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Solution of Non-linear Fractional Burger's Type Equations Using The Laplace Transform Decomposition Method

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Abstract

Our goal in this paper is to use combined Laplace transform (CLT) and Adomian decomposition method (ADM) (that will be explained in section 3), to study approximate solutions for non-linear time-fractional Burger's equation, fractional Burger's Kdv equation and the fractional modified Burger's equation for the Caputo and Conformable derivatives. Comparison between the two solutions and the exact solution is made. Here we report that the Laplace transform decomposition method (LTDM) proved to be efficient and be used to obtain new analytical solutions of nonlinear fractional differential equations (FDEs).

Keywords: Conformable fractional derivative Caputo fractional derivative conformable differential equations Burger's equation modified Burger's equation Burger's Kdv equation Laplace transform Adomian decomposition method.

1. Introduction

Fractional partial differential equations (FPDEs) proved to be important in applications in physics, engineering, chemistry, electromagnetic, acoustics, electrochemistry, and material science,..., etc [5]. In general, there exists no method that yields an exact solution for fractional partial differential equations. Only approximate and numerical solutions can be derived.
Burger's equation :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

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is a fundamental nonlinear partial differential equation in fluid mechanics. It is also a very important model encountered in several areas of applied mathematics, such as heat conduction, acoustic waves, gas dynamics, and traffic flow. It was actually first introduced by Bateman (1915), [6], when he mentioned it as worthy of study and gave its steady solutions. It was later proposed by Burger (1948), [7], as one of a class of equations describing mathematical models of turbulence. In (1972), Benton and Platzman [8] surveyed the exact solution of one-dimensional Burger's equation. Investigating and reaching exact or numerical solutions to these types of equations has great importance in applied mathematics. Many studies were conducted by scientists in order to determine the numerical or analytical solution to Eq.(1). For example, T. Ozis and A. Ozdes [10] used a direct variational method to solve Burger's equation. E. Aksan and A. Ozdes [11] used a variational method constructed on the method of discretization in time to solve Burger's equation. S. Kutluay et al. [12] obtained a numerical solution of Burger's equation by using finite difference methods. E. Varoglu and L. Finn [13] made use of a weighted residue method. J. Caldwell and P. Wanless [14] used finite elements. D.J. Evans and A.R. Abdullah [15] used the group explicit method. R.C. Mittal and P. Singhal [16] used the Galerkin method to determine numerical solutions to Burger's equation. Therefore, it can be concluded that scientists have devoted much attention to obtaining the numerical and/or analytical solution for the fractional Burger's equation.
For instance, in (2005), Gorguis [9] gave a comparison between the Cole-Hopf transformation and the decomposition method for solving Burger's equation.

In this paper, we use the Laplace transform decomposition method (LTDM), which is a known method for finding approximate solutions of nonlinear equations. A comparative analysis of the temperature distributions obtained in both cases will be established. The following equations are of great interest in applied sciences :

Non-homogenous fractional Burger's equation :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} - 2u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial u^2(x,t)}{\partial x} = f(x,t), \tag{2}$$

time-fractional Burger's Kdv equation :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^3 u(x,t)}{\partial x^3} + 6u(x,t)\frac{\partial u(x,t)}{\partial x} = 0, \tag{3}$$

and fractional modified Burger's equation :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u^2(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \tag{4}$$

where $0 < \alpha \leq 1$, $u(x,t)$ represents the velocity for spatial dimension x and time t.
As a consequence of this importance, scientists paid great attention to obtain the exact or numerical solutions of Burger's type equations.

2. Preliminaries

Here we present basic known material needed in this paper.

Definition 2.1. *[2] Given a function $f : [0,\infty) \to \mathbb{R}$, $t > 0$ and $\alpha \in (0,1)$, the conformable derivative of f with respect to t of order α is defined by :*

$$D_t^\alpha(f)(t) = \lim_{\epsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\epsilon}, \tag{5}$$

If f is α-differentiable in some $(0,a)$, $a>0$, and $\lim\limits_{t\to 0^+} D_t^\alpha(f)(t)$ exists, then $D_t^\alpha(f)(0)=\lim\limits_{t\to 0^+} D_t^\alpha(f)(t)$.

Definition 2.2. *[18] Let $0<\alpha\leq 1$ and $f:[0,\infty)\to\mathbb{R}$ be a real valued function. Then the conformable Laplace transform of order α is defined by :*

$$L_\alpha\{f(t)\}(s)=\int_0^\infty \exp(-s\frac{t^\alpha}{\alpha})f(t)d^\alpha t=F_\alpha(s). \tag{6}$$

Definition 2.3. *[18] The function $f(t)$, $t\geq 0$ is said to be of conformable exponential order m if there exists $K>0$ and $T>0$ such that :*

$$|f(t)|\leq Ke^{m\frac{t^\alpha}{\alpha}}, \qquad \textit{for all } t\geq T.$$

Theorem 2.4. *[1] Let $f:[0,\infty)\to\mathbb{R}$ be a continuous function, $f^{(\alpha)}$ is piecewise continuous real valued function and $0<\alpha\leq 1$. If f is of conformable exponential order m, then*

$$L_\alpha\{D_t^\alpha f(t)\}(s)=sF_\alpha(s)-f(0), \quad s>m. \tag{7}$$

For more on conformable Laplace transform (CLT), we refer to [1, 18, 21].

Definition 2.5. *[3] The Caputo fractional derivative of order α, $0<\alpha<1$, is defined by :*

$$D_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}f'(\tau)d\tau, \tag{8}$$

where $\Gamma(.)$ is the gamma function with the following integral representation :

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt.$$

Definition 2.6. *[3] The Laplace transform for the Caputo's fractional derivative is given by the following form :*

$$L_\alpha\{D_t^\alpha f(t)\}=s^\alpha F_\alpha(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}f^k(0),\ n-1<\alpha<n \tag{9}$$

Definition 2.7. *[3] The Mittag-Leffler function that arises in fractional calculus is defined for complex t and $\alpha>0$ as :*

$$E_\alpha(t)=\sum_{k=0}^\infty\frac{t^k}{\Gamma(\alpha k+1)}. \tag{10}$$

3. Laplace Transform Decomposition Method (LTDM)

To show the basic idea of LTDM, we consider the following fractional PDE's in its general operator form :

$$D_t^{\alpha} u(x,t) + D_x^{\alpha} u(x,t) + R(u(x,t)) + N(u(x,t)) = f(x,t), \tag{11}$$

with the initial condition

$$u(x,0) = g(x), \qquad x > 0,\ t > 0,\ 0 < \alpha \leq 1, \tag{12}$$

where, D_t^{α} is the linear derivative operator in conformable sense of order α in t, D_x^{α} is the highest order linear differential operator in x, R is a linear term with lower derivative, N is a nonlinear term and $f(x,t)$ is the nonhomogeneous part.

In order to solve equation (11), we follow the following steps :

Step 1 : Applying the conformable Laplace transform on both sides of equation (11) with respect to t, it becomes :

$$L_{\alpha}\left\{D_t^{\alpha} u(x,t)\right\} + L_{\alpha}\left\{D_x^{\alpha} u(x,t)\right\} + L_{\alpha}\left\{R(u(x,t)) + N(u(x,t))\right\} = L_{\alpha}\left\{f(x,t)\right\}. \tag{13}$$

Using Theorem (2.1) and equation (12), in equation (13), we obtain :

$$sL_{\alpha}\left\{u(x,t)\right\} - u(x,0) + L_{\alpha}\left\{D_x^{\alpha} u(x,t)\right\} + L_{\alpha}\left\{R(u(x,t)) + N(u(x,t))\right\} = L_{\alpha}\left\{f(x,t)\right\}. \tag{14}$$

Step 2 : Devide by s, and apply the conformable inverse Laplace transform to equation (14). This gives :

$$u(x,t) = L_{\alpha}^{-1}\left\{\frac{1}{s}[u(x,0) + L_{\alpha}\left\{f(x,t)\right\}]\right\} - L_{\alpha}^{-1}\left\{\frac{1}{s}L_{\alpha}\left\{D_x^{\alpha} u\right\}\right\} - L_{\alpha}^{-1}\left\{\frac{1}{s}L_{\alpha}[R(u) + N(u)]\right\}. \tag{15}$$

Step 3 : Considering the conformable Laplace transform decomposition method, let the solution of equation (11) be an infinite series :

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{16}$$

and the nonlinear term can be decomposed as follows :

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n, \tag{17}$$

where A_n is called Adomian polynomials of $u_0, u_1, u_2, ..., u_n$, and it can be calculated by the following formula :

$$A_n = \frac{1}{n!}\left\{\frac{\partial^n}{\partial\lambda^n}(N(\sum_{i=0}^{n}\lambda^i u_i))\right\}_{\lambda=0}, \quad where \;\; n = 0,1,2,3,4,... \tag{18}$$

Substituting equations (16) and (17) in equation (15), we obtain :

$$\sum_{n=0}^{\infty} u_n(x,t) = L_\alpha^{-1}\left\{\frac{1}{s}[u(x,0) + L_\alpha\{f(x,t)\}]\right\} - L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{D_x^\alpha(\sum_{n=0}^{\infty}u_n(x,t))\right\}\right\}$$
$$-L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha[R(\sum_{n=0}^{\infty}u_n(x,t)) + \sum_{n=0}^{\infty}A_n]\right\}. \tag{19}$$

Step 4 : Now, by comparing both sides of equation (19), we get the following iterative algorithm :

$$u_0(x,t) = L_\alpha^{-1}\left\{\frac{1}{s}[u(x,0) + L_\alpha\{f(x,t)\}]\right\}. \tag{20}$$

$$u_{n+1}(x,t) = -L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{D_x^\alpha(\sum_{n=0}^{\infty}u_n)\right\}\right\} - L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha[R(\sum_{n=0}^{\infty}u_n) + \sum_{n=0}^{\infty}A_n]\right\}. \tag{21}$$

Hence by calculating as many u_n components as needed, the solution $u(x,t)$ can be obtained from equation (16).

When α is in Caputo's derivative sense with the same process, we get :

$$u_0 = u(x,0) + L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\{f(x,t)\}\right\}. \tag{22}$$

$$u_{n+1} = -L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\left\{D_x^\alpha(\sum_{n=0}^{\infty}u_n)\right\}\right\} - L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha[R(\sum_{n=0}^{\infty}u_n) + \sum_{n=0}^{\infty}A_n]\right\}. \tag{23}$$

Thus, the approximate analytical solution of equations (11)-(12) is determined by the series given in (16).

4. Main Results

In this section, we discuss three types of Burger's equations involving nonlinear partial differential equations using the LTDM described above with the fractional derivatives under consideration.

4.1. A Numerical Solution of the Fractional Burger's equation

Consider the following homogenous fractional Burger's equation :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} - 2u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial u^2(x,t)}{\partial x} = 0, \tag{24}$$

with initial condition :

$$u(x,0) = \sin(x), \tag{25}$$

where $0 < \alpha \leq 1$, $t > 0$, $x \in \mathbb{R}$, and $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ means the conformable derivative of the function $u(x,t)$ with respect to t.

By applying the above steps, we obtain :

$$u_0(x,t) = \sin(x), \tag{26}$$

$$u_{n+1}(x,t) = L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_n}{\partial x^2} - \frac{\partial A_n}{\partial x}\right\}\right\} + L_\alpha^{-1}\left\{\frac{2}{s}L_\alpha\left\{B_n\right\}\right\}. \tag{27}$$

where A_n and B_n are the so-called Adomian polynomials, given by

$$\begin{aligned} A_n &= \sum_{n=0}^{\infty} u_n^2, \\ B_n &= \sum_{n=0}^{\infty} u_n u_{nx}. \end{aligned} \tag{28}$$

The nonlinear terms uu_x and u^2 are represented as :

$$\begin{aligned} A_0 =& u_0^2, \\ A_1 =& 2u_0u_1, \\ A_2 =& 2u_0u_2 + u_1^2, \text{ and so on...} \\ B_0 =& u_0u_{0x}, \\ B_1 =& u_0u_{1x} + u_1u_{0x}, \\ B_2 =& u_0u_{2x} + u_1u_{1x} + u_2u_{0x}, \text{ and so on...} \end{aligned} \tag{29}$$

Based on the LTDM, we obtain :

$$\begin{aligned} u_1(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial A_0}{\partial x} + 2B_0\right\}\right\}, \\ =& -\frac{t^\alpha}{\alpha}\sin(x). \end{aligned} \tag{30}$$

In a similar manner, we obtain that :

$$\begin{aligned} u_2(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial A_1}{\partial x} + 2B_1\right\}\right\}, \\ =& \frac{t^{2\alpha}}{2!\alpha^2}\sin(x). \end{aligned} \tag{31}$$

Also,

$$\begin{aligned} u_3(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial A_2}{\partial x} + 2B_2\right\}\right\}, \\ =& -\frac{t^{3\alpha}}{3!\alpha^3}\sin(x). \end{aligned} \tag{32}$$

By adding all the terms, the series solution of equation (24) can be found as :

$$\begin{aligned} u(x,t) =& u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ... \\ =& \sin(x)[1 - \frac{t^\alpha}{\alpha} + \frac{t^{2\alpha}}{2!\alpha^2} - \frac{t^{3\alpha}}{3!\alpha^3} + ...] \\ =& e^{-\frac{t^\alpha}{\alpha}} \sin(x). \end{aligned} \tag{33}$$

By taking $\alpha = 1$, the exact solution of the classical form of equation (24) is given by:

$$u(x,t) = e^{-t} \sin(x). \tag{34}$$

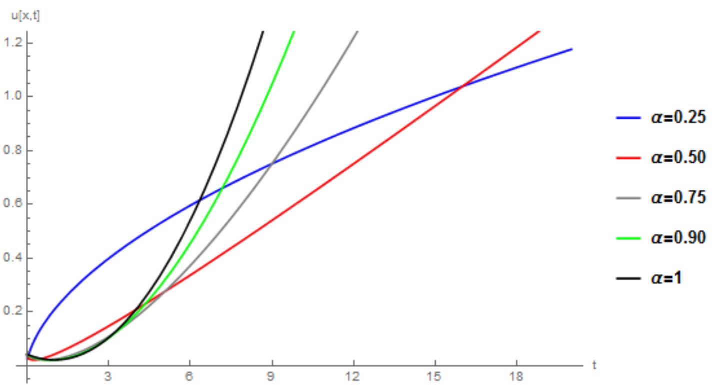

Figure 1: Temperature plots for equation (24) at various α with $x = 3.1$, $M = 5$
.

When α is in Caputo sense, using equation (22)-(23) with the same process, we get :

$$u_0(x,t) = \sin(x), \tag{35}$$

$$u_{n+1}(x,t) = L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha \left\{ \frac{\partial^2 u_n}{\partial x^2} - \frac{\partial A_n}{\partial x} + 2B_n \right\} \right\}. \tag{36}$$

Thus, we get :

$$\begin{aligned} u_1(x,t) =& L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha \left\{ \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial A_0}{\partial x} + 2B_0 \right\} \right\}, \\ =& -\frac{t^\alpha}{\Gamma(\alpha+1)} \sin(x). \end{aligned} \tag{37}$$

$$\begin{aligned} u_2(x,t) =& L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha \left\{ \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial A_1}{\partial x} + 2B_1 \right\} \right\}, \\ =& \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \sin(x). \end{aligned} \tag{38}$$

$$\begin{aligned} u_3(x,t) =& L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha \left\{ \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial A_2}{\partial x} + 2B_2 \right\} \right\}, \\ =& -\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \sin(x). \end{aligned} \tag{39}$$

Thus, we get the numerical solution of equation (24) in a series form :

$$
\begin{aligned}
u(x,t) =& u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots, \\
=& \sin(x)[1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots], \\
=& \sin(x) \sum_{n=0}^{\infty} \frac{(-t)^n}{\Gamma(n\alpha+1)}, \\
=& \sin(x) E_{\alpha}(-t).
\end{aligned}
\tag{40}
$$

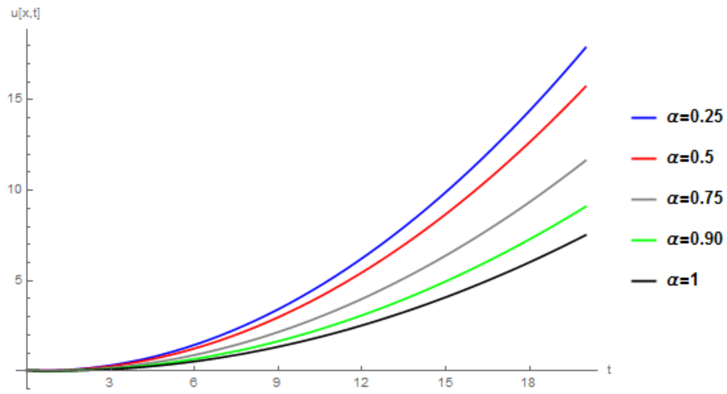

Figure 2: Temperature plots for equation (24) at various α with $x = 3.1$, $M = 5$

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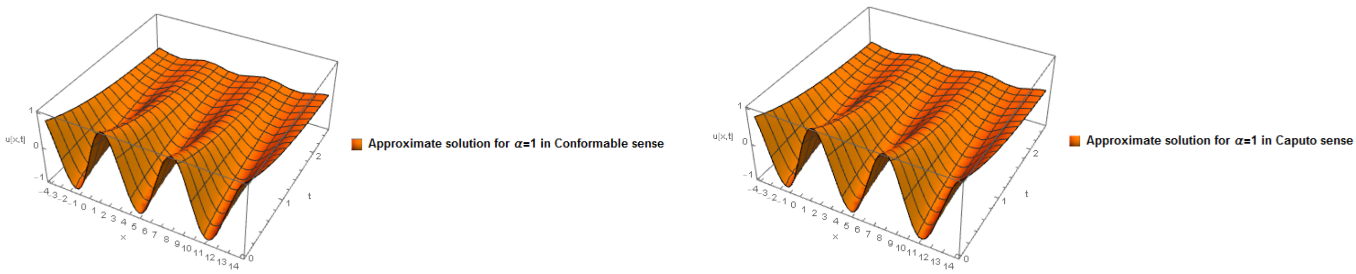

(a) (b)

Figure 3: The fractional solutions $u(x,t)$ for equation (24) at $\alpha = 1$ for $-4 \le x \le 14$ and $0 \le t \le 2.5$

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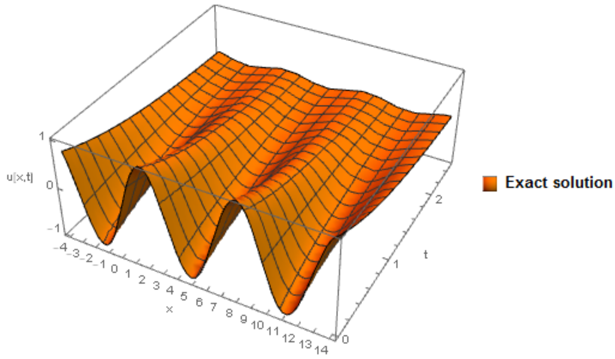

Figure 4: The exact solution for equation (24) for $-4 \leq x \leq 14$ and $0 \leq t \leq 2.5$

.

Remark 4.1. *Caputo fractional differential problem (24)-(25) has similar shaped solutions to the classical one compared to those obtained for the conformable differential problem.*

Let us now consider the following non-homogenous fractional Burger's equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} - 2u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial u^2(x,t)}{\partial x} = x^2 - 2\frac{t^\alpha}{\alpha}, \tag{41}$$

with initial condition :

$$u(x,0) = 0, \tag{42}$$

where $0 < \alpha \leq 1$, $t > 0$, $x \in \mathbb{R}$, and $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ means the conformable derivative of the function $u(x,t)$ with respect to t.

By applying the above steps, we obtain :

$$u_0(x,t) = x^2\frac{t^\alpha}{\alpha} - \frac{t^{2\alpha}}{\alpha^2}, \tag{43}$$

$$u_{n+1}(x,t) = L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_n}{\partial x^2} - \frac{\partial A_n}{\partial x} + 2B_n\right\}\right\}. \tag{44}$$

Based on the LTDM, we obtain :

$$\begin{aligned} u_1(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial A_0}{\partial x} + 2B_0\right\}\right\}, \\ =& \frac{t^{2\alpha}}{\alpha^2}. \end{aligned} \tag{45}$$

In a similar manner, we obtain that :

$$\begin{aligned} u_2(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial A_1}{\partial x} + B_1\right\}\right\} = 0. \\ u_3(x,t) =& 0, \ldots . \end{aligned} \tag{46}$$

Thus,

$$\begin{aligned} u(x,t) =& u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ..., \\ =& x^2 \frac{t^\alpha}{\alpha} - \frac{t^{2\alpha}}{\alpha^2} + \frac{t^{2\alpha}}{\alpha^2} + 0 + 0 + ..., \\ =& x^2 \frac{t^\alpha}{\alpha}. \end{aligned} \tag{47}$$

By taking $\alpha = 1$, the exact solution of the classical form of equation (41) is given by:

$$u(x,t) = x^2 t. \tag{48}$$

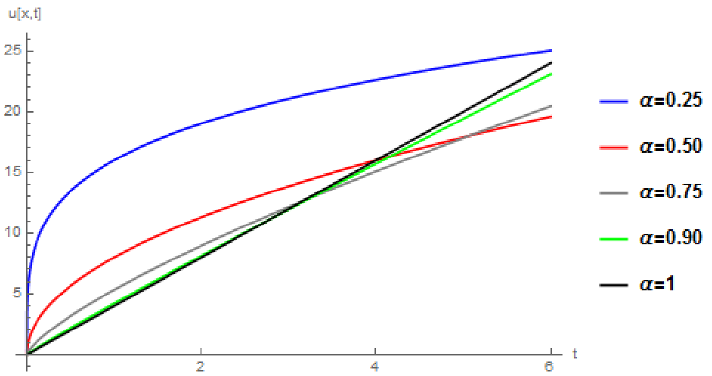

Figure 5: Temperature plots for equation (41) at various α with $x = 2$.

When α is in Caputo sense, using equation (41)-(42) with the same process, we get :

$$\begin{aligned} u_0(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s^\alpha} L_\alpha \left\{x^2 - 2\frac{t^\alpha}{\alpha}\right\}\right\}, \\ =& x^2 \frac{t^\alpha}{\Gamma(\alpha+1)} - 2\Gamma(\alpha)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned} \tag{49}$$

$$\begin{aligned} u_1(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s^\alpha} L_\alpha \left\{\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial A_0}{\partial x} + 2B_0\right\}\right\}. \\ =& \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned} \tag{50}$$

$$\begin{aligned} u_2(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s^\alpha} L_\alpha \left\{\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial A_1}{\partial x} + B_1\right\}\right\} = 0. \\ u_3(x,t) =& 0, ... \end{aligned} \tag{51}$$

By adding all the terms, the series solution of equation (41) can be found as :

$$
\begin{aligned}
u(x,t) =& u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ..., \\
=& x^2 \frac{t^\alpha}{\Gamma(\alpha+1)} - 2\Gamma(\alpha)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + 0 + 0 + ..., \\
=& x^2 \frac{t^\alpha}{\Gamma(\alpha+1)} + 2(1-\Gamma(\alpha))\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned}
\tag{52}
$$

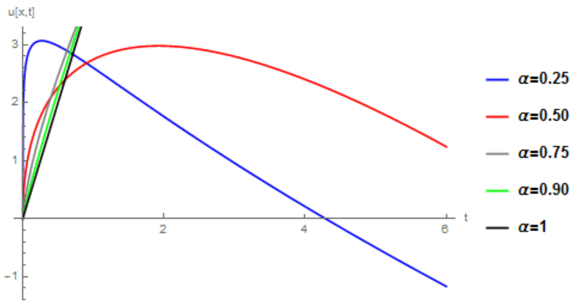

Figure 6: Temperature plots for equation (41) at various α with $x = 2$

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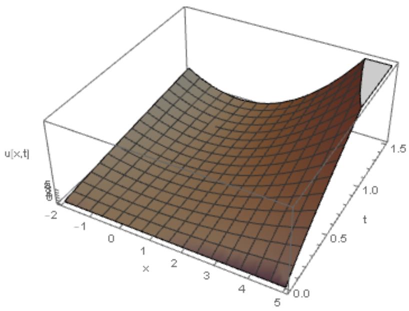
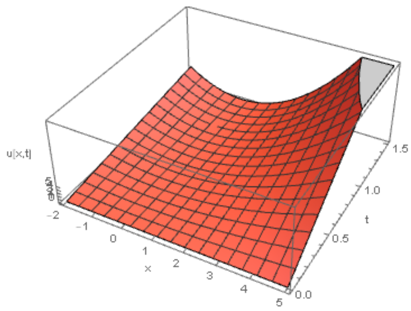
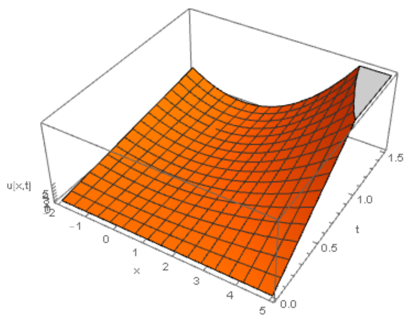


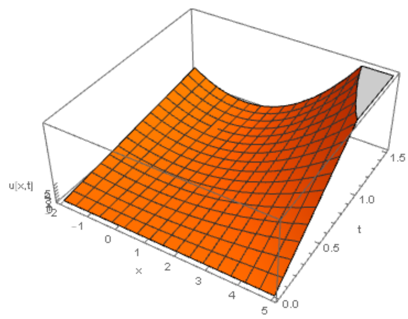
(a) Approximate solution $u(x,t)$ in Conformable sense for $\alpha = 0.85$.



(b) Approximate solution $u(x,t)$ in Conformable sense for $\alpha = 1$.



(c) Exact solution $u(x,t)$.



(d) Approximate solution $u(x,t)$ in Caputo sense for $\alpha = 1$.

Figure 7: The classical ($\alpha = 1$) and fractional solutions $u(x,t)$ of equation (41) for $-2 \leq x \leq 5$ and $0 \leq t \leq 1.5$.

Remark 4.2. *The nonhomogeneous conformable differential problem (41)-(42) has similar shaped solutions to the classical one compared to those obtained for the Caputo differential problem.*

Another equation of interest is the following fractional Burger's equation :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+u(x,t)\frac{\partial u(x,t)}{\partial x}-\frac{\partial^2 u(x,t)}{\partial x^2}=0, \tag{53}$$

with initial condition :

$$u(x,0)=x, \tag{54}$$

where, $0<\alpha\le 1$, $t>0$, $0<x<1$, and $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ means the conformable derivative of the function $u(x,t)$.

By applying the above steps, we obtain :

$$u_0(x,t)=x. \tag{55}$$

$$u_{n+1}(x,t)=L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_n}{\partial x^2}-A_n\right\}\right\}. \tag{56}$$

Based on the LTDM, we obtain :

$$\begin{aligned} u_1(x,t)=&L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_0}{\partial x^2}-A_0\right\}\right\},\\ =&-x\frac{t^\alpha}{\alpha}. \end{aligned} \tag{57}$$

In a similar manner, we obtain that :

$$\begin{aligned} u_2(x,t)=&L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_1}{\partial x^2}-A_1\right\}\right\},\\ =&x\frac{t^{2\alpha}}{\alpha^2}, \end{aligned} \tag{58}$$

and,

$$\begin{aligned} u_3(x,t)=&L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_2}{\partial x^2}-A_2\right\}\right\},\\ =&-x\frac{t^{3\alpha}}{\alpha^3}. \end{aligned} \tag{59}$$

Thus, we have :

$$\begin{aligned} u(x,t) =& u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ..., \\ =& x - x\frac{t^\alpha}{\alpha} + x\frac{t^{2\alpha}}{\alpha^2} - x\frac{t^{3\alpha}}{\alpha^3} + ... \\ =& x(1 - \frac{t^\alpha}{\alpha} + \frac{t^{2\alpha}}{\alpha^2} - \frac{t^{3\alpha}}{\alpha^3} + ...) \\ =& x(1 - \frac{t^\alpha}{\alpha} + (\frac{t^\alpha}{\alpha})^2 - (\frac{t^\alpha}{\alpha})^3 + ...) \\ =& x\sum_{n=0}^{\infty}(\frac{-t^\alpha}{\alpha})^n \\ =& \frac{x}{1+\frac{t^\alpha}{\alpha}}. \end{aligned} \tag{60}$$

For $\alpha = 1$, the exact solution of the classical form of equation (53) is given by :

$$u(x,t) = \frac{x}{1+t}. \tag{61}$$

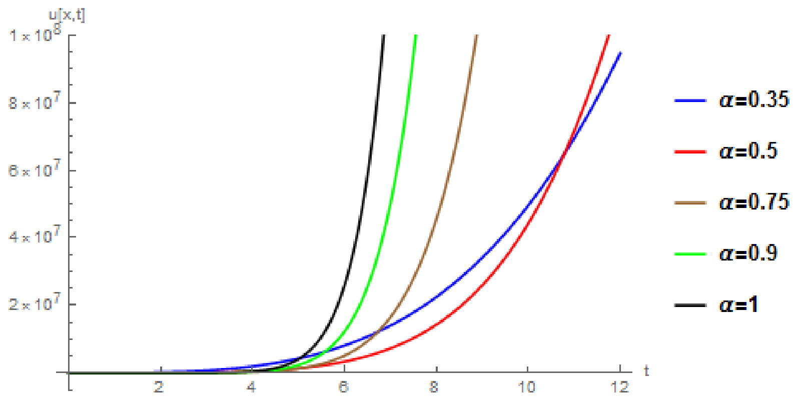


Figure 8: Temperature plots for equation (53) at various α in conformable sense for $x = 0.5$, $M = 10$.

When α is in Caputo sense, using equation (53)-(54) with the same process, we get :

$$u_0(x,t) = x. \tag{62}$$

$$u_{n+1}(x,t) = L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\left\{\frac{\partial^2 u_n}{\partial x^2} - A_n\right\}\right\}. \tag{63}$$

Thus,

$$\begin{aligned} u_1(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\left\{\frac{\partial^2 u_0}{\partial x^2} - A_0\right\}\right\}. \\ =& -x\frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{64}$$

$$\begin{aligned} u_2(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\left\{\frac{\partial^2 u_1}{\partial x^2} - A_1\right\}\right\}. \\ =& 2x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned} \tag{65}$$

$$\begin{aligned} u_3(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\left\{\frac{\partial^2 u_2}{\partial x^2} - A_2\right\}\right\}. \\ =& -6x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \end{aligned} \tag{66}$$

Summing the above terms yields :

$$\begin{aligned} u(x,t) =& u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ..., \\ =& x - x\frac{t^\alpha}{\Gamma(\alpha+1)} + 2x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 6x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ..., \\ =& x(1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 6\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...), \\ =& x(1 + \sum_{n=1}^{\infty}\frac{n!(-1)^n t^{n\alpha}}{\Gamma(n\alpha+1)}). \end{aligned} \tag{67}$$

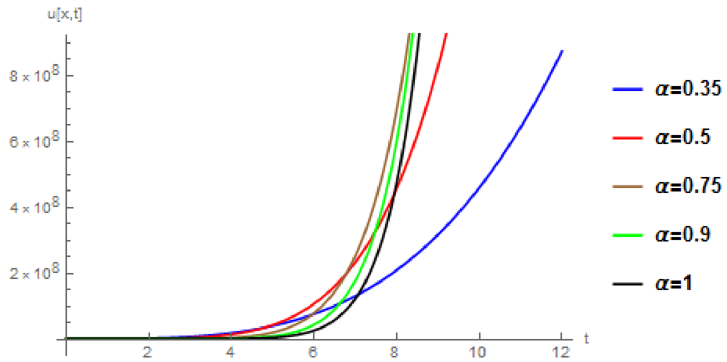


Figure 9: Temperature plots for equation (53) at various α in Caputo sense for $x = 0.5$, $M = 10$

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Remark 4.3. *The solutions of the Caputo differential problem (53)-(54) converge (as $\alpha \to 1$) to the classical one better than those obtained from the conformable differential problem.*

4.2. A Numerical Solution of the time-fractional Burger's Kdv Equation

Consider the time-fractional Burger's Kdv equation :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^3 u(x,t)}{\partial x^3} + 6u(x,t)\frac{\partial u(x,t)}{\partial x} = 0, \tag{68}$$

with initial condition :

$$u(x,0) = \frac{1}{2}sech^2(\frac{1}{2}x), \tag{69}$$

where, $0 < \alpha \leq 1$, $t > 0$, $x \in \mathbb{R}$, and $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ means the conformable derivative of the function $u(x,t)$ with respect to t.

By applying the above steps, we obtain :

$$u_0(x,t) = \frac{1}{2}sech^2(\frac{1}{2}x), \tag{70}$$

$$u_{n+1} = -L_{\alpha}^{-1}\left\{\frac{1}{s}L_{\alpha}\left\{\frac{\partial^3 u_n}{\partial x^3} + 6A_n\right\}\right\}. \tag{71}$$

Based on the LTDM, we obtain :

$$\begin{aligned} u_1(x,t) =& -L_{\alpha}^{-1}\left\{\frac{1}{s}L_{\alpha}\left\{\frac{\partial^3 u_0}{\partial x^3} + 6A_0\right\}\right\}, \\ =&(\frac{1}{2}sech^4(\frac{1}{2}x)\tanh(\frac{1}{2}x) + \frac{1}{2}sech^2(\frac{1}{2}x)\tanh^3(\frac{1}{2}x))\frac{t^{\alpha}}{\alpha}. \end{aligned} \tag{72}$$

In a similar manner, we obtain that :

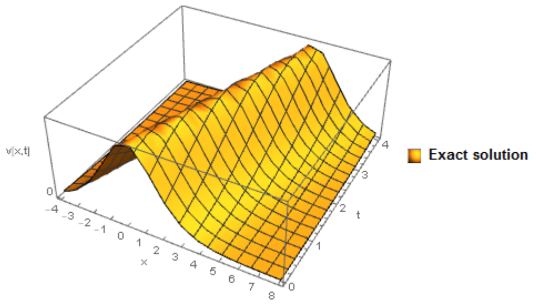
$$\begin{aligned} u_2(x,t) =& -L_{\alpha}^{-1}\left\{\frac{1}{s}L_{\alpha}\left\{\frac{\partial^3 u_1}{\partial x^3} + 6A_1\right\}\right\}, \\ =&\frac{1}{8}(-2+\cosh(x))sech^4(\frac{1}{2}x)\frac{t^{2\alpha}}{\alpha}. \end{aligned} \tag{73}$$

Therefore,

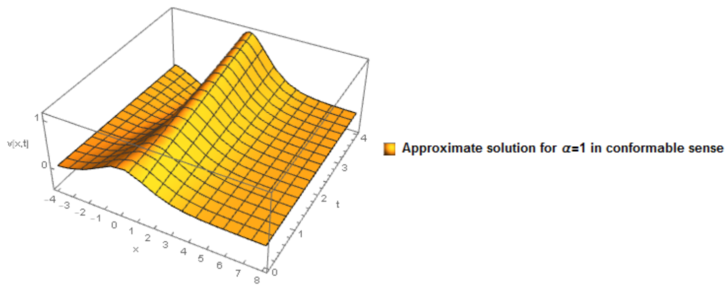
$$\begin{aligned} u(x,t) =&u_0(x,t) + u_1(x,t) + u_2(x,t) + ..., \\ =&\frac{1}{2}sech^2(\frac{1}{2}x) + \frac{1}{2}sech^2(\frac{1}{2}x)\tanh^3(\frac{1}{2}x)\frac{t^{\alpha}}{\alpha} + \frac{1}{2}sech^4(\frac{1}{2}x)\tanh(\frac{1}{2}x))\frac{t^{\alpha}}{\alpha} + ... \end{aligned} \tag{74}$$

By taking $\alpha = 1$, the exact solution of equation (68) is given by :

$$u(x,t) = \frac{1}{2}sech^2(\frac{1}{2}(x-t)). \tag{75}$$



(a) Exact solution $u(x,t)$.



(b) Approximate solution $u(x,t)$ in conformable sense for $\alpha = 1$.

Figure 10: The exact and approximate solution $u(x,t)$ of equation (68) for $\alpha = 1$, $-4 \le x \le 8$ and $0 \le t \le 4$.

When α is in Caputo sense, using equation (68)-(69) with the same process, we get :

$$u_0(x,t) = \frac{1}{2}sech^2(\frac{1}{2}x), \tag{76}$$

$$u_{n+1}(x,t) = -L_{\alpha}^{-1}\left\{\frac{1}{s^{\alpha}}L_{\alpha}\left\{\frac{\partial^3 u_n}{\partial x^3} + 6A_n\right\}\right\}. \tag{77}$$

Thus, we obtain :

$$\begin{aligned} u_1(x,t) =& -L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\left\{\frac{\partial^3 u_0}{\partial x^3}+6A_0\right\}\right\}, \\ =&(\frac{1}{2}sech^4(\frac{1}{2}x)\tanh(\frac{1}{2}x)+\frac{1}{2}sech^2(\frac{1}{2}x)\tanh^3(\frac{1}{2}x))\frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{78}$$

In a similar manner, we obtain that :

$$\begin{aligned} u_2(x,t) =& -L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\left\{\frac{\partial^3 u_1}{\partial x^3}+6A_1\right\}\right\}, \\ =&\frac{1}{4}(-2+\cosh(x))sech^4(\frac{1}{2}x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned} \tag{79}$$

Therefore,

$$\begin{aligned} u(x,t) =& u_0(x,t)+u_1(x,t)+u_2(x,t)+..., \\ =&\frac{1}{2}sech^2(\frac{1}{2}x)+\frac{1}{2}sech^2(\frac{1}{2}x)\tanh^3(\frac{1}{2}x)\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{1}{2}sech^4(\frac{1}{2}x)\tanh(\frac{1}{2}x))\frac{t^\alpha}{\Gamma(\alpha+1)}+... \end{aligned} \tag{80}$$

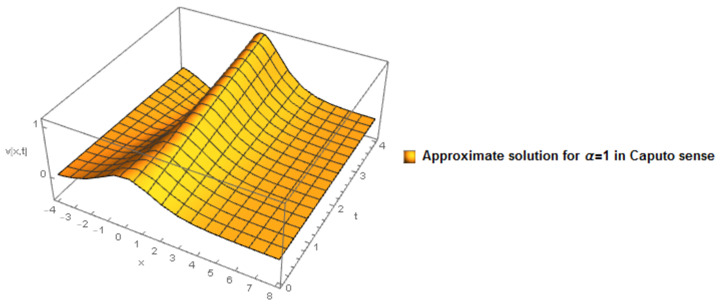


Figure 11: The approximate solution $u(x,t)$ for $\alpha=1$, $-4\le x\le 8$ and $0\le t\le 4$.

Remark 4.4. *Both conformable and Caputo fractional differential problem (68)-(69) generate similar approximate solutions to the classical one when* $\alpha\to 1$.

4.3. A Numerical Solution of the Fractional Modified Burger's Equation

Consider the following fractional modified Burger's equation :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+u^2(x,t)\frac{\partial u(x,t)}{\partial x}-\frac{\partial^2 u(x,t)}{\partial x^2}=0, \tag{81}$$

with initial condition :

$$u(x,0)=\frac{-\sqrt{3}}{\sqrt{1-\cosh(2x)-\sinh(2x)}}, \tag{82}$$

where, $0<\alpha\le 1$, $t>0$, $x\in\mathbb{R}$, and $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ means the conformable derivative of the function $u(x,t)$.

By applying the above steps, we obtain :

$$u_0(x,t) = \frac{-\sqrt{3}}{\sqrt{1-\cosh(2x)-\sinh(2x)}}. \tag{83}$$

$$u_{n+1}(x,t) = L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_n}{\partial x^2} - A_n\frac{\partial u_n}{\partial x}\right\}\right\}. \tag{84}$$

Based on the LTDM, we obtain :

$$\begin{aligned} u_1(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_0}{\partial x^2} - A_0\frac{\partial u_0}{\partial x}\right\}\right\} \\ =& \frac{-\sqrt{3}}{(1-e^{2x})^{\frac{3}{2}}}\frac{t^\alpha}{\alpha}, \end{aligned} \tag{85}$$

and,

$$\begin{aligned} u_2(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s}L_\alpha\left\{\frac{\partial^2 u_1}{\partial x^2} - A_1\frac{\partial u_1}{\partial x}\right\}\right\} \\ =& \frac{18\sqrt{3}e^{2x}}{(1-e^{2x})^{\frac{9}{2}}}\frac{t^{3\alpha}}{3\alpha^3} + \frac{9\sqrt{3}e^{6x} - 3\sqrt{3}e^{4x} - 6\sqrt{3}e^{2x}}{(1-e^{2x})^{\frac{9}{2}}}\frac{t^{2\alpha}}{2\alpha^2}. \end{aligned} \tag{86}$$

Therefore,

$$\begin{aligned} u(x,t) =& u_0(x,t) + u_1(x,t) + u_2(x,t) + ..., \\ =& \frac{-\sqrt{3}}{\sqrt{1-\cosh(2x)-\sinh(2x)}} + \frac{-\sqrt{3}}{(1-e^{2x})^{\frac{3}{2}}}\frac{t^\alpha}{\alpha} + \frac{18\sqrt{3}e^{2x}}{(1-e^{2x})^{\frac{9}{2}}}\frac{t^{3\alpha}}{3\alpha^3} \\ &+ \frac{9\sqrt{3}e^{6x} - 3\sqrt{3}e^{4x} - 6\sqrt{3}e^{2x}}{(1-e^{2x})^{\frac{9}{2}}}\frac{t^{2\alpha}}{2\alpha^2} + ... \end{aligned} \tag{87}$$

For $\alpha = 1$, the exact solution of equation (81) is given by :

$$u(x,t) = \frac{-\sqrt{3}}{\sqrt{1-\cosh(2(x-t))-\sinh(2(x-t))}}. \tag{88}$$

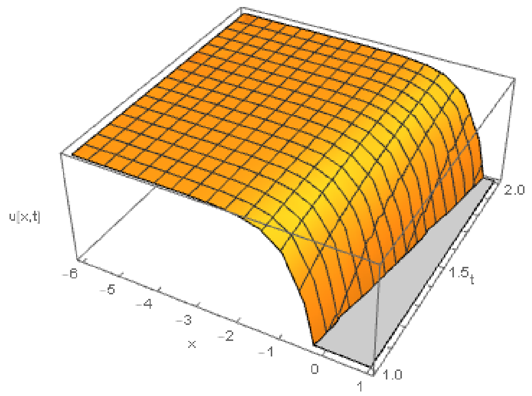


Figure 12: The classical solution $u(x,t)$ for $-6 \leq x \leq 1$ and $1 \leq t \leq 2$.

When α is in Caputo sense with the same process, we get :

$$u_0(x,t) = \frac{-\sqrt{3}}{\sqrt{1-\cosh(2x)-\sinh(2x)}}. \tag{89}$$

$$u_{n+1}(x,t) = L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\left\{\frac{\partial^2 u_n}{\partial x^2} - A_n\frac{\partial u_n}{\partial x}\right\}\right\}. \tag{90}$$

Thus, we obtain :

$$\begin{aligned} u_1(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\left\{\frac{\partial^2 u_0}{\partial x^2} - A_0\frac{\partial u_0}{\partial x}\right\}\right\} \\ =& \frac{-\sqrt{3}}{(1-e^{2x})^{\frac{3}{2}}}\frac{t^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{91}$$

and,

$$\begin{aligned} u_2(x,t) =& L_\alpha^{-1}\left\{\frac{1}{s^\alpha}L_\alpha\left\{\frac{\partial^2 u_1}{\partial x^2} - A_1\frac{\partial u_1}{\partial x}\right\}\right\} \\ =& \frac{9\sqrt{3}e^{6x} - 3\sqrt{3}e^{4x} - 6\sqrt{3}e^{2x}}{(1-e^{2x})^{\frac{9}{2}}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{18\sqrt{3}e^{2x}}{(1-e^{2x})^{\frac{9}{2}}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \end{aligned} \tag{92}$$

Therefore,

$$\begin{aligned} u(x,t) =& u_0(x,t) + u_1(x,t) + u_2(x,t) + ..., \\ =& \frac{-\sqrt{3}}{\sqrt{1-\cosh(2x)-\sinh(2x)}} + \frac{-\sqrt{3}}{(1-e^{2x})^{\frac{3}{2}}}\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{9\sqrt{3}e^{6x} - 3\sqrt{3}e^{4x} - 6\sqrt{3}e^{2x}}{(1-e^{2x})^{\frac{9}{2}}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & + \frac{18\sqrt{3}e^{2x}}{(1-e^{2x})^{\frac{9}{2}}}\frac{t^{3\alpha}}{3\alpha^3} + ... \end{aligned}$$

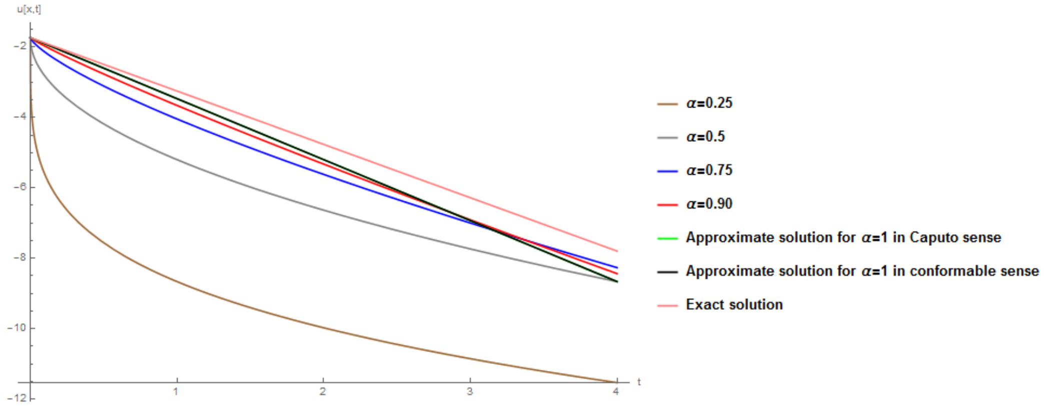


Figure 13: The Exact and approximate solution $u(x,t)$ of equation (81) for $x = -4$ and $0 \leq t \leq 4$.

Remark 4.5. *Both conformable and Caputo fractional differential problem (81)-(82) generate the same approximate solution to the classical one when* $\alpha \to 1$.

5. Conclusion

In this work, we apply the LTDM method to solve fractional Burger's equation, fractional Burger's Kdv equation, and fractional modified Burger's equation. Some applications are given to show that LTDM is an effective and easy method for obtaining exact and approximated solutions to such previous equations. We observe, from the graphs studied, that the different values of the fractional-order of the derivative allow very different behaviors of the solution. In addition, if we let $\alpha = 1$ in the given examples, we obtain the exact solutions that are studied in [26, 27]. There are a few important points to make here. Firstly, LT and ADM provide the solution in terms of easily computable components. They provide more realistic solutions that have very rapid convergence on real physical problems. The analytic solutions of these three applications, found by these two methods, are compared with exact solutions for both conformable fractional derivative and Caputo fractional derivative. The numerical results show that the solutions are in good agreement with their respective exact solutions. Secondly, the methods were used in a direct way without using linearization, perturbation, or restrictive assumptions. The results show that this technique can be extended to solve various linear and nonlinear fractional problems in applied science. Mathematica has been used, in this paper, for presenting graphs of solutions.

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