.41	0.3324	0.3340	0.49
2.43	0.8058	0.7943	-1.43	17.21	0.2850	0.2867	0.58
2.91	0.7584	0.7495	-1.17	20.54	0.2377	0.2393	0.68
3.48	0.7111	0.7051	-0.84	24.53	0.1903	0.1917	0.75
4.17	0.6637	0.6592	-0.68	29.28	0.1430	0.1442	0.82
4.97	0.6164	0.6137	-0.44	34.95	0.0957	0.0965	0.88
5.94	0.5691	0.5675	-0.27	41.70	0.0483	0.0490	1.37
7.10	0.5217	0.5213	-0.07	50.00	0.0000	0.0000	0.00

5. Conclusion

In this study, a general formula that describes the circumferences of super-ellipses is proposed and an approximate solution method of the Laplace equation to obtain electric potential in square boundary conditions is proposed with this formulation. The mean absolute error of the perimeters calculated by this approximated perimeter formula is 0.0243%. The average of the absolute errors in computing potential is 0.74%.

With the information obtained in this study, the circumferences of super-ellipses can be calculated more precisely numerically or can be obtained with a good approximation using a simple formulation. Also, in an angular boundary condition, boundary value problems can be simplified. The methods proposed in the study are open to development and can be generalized to methods that can yield appropriate results even under rounded-corner boundary conditions. Considering that super-elliptical forms will be used more widely in the future due to the negative aspects of angular devices or materials, further development of the work should be expected in the future.

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APPENDIX

```
% Numerical Integration 1 for MATLAB R2021b

n=4; k=0.5;

xmax=1; fun5 = @(x) sqrt(1 + (((1-x.^n).^(1./n-1)).*k.*x.^(n-1)).^2 );

P_1 = 4*integral(fun5,0,xmax);


% Numerical Integration 2

xmax=2.^(-1/n);

fun5 = @(x) sqrt(1 + (((1-x.^n).^(1./n-1)).*k.*x.^(n-1)).^2 ); % Integrand of L_n

fun6 = @(x) k*sqrt(1 + (((1-x.^n).^(1./n-1)).*(1/k).*x.^(n-1)).^2 );  % Integrand of S_n

L_n = integral(fun5,0,xmax); S_n = integral(fun6,0,xmax);

P_2= 4*(L_n + S_n);
```