*International Journal of Engineering and Applied Sciences (IJEAS)*
*Vol.1, Issue 4(2009)1-25*

# ON THE ACCURACY AND STABILITY OF A VARIETY OF DIFFERENTIAL QUADRATURE FORMULATIONS FOR THE VIBRATION ANALYSIS OF BEAMS

***C. H. W. Ng[1], Y. B. Zhao[2], Y. Xiang[3] and G. W. Wei[4,5]***
*[1]Department of Computational Science, National University of Singapore, Singapore 117543*
*[2]Lee Kong Chian School of Business, Singapore Management University, 50 Stamford Road, Singapore 178899*
*[3]School of Engineering and Industrial Design, University of Western Sydney, Penrith South DC NSW 1797 Australia*
*[4]Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA*
*[5]Department of Electrical and Computer Engineering, Michigan State University, East Lansing, MI 48824, USA*

Accepted Date: 07 June 2009

**Abstract**

*The occurrence of spurious complex eigenvalues is a serious stability problem in many differential quadrature methods (DQMs). This paper studies the accuracy and stability of a variety of different differential quadrature formulations. Special emphasis is given to two local DQMs. One utilizes both fictitious grids and banded matrices, called local adaptive differential quadrature method (LaDQM). The other has banded matrices without using fictitious grids to facilitate boundary conditions, called finite difference differential quadrature methods (FDDQMs). These local DQMs include the classic DQMs as special cases given by extending their banded matrices to full matrices. LaDQMs and FDDQMs are implemented on a variety of treatments of boundary conditions, distributions of grids (i.e., uniform grids and Chebyshev grids), and lengths of stencils. A comprehensive comparison among these methods over beams of six different combinations of supporting edges sheds light on the stability and accuracy of DQMs.*

**Keywords**: Stability analysis; Local adaptive differential quadrature method; Differential quadrature method; Beam; Vibration analysis.

## 1. Introduction

Beams are fundamental components in structures and machines. They also prove to be useful for modeling more complex structural behavior. A large number of approximation methods have been developed to obtain solutions for beams of practical applications. As the Euler-Bernoulli beam has analytical solution for simply-supported, clamped and free edges, it is an ideal model to analyze the accuracy and stability of different numerical methods.

Differential quadrature method (DQM) is a collocation scheme and was first introduced by Bellman et al. [1, 2] as a numerical method to solve partial differential equations. Its early applications include the Hodgkin-Huxley model by Bellman et al. [3], chemical reactor analysis by Naadimuthu et al. [4] and transient nonlinear diffusion by Mingle [5]. Civan has also applied the DQM successfully to the Poisson equation [7] and multidimensional problems [8]. Due to certain difficulties in implementing boundary conditions, the DQM was virtually ignored for many years until Bert and co-workers [9, 10] brought in back to life in late 1980s. They proposed the $\delta$-technique to implement multiple boundary conditions [9, 10].

The DQM using the δ-technique to implement boundary conditions has been applied successfully to many problems such as nonlinear bending analysis of thin circular plates by Striz et al. [11], nonlinear bending analysis of orthotropic plates by Bert et al. [12], free vibration and buckling analysis of beams, membrane and thin plates by Bert et al. [10] and free vibration analysis of rectangular plates with non-uniform boundary conditions by Laura et al. [13].

The δ-technique consists of placing two grid points separated from each other by a small distance δ near the boundary edge. For two boundary conditions to be satisfied at the boundary edge, one is located at the grid points on the boundary edge while the other is placed to the adjacent δ-grid point. In order to obtain an accurate numerical solution, δ must be very small (possibly not greater than $10^{-4}$L where L is the length of the computational domain). However, when one mesh spacing (δ) is much smaller than the others mesh spacing, the DQ weighting coefficients matrices may become highly ill-conditioned, which then causes the solution to oscillate. Hence, the numerical solution becomes less accurate.

To overcome some drawback of the δ-technique, Shu and Hu proposed a new approach to implement simply supported and clamped boundary conditions for the free vibration analysis of beams and plates using the GDQ method. The proposed approach directly substitutes the boundary conditions into the governing equations (SBCGE) [14]. The SBCGE was subsequently generalized to directly couple the boundary conditions with the governing equations (CBCGE) [15]. The CBCGE can be used to implement in any general boundary equations including free edges for plate vibration analysis. The key difference between the δ-technique and CBCGE is that the boundary conditions are satisfied on the boundaries in the CBCGE while only one boundary condition is exactly satisfied in the δ-technique. The GDQ using the SBCGE or CBCGE to implement boundary conditions has been applied successfully to many problems such as the free vibration analysis of plates with mixed and nonuniform boundary conditions by Shu and Wang [16], free vibration analysis of laminated conical shells with variable stiffness by Wu [17] and analysis of elliptical waveguides by Shu [18]. The detail of the CBCGE is given in Section II. Wu and Liu [19] proposed the generalized differential quadrature rule (GDQR) to improve the drawback of the δ-technique. The GDQR chooses the function values and some of the derivatives whenever necessary as independent variables. The GDQR's weighting coefficients can be obtained using the Hermite interpolation function. Hence, the δ-grid points arrangement used in the δ-technique is exempted while applying the boundary conditions exactly on the boundaries. Wu and Liu have applied the GDQR successfully to many fourth-order differential equations including vibration analysis of beams, circular plates and arches [20]. Recently, Wu and Liu further extend the GDQR to solving sixth-order differential equations [21].

The DQM is based on the idea that the derivative of a function with respect to a spatial variable at a given discrete point can be expressed as a weighted linear sum of the function values at all the discrete points in the computational domain. One important aspect of the DQM is to determine the weighting coefficients for the first derivative approximation. Bellman et al. [2] proposed two methods to compute the weighting coefficients. In the first method, the weighting coefficients are determined by solving an algebraic equation system. In the second method, the weighting coefficients are determined by a simple algebraic formula and the grid points are chosen as the roots of shifted Legendre polynomial. Between these two methods, the first method is usually used as it allows the coordinate of the grid points to be chosen arbitrary. However, as the order of the algebraic equation system is enlarged, the condition number of the corresponding differential matrix becomes increasely large, which leads to an ill-conditioned matrix. Thus, using the first method, it is very difficult to obtain the weighting coefficients for a large number of grid points. In early practical applications of the first method, the number of grid points is usually chosen to be equal to or less than 13 [22]. A mian contribution to this field was due to

Shu et al. [23]. These authors proposed the generalized differential quadrature (GDQ) method which uses the Lagrange interpolation polynomial as the basis function. An algebraic expression is presented to compute the weighting coefficients of the first order derivative approximation directly. A convenient recurrence formula is also presented to compute the weighting coefficients for higher order derivatives without restriction on the choice of the grid points [23]. Shu and Chew [24] show that the GDQ is equivalent to the highest order finite difference scheme.

Recently, the discrete singular convolution (DSC) algorithm has been proposed as a wavelet collocation scheme for the computer realization of singular convolutions [25, 26, 41]. The underlying mathematical structure of the DSC algorithm is the theory of distributions [42] and wavelet analysis. The DSC algorithm has global methods' accuracy and local methods' flexibility for handling complex geometry and boundary conditions in fluid dynamics [43] and electrodynamics [44]. The DSC algorithm has found its success in structural analysis, including the vibration and buckling of beams [45], plate vibration under various edge and internal supports [46, 47, 48, 49, 50, 51, 52, 53, 54]. It provides a unified framework for Galerkin, collocation and generalized finite difference methods [26]. The good performance of the DSC algorithm for the vibration analysis has been independently verified by Civalek [27, 28, 29, 30, 31, 32, 33, 34, 35]. One of the important features of the DSC algorithm is that it admits a banded matrix. Another important feature of the DSC is that it utilizes fictitious grid points to accommondate boundary conditions. The use of banded Lagrange kernel with fictitious grid points for treating boundary conditions in the collocation formulation was proposed in an earlier work by Wei and his co-workers [55]. The matched interface and boundary (MIB) method [36, 37, 38, 39] has been utlized to improve the boundary treatment of the DSC method [40]. Stability analysis has been presented. The combination of local Lagrange kernel and fictitious grids was implemented for plate vibration analysis [47, 56]. Recently, this combination has been extended for treating multiple boundary conditions raised in high-order differential equations [57] via the use of the CBCGE [15]. To accommodate multiple boundary conditions, the number of fictitious grid points was restricted to the number of nontrivial boundary conditions, and the scheme was called local adaptive differential quadrature method (LaDQM) to acknowledge its similarity to the DQM or GDQ. In general, the LaDQM generates a banded matrix which has a lower condition number and improves the numerical stability. In this work, another local DQM which utilizes the local Lagrange kernel while without using the fictitious grid points is denoted as finite difference differential quadrature method (FDDQM). The length of stencils in both the LaDQM and FDDQM can vary from 3 to the size of the whole domain.

Since there exist a variety of different local differential quadrature formulations based on Langrange polynomials by varying the degree of the polynomials, treatment of boundary conditions, and use of regular/irregular grid points, it is often very confusing for researchers and engineers to make a choice of a DQM for solving practical problems. Such a choice can be quite dangerous as DQMs can admit spurious complex eigenvalues [56], i.e., the stability problem, whose occurrence is not well understood in general. The main objective of the present work is to analyze the accuracy and stability of different local differential quadrature formulations, varying from the treatment of boundary conditions, the length of stencils, to the distribution of grid meshes.

The rest of this paper is organized as the follows. Section II is devoted to the formulation of beam vibration, the FDDQM and LaDQM formalism. The numerical results of the FDDQM and LaDQM are presented and discussed in Section III. Finally, conclusion is given in Section IV.

## 2. Theory and Algorithm

For integrity and completeness, this section accounts for the theory of beam analysis and methods of solution. Particular emphasis is given to a comparison of both the LaDQM and FDDQM algorithms.

### *2.1. Theory of beam vibration*

For an Euler-Bernoulli beam of length *L*, the governing equation of free vibration is

$$\frac{d^4 w}{dX^4} = \frac{\rho A \omega^2}{EI} w, \tag{1}$$

where *w* is the transverse displacement of the beam, *X* is the Cartesian coordinate in the middle axis of the beam, $\rho$ is the mass density of the beam, *A* is the cross-section area of the beam, $\omega$ is the angular frequency of the beam, *E* is Young's modulus of elasticity and I is the constant area moment of inertia about the neutral axis. For generality, dimensionless governing equation is used

$$\frac{d^4 W}{dx^4} = \Omega^2 W, \tag{2}$$

where *W* is the dimensionless displacement $(W = w/L)$, $\Omega$ is the dimensionless frequency parameter $(\Omega = \omega L^2(\rho A/EI)^{1/2})$, *x* is the dimensionless coordinate along the X-direction $(x = X/L)$. In this paper, three types of boundary

- Simply-supported edge (S)

$$W = 0, \quad \frac{d^2 W}{dx^2} = 0; \tag{3}$$

- Clamped edge (C)

$$W = 0, \quad \frac{dW}{dx} = 0; \tag{4}$$

- Free edge (F)

$$\frac{d^2 W}{dx^2} = 0, \quad \frac{d^3 W}{dx^3} = 0. \tag{5}$$

Six possible combinations of these three edge supports are used in numerical experiments.

### *2.2. Finite difference type of differential quadrature methods*

The DQ approximates the derivative of a function with respect to a spatial variable at a given discrete point by a weighted linear sum of the function values at all the discrete points in the computational domain. For example, the mth derivative of a function *u(x)* at the *ith* point, $x_i$, is approximated as

$$u^{(m)}(x_i) = \sum_{j=1}^{N} c_{i,j}^{(m)} u(x_j), \qquad i = 1, 2, \ldots, N, \tag{6}$$

where $u^{(m)}(x_i)$ is the mth order derivative of $u(x)$ at $x_i$, $N$ is the *total* number of grid points employs to discretize the beam. $c_{ij,}^{(m)}$ $(j = 1, \ldots, N)$, are the weighting coefficients for the mth derivative approximation of the i*th* point. These coefficients have to be pre-determined. In the generalized differential quadrature (GDQ) method [23], the global Lagrange interpolation polynomial is used as the test function

$$g_j(x) = \frac{l(x)}{(x - x_j) l^{(1)}(x_j)}, \qquad j = 1, 2, \ldots, N, \tag{7}$$

where

$$l(x) = \prod_{k=1}^{N} (x - x_k), \qquad l^{(1)}(x_j) = \prod_{k=1, k \neq j}^{N} (x_j - x_k), \tag{8}$$

where $l^{(1)}(x)$ is the first derivative of $l(x)$. Thus, the weighting coefficients $c_{ij}^{(1)}$ $(i,j = 1, 2, \ldots, N)$ can be obtained analytically from the differentiation of Eq. (7) to obtain

$$c_{i,j}^{(1)} = \frac{l^{(1)}(x_i)}{(x_i - x_j) l^{(1)}(x_j)}, \qquad \text{for } i, j = 1, 2, \ldots, N; j \neq i, \tag{9}$$

$$c_{i,i}^{(1)} = - \sum_{j=1, j \neq i}^{N} c_{i,j}^{(m)}, \qquad \text{for } i = 1, 2, \ldots, N. \tag{10}$$

The weighting coefficients for higher order derivatives can be obtained in the same manner. Actually, a recurrence relationship had been found for the m*th* order weighting coefficients $c_{ij}^{(m)}$ (See Ref. [23])

$$c_{i,j}^{(m)} = \begin{cases} m \left( c_{i,j}^{(1)} c_{i,i}^{(m-1)} - \dfrac{c_{i,j}^{(m-1)}}{x_i - x_j} \right) & j \neq i, \\ - \displaystyle\sum_{j=1, j \neq i}^{N} c_{i,j}^{(m)} & j = i. \end{cases} \tag{11}$$

Eqs. (6-11) give a convenient and general form for determining the weighting coefficients for the first to (N - 1)-th order derivatives. There is no restriction on the choice of the coordinates of the grid points. There is no need to solve the weighting coefficients from a set of algebraic equations like in the original DQ method. Thus, the ill conditioned problem in the determination of weighting coefficients in the original DQM no longer exists in the GDQ.

The DQ discretization, Eq. (6), described above uses all the grid points to approximate the derivative of a function. It is possible to use just part of the grid points to achieve the approximation. For example, the mth derivative of a function u(x) at the ith point, $x_i$, can be approximated with $x_{S1}, x_{S1+1}, \ldots, x_{S2}$, instead of $x_1, x_2, \ldots, x_N$. Thus, Eqs. (6-11) can be rewrite as

$$u^{(m)}(x_i) = \sum_{j=S1}^{S2} C_{i,j}^{(m)} u(x_j), \tag{12}$$

$$g_{S1,S2,j}(x) = \begin{cases} \dfrac{l_{S1,S2}(x)}{(x-x_j)\, l_{S1,S2}^{(1)}(x_j)} & S1 \leq j \leq S2, \\ 0 & \text{otherwise}; \end{cases} \tag{13}$$

$$l_{S1,S2}(x) = \begin{cases} \prod_{k=S1}^{S2} (x - x_k) & x_{S1} \leq x \leq x_{S2}, \\ 0 & \text{otherwise}; \end{cases} \tag{14}$$

$$l_{S1,S2}^{(1)}(x_j) = \begin{cases} \prod_{k=S1,k\neq j}^{S2} (x_j - x_k), & S1 \leq j \leq S2, \\ 0 & \text{otherwise}; \end{cases} \tag{15}$$

$$C_{i,j}^{(1)} = \begin{cases} \dfrac{M_{S1,S2}^{(1)}(x_i)}{(x_i - x_j) M_{S1,S2}^{(1)}(x_j)}, & S1 \leq j \leq S2, j \neq i, \\ -\sum_{j=S1,j\neq i}^{S2} C_{i,j}^{(1)} & S1 \leq j \leq S2, j = i \\ 0 & \text{otherwise}; \end{cases} \tag{16}$$

$$C_{i,j}^{(m)} = \begin{cases} m\left( C_{i,j}^{(1)} C_{i,i}^{(m-1)} - \dfrac{C_{i,j}^{(m-1)}}{x_i - x_j} \right) & S1 \leq j \leq S2, j \neq i, \\ -\sum_{j=S1,j\neq i}^{S2} C_{i,j}^{(1)} & S1 \leq j \leq S2, j = i \\ 0 & \text{otherwise}. \end{cases} \tag{17}$$

For simplicity, Eq. (12) is expressed as

$$u^{(m)}(x_i) = \sum_{j=1}^{N} C_{i,j}^{(m)} u(x_j). \tag{18}$$

Eq. (18) is equivalence to Eq. (12) as C(m) i,j is zero for j < S1 and j > S2. When S1 = 1 and S2 = N, where N is the total number of grid points employs to discretize the beam, Eq. (12) will reduce to the GDQ Eq. (6).

This paper presents two methods to select the value of S1 and S2 in Eq. (12). The first method uses 2M +1 grid points to approximate the derivative at a discrete point, i.e., a central scheme. This method generally produces a box-banded differential matrix and is referred to as a DQ box banded, i.e., DQ(BB). The second method uses M + 1 grid points to approximate the derivative at the boundaries, i.e., one side approximation. For discrete points further away from the boundaries, the number of grid points used in the approximation of its derivative will linearly increase to a maximum of 2M + 1 grid points. This method generally produces uniformly-banded differential matrix and is referred to as DQ uniformly banded, i.e., DQ(UB).
Thus, the value S1 and S2 for the approximation of the derivative at xi (see Eq. (12)) are

- DQ(BB)

$$S1 = \max(\min(i - M, N - 2M), 1), \qquad S2 = \min((\max(1 + 2M, i + M), N), \tag{19}$$

- DQ(UB)

$$S1 = \max(i - M, 1), \qquad S2 = \min(i + M, N), \tag{20}$$

where N is the number of grid points employed to discretize the beam, 1 ≤ i ≤ N and 2M+1 ≤ N. The DQ(BB) and DQ(UB) produce box-banded and banded matrix respectively while the GDQ method produces dense (full) differential matrix. Thus, the DQ(BB) and DQ(UB) have a distinct advantage over the GDQ in large scale computation. For 2M + 1 = N, the DQ(BB) reduces to the GDQ method.

Finally, the DQ(BB) and DQ(UB) approximations can be applied to Eq. (2) at each discrete point on the grid and the discretized governing equation at (xi) is given by

$$\sum_{j=1}^{N} C_{i,j}^{(4)} W_j = \Omega^2 W_i, \qquad i = 1, 2, \ldots, N. \tag{21}$$

The boundary conditions can be similarly discretized as

- At $x_1$
  - Simply-supported edge

$$W_1 = 0, \qquad \sum_{j=1}^{N} C_{1,j}^{(2)} W_j = 0, \tag{22}$$

  - Clamped edge

$$W_1 = 0, \qquad \sum_{j=1}^{N} C_{1,j}^{(1)} W_j = 0, \tag{23}$$

  - Free edge

$$\sum_{j=1}^{N} C_{1,j}^{(2)} W_j = 0, \qquad \sum_{j=1}^{N} C_{1,j}^{(3)} W_j = 0, \tag{24}$$

- At $x_N$
  - Simply-supported edge

$$W_N = 0, \qquad \sum_{j=1}^{N} C_{N,j}^{(2)} W_j = 0, \tag{25}$$

  - Clamped edge

$$W_N = 0, \qquad \sum_{j=1}^{N} C_{N,j}^{(1)} W_j = 0, \tag{26}$$

  - Free edge

$$\sum_{j=1}^{N} C_{N,j}^{(2)} W_j = 0, \qquad \sum_{j=1}^{N} C_{N,j}^{(3)} W_j = 0. \tag{27}$$

The boundary conditions are implemented by coupling all the boundary conditions with the discretized governing equations, a technique referred to as the CBCGE [14, 15, 16, 17, 18]. Taking an FF beam as an example, the boundary equations can be discretized as

$$\sum_{j=1}^{N} C_{1,j}^{(2)} W_j = 0, \tag{28}$$

$$\sum_{j=1}^{N} C_{1,j}^{(3)} W_j = 0, \tag{29}$$

$$\sum_{j=1}^{N} C_{N,j}^{(2)} W_j = 0, \tag{30}$$

$$\sum_{j=1}^{N} C_{N,j}^{(3)} W_j = 0. \tag{31}$$

Eqs. (28-31) can be expressed in the matrix form such that

$$B_B W^{(S)} + B_D W^{(I)} = 0, \tag{32}$$

where W(S) = [W1,W2,WN−1,WN]T , W(I) = [W3,W4, . . . ,WN−4,WN−2]T . BB and BD are 4×4 and 4 × (N − 4) matrix respectively. Similarly, Eq. (21) can be expressed in the matrix form

$$D_B W^{(S)} + D_D W^{(I)} = \Omega^2 W^{(I)}, \tag{33}$$

where $D_B$ and $D_D$ are N × 4 and N × N matrix respectively. Eq. (32) can be coupled with Eq. (33) to give

$$D_D W^{(I)} - D_B B_B^{-1} B_D W^{(I)} = \Omega^2 W^{(I)}, \tag{34}$$

$$(D_D - D_B B_B^{-1} B_D) W^{(I)} = \Omega^2 W^{(I)}. \tag{35}$$

Finally, Eq. (35) can be expressed as an eigenvalue and eigenvector problem

$$[A][W^{(I)}] = \Omega^2[W^{(I)}]. \tag{36}$$

The eigenvalue $\Omega^2$ can be obtained by solving Eq. (36) with a standard eigenvalue solver.

### *2.3. Local adaptive differential quadratural Methods*

In this work, we employ the local adaptive differential quadrature method (LaDQM) for beam vibration analysis and compare its performance with the aforementioned two DQ formulations. The LaDQM uses the Lagrange polynomial

$$L_k(x) = \begin{cases} \prod_{j=S1, j\neq k}^{S2} \dfrac{x - x_j}{x_k - x_j} & x_{S1} \leq x \leq x_{S2}, \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

where {xj} is a set of discrete points, S2 − S1 + 1 is the degree of the Lagrange polynomial. The value for S1 and S2 will be subsequently defined in Eq. (45).

The LaDQM can be implemented as a discrete convolution scheme such as the mth order derivative of a function f(x) on a grid point xi is approximated as

$$f^{(m)}(x_i) \approx \sum_{k=S1}^{S2} L_k^{(m)}(x_i) f(x_k). \tag{38}$$

Here, as for many other DSC kernels[25, 26, 51], the derivative kernel L(m) k (x) is obtained Analytically

$$L_k^{(m)}(x_i) = \begin{cases} C_{i,k}^{(m)}, & S1 \leq k \leq S2, \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

In the DSC algorithm, the domain of the definition of the system is extended with fictitious grid points so as to ensure "boundary condition" is well defined. It is well known that in the continuous case, a derivative at the boundary points exists if and only if both the left derivative and right derivative exist and are equal. Therefore, with a finite computational domain, a boundary condition involving differentiation, such as the Neumann type of boundary conditions needs special care. We handle this problem by extending the domain of definition of the system so that the "boundary condition" is appropriately accounted. At each boundary, 1 or 2 ghost points are extended from one side of the boundary. The number of ghost points extends from the boundary is equal to the number of Neumann boundary conditions at the boundaries. Let $\beta_1$ and $\beta_N$ be the number of Neumann type of boundary conditions at the boundary $x_1$ and $x_N$ respectively. Thus $\beta_1$ and $\beta_N$ are given as

- At the boundary $x_1$
  - Simply supported (S) or Clamped edge (C)

$$\beta_1 = 1, \tag{40}$$

  - Free edge (F)

$$\beta_1 = 2, \tag{41}$$

- At the boundary $x_N$
  - Simply supported (S) or Clamped edge (C)

$$\beta_N = 1, \tag{42}$$

  - Free edge (F)

$$\beta_N = 2. \tag{43}$$

Take note that the beam is discretized using grid points $\{x_1, x_2, \ldots, x_{N-1}, x_N\}$. The ghost points (which are not part of the beam) are $\{x_{1-\beta_1}, \ldots, x_0, x_{N+1}, \ldots, x_{N+\beta_N}\}$. Furthermore, we set $x_1 = 0$ and $x_N = 1$ as our governing equation for beam vibration Eq. (2) is a dimensionless equation.

Just as in the DQM, for simplicity, Eq. (38) is expressed as

$$f^{(m)}(x_i) \approx \sum_{k=1-\beta_1}^{N+\beta_N} L_k^{(m)}(x_i) f(x_k). \tag{44}$$

Similar to the DQ(BB) and DQ(UB), we have two ways to select the value of S1 and S2 in Eq. (37). These two methods are referred as LaDQM box banded (LaDQM(BB)) and LaDQM uniformly banded (LaDQM(UB)). Thus, the value S1 and S2 for approximation of the derivative at $x_i$ (see Eq. (38)) is given by

- LaDQM(BB)

$$S1 = \max(i - M, 1 - \beta_1), \qquad S2 = \min(S1 + 2M, N + \beta_N), \tag{45}$$

- LaDQM(UB)

$$S1 = \max(i - M, 1 - \beta_1), \qquad S2 = \min(i + M, N + \beta_N), \tag{46}$$

where $1 - \beta_1 \leq i \leq N + \beta_N$ and $2M + 1 \leq N + \beta_1 + \beta_N$. The LaDQM approximation can be finally applied to Eq. (2) at each discrete point on the grid and the discretized governing equation at (xi) is given by

$$\sum_{j=1-\beta_1}^{N+\beta_N} C_{i,j}^{(4)} W_j = \Omega^2 W_i, \qquad i = 1, 2, \ldots, N. \tag{47}$$

The boundary conditions can be similarly discretized as

- At $x_1$
  - Simply-supported edge

$$W_1 = 0, \qquad \sum_{j=1-\beta_1}^{N+\beta_N} C_{1,j}^{(2)} W_j = 0, \tag{48}$$

  - Clamped edge

$$W_1 = 0, \qquad \sum_{j=1-\beta_1}^{N+\beta_N} C_{1,j}^{(1)} W_j = 0, \tag{49}$$

  - Free edge

$$\sum_{j=1-\beta_1}^{N+\beta_N} C_{1,j}^{(2)} W_j = 0, \qquad \sum_{j=1-\beta_1}^{N+\beta_N} C_{1,j}^{(3)} W_j = 0, \tag{50}$$

- At $x_N$
  - Simply-supported edge

$$W_N = 0, \qquad \sum_{j=1-\beta_1}^{N+\beta_N} C_{N,j}^{(2)} W_j = 0, \tag{51}$$

  - Clamped edge

$$W_N = 0, \sum_{j=1-\beta_1}^{N+\beta_N} C_{N,j}^{(1)} W_j = 0, \tag{52}$$

  - Free edge

$$\sum_{j=1-\beta_1}^{N+\beta_N} C_{N,j}^{(2)} W_j = 0, \qquad \sum_{j=1-\beta_1}^{N+\beta_N} C_{N,j}^{(3)} W_j = 0. \tag{53}$$

The boundary conditions are implemented by the CBCGE[14, 15, 16, 17, 18]. However, unlike the DQM, the ghost points (instead of the interior points) are used to couple the boundary conditions with the governing equations. Taking the FF beam ($\beta_1 = \beta_N = 2$) as an example, the boundary equations can be discretized as

$$\sum_{j=1-2}^{N+2} C_{1,j}^{(2)} W_j = 0, \tag{54}$$

$$\sum_{j=1-2}^{N+2} C_{1,j}^{(3)} W_j = 0, \tag{55}$$

$$\sum_{j=1-2}^{N+2} C_{N,j}^{(2)} W_j = 0, \tag{56}$$

$$\sum_{j=1-2}^{N+2} C_{N,j}^{(3)} W_j = 0. \tag{57}$$

Eqs. (54-57) can be expressed in the matrix form such that

$$B_B W^{(G)} + B_D W^{(I)} = 0, \tag{58}$$

where W(G) are the ghost points ($W^{(G)} = [W_{-1}, W_0, W_{N+1}, W_{N+2}]^T$ ), $W^{(I)}$ are the interior points ($W^{(I)} = [W_1, W_2, \ldots, W_{N-1}, W_N]^T$ ). $B_B$ and $B_D$ are 4×4 and 4×N matrix, respectively. Similarly, Eq. (47) can be expressed in the matrix form

$$D_B W^{(G)} + D_D W^{(I)} = \Omega^2 W^{(I)}, \tag{59}$$

where $D_B$ and $D_D$ are N ×4 and N ×N matrix respectively. Eq. (58) can be coupled with Eq. (59) to give

$$\begin{aligned} D_D W^{(I)} - D_B B_B^{-1} B_D W^{(I)} &= \Omega^2 W^{(I)}, \qquad (60) \\ (D_D - D_B B_B^{-1} B_D) W^{(I)} &= \Omega^2 W^{(I)}. \qquad (61) \end{aligned}$$

Finally, Eq. (61) can be expressed as a problem of eigenvalue and eigenvector

$$[A][W^{(I)}] = \Omega^2 [W^{(I)}]. \tag{62}$$

The vector, $W^{(I)}$, for other supported beams are

$$[W^{(I)}] = \begin{cases} [W_2, \ldots, W_{N-1}]^T & \text{for SS, SC and CC beams,} \\ [W_2, \ldots, W_N]^T & \text{for SF, CF beams,} \\ [W_1, \ldots, W_N]^T & \text{for FF beam.} \end{cases} \tag{63}$$

The eigenvalue $\Omega^2$ can be obtained by solving Eq. (62) with a standard eigenvalue solver.

## 3. Results and discussions

Four different schemes, the DQ(BB), DQ(UB), LaDQM(BB) and LaDQM(UB) are implemented on both uniform grids and redistributed Chebyshev-Gauss-Lobatto grids[14, 15, 16, 17, 18]

- Uniform grids

$$x_i = \frac{i-1}{N-1} \qquad \text{for } i = 1, 2, \ldots, N. \tag{64}$$

- Redistributed Chebyshev-Gauss-Lobatto grids

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right] \qquad \text{for } i = 1, 2, \ldots, N. \tag{65}$$

For convenience of discussion, redistributed Chebyshev-Gauss-Lobatto grids are simply referred as Chebyshev grids. For the LaDQM(BB) and LaDQM(UB), there are two methods to extend the ghost grid points. In the first methods, ghost points are symmetrically extended (SymExt) from the boundary

$$x_{1+i} - x_1 = x_1 - x_{1-i} \qquad \text{for } 1 \leq i \leq \beta_1, \tag{66}$$

$$x_{N+i} - x_N = x_N - x_{N-i} \qquad \text{for } 1 \leq i \leq \beta_2. \tag{67}$$

In the second method, Chebyshev grids is constructed with $x_{1-\beta 1}$ to $x_{N+\beta 2}$ . Subsequently, a linear transformation is carried out to fulfil the requirement of $x_1 = 0$ and $x_N = 1$. This method of ghost points extension is referred to as linearly transformed extensions (LTExt). Thus

$$x_i = \frac{x^*_{i+\beta_1} - x^*_{1+\beta_1}}{x^*_{N+\beta_1} - x^*_{1+\beta_1}} \qquad \text{for } i = 1-\beta_1, \ldots, 1, \ldots, N, \ldots, N+\beta_2, \tag{68}$$

where

$$x^*_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{\eta-1}\pi\right)\right] \qquad \text{for } i = 1, 2, \ldots, \eta; \quad \eta = N + \beta_1 + \beta_2. \tag{69}$$

$$\tag{70}$$

In the following subsections, we compare the stability and accuracy of DQ(BB), DQ(UB), LaDQM(BB) and LaDQM(UB) methods. The number of real eigenvalues is used to gauge the stability of the FDDQM and LaDQM methods. For the convenience of discussion and comparison, the following terminology is defined for the frequency parameter $\Omega^2 = \omega L^2(\rho A/EI)^{1/2}$

$$\text{Error(DQ(BB))} = \left|\frac{\text{DQ(BB)'s result} - \text{Exact result}}{\text{Exact result}}\right| \times 100\% \tag{71}$$

$$\text{Error(DQ(UB))} = \left|\frac{\text{DQ(UB)'s result} - \text{Exact result}}{\text{Exact result}}\right| \times 100\% \tag{72}$$

$$\text{Error(LaDQM(BB))} = \left|\frac{\text{LaDQM(BB)'s result} - \text{Exact result}}{\text{Exact result}}\right| \times 100\% \tag{73}$$

$$\text{Error(LaDQM(UB))} = \left|\frac{\text{LaDQM(UB)'s result} - \text{Exact result}}{\text{Exact result}}\right| \times 100\% \tag{74}$$

$$\text{Error(SymExt)} = \left|\frac{\text{LaDQM's result(with SymExt)} - \text{Exact result}}{\text{Exact result}}\right| \times 100\% \tag{75}$$

$$\text{Error(TLExt)} = \left|\frac{\text{LaDQM's result(with TLExt)} - \text{Exact result}}{\text{Exact result}}\right| \times 100\% \tag{76}$$

$$\text{ErrorRatio(BB)} = \frac{\text{Error(DQ(BB))}}{\text{Error(LaDQM(BB))}} \tag{77}$$

### *3.1. Comparison study on different types of grids*

With reference to Fig 1, the number of real eigenvalues (which is used as a gauge for stability) for the DQ(BB) (implemented on both uniform grids and Chebyshev) grids reduces as M increases. It is also observed that the DQ(BB) implemented on Chebyshev grids is more stable than on uniform grids. Fig 1 shows that the number of real eigenvalues of the DQ(UB) for simply supported beam implemented on uniform grids also reduces as M increases. However, when the DQ(UB) is implemented on Chebyshev grids, the number of real eigenvalues remains constant as M increases. The above comparisons suggest that Chebyshev grids is more stable than uniform grids. In order to verify that Chebyshev grids are indeed more stable than uniform grids, the total number of real eigenvalues produced by 121 beams of different combinations of N and M (M = 5 to 15, N = 2M + 1 to 31) is tabulated in Table 1. From Table 1, it is observed that Chebyshev grids have more real eigenvalues than uniform grids for both the DQ(BB) and DQ(UB). Thus, we can conclude that Chebyshev grids are more stable than uniform grids. Fig 2 shows the log10 plot of absolute relative errors of the DQ(BB) and DQ(UB) for simply supported beam's fundamental (lowest) eigenfrequency. Figs 2a and 2b illustrate. Chebyshev grids are more accurate than uniform grids. Figs 2c and 2d further illustrate that Chebyshev grids are more accurate than uniform grids over a wide range of M (M = 5 to 15). Thus, we can conclude that Chebyshev grids are more accurate than uniform grids. This conclusion is expected as Shu had shown that Chebyshev grids help to enhance the stability and accuracy of the GDQ with a full matrix.

It is also observed in Figs 2c and 2d that the DQM implemented on uniform grids will diverge when M is too big. The DQ (BB) and DQ (UB), implemented on uniform grids, will diverge at M = 7 and M = 11 respectively. The divergence may due to the lower stability of the uniform grid. As stability decreases with M, it is expected that numerical results will diverge for a large M.

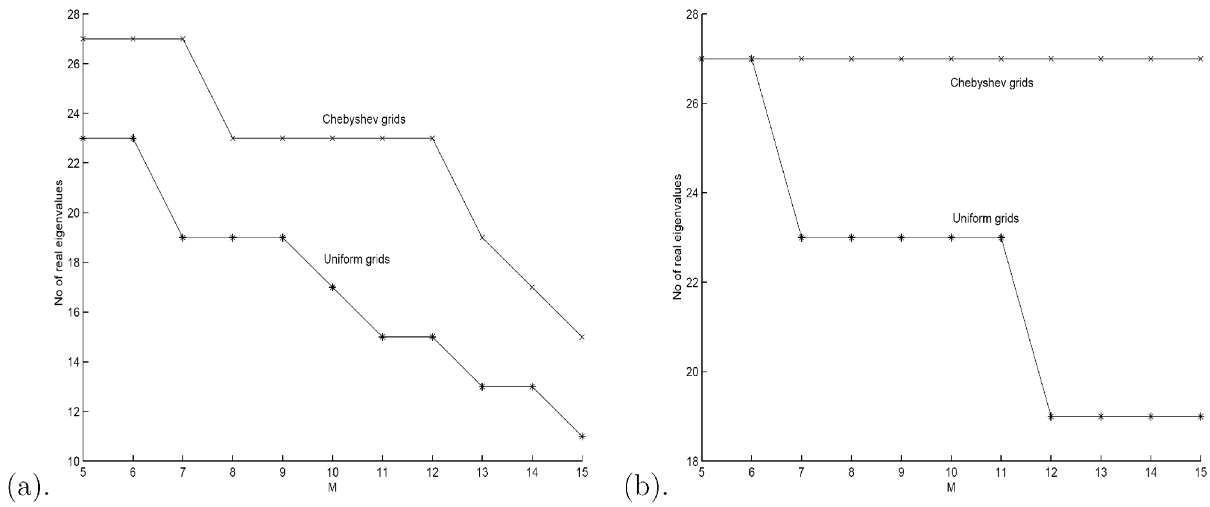


Fig. 1. Comparison of the stability of Chebyshev and uniform grids for an SS beam (N=31).
(a) DQ(BB) method; (b) DQ(UB) method.

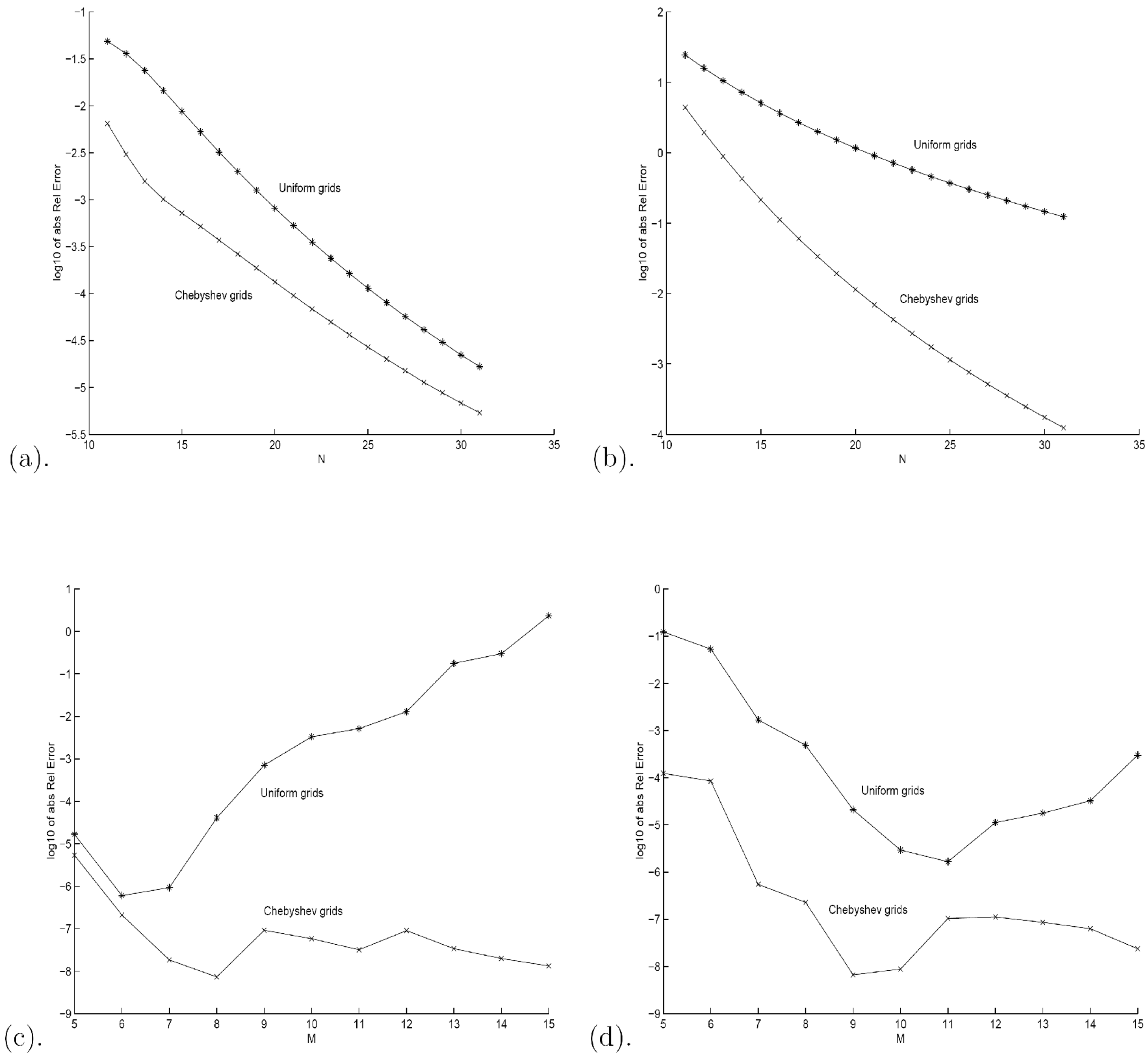


Fig. 2. Comparison of accuracy for 1 by Chebyshev and uniform grids for an SS beam. (a) M = 5, DQ(BB) method; (b) M = 5, DQ(UB) method; (c) N = 31, DQ(BB) method (d) N = 31, DQ (UB) method.

### *3.2. Comparison study on DQ(BB) and DQ(UB)*

The number of real eigenvalues produces by the DQ(BB) and DQ(UB) is shown in Table 1. It is observed that the DQ(UB) has more real eigenvalues than the DQ(BB). One possible explanation is that the DQ(UB) uses lower order approximations near the boundaries which will in turn help to enhance stability. Although the DQ(UB) is more stable, Fig 3a shows that the DQ(UB) is more accurate than DQ(BB) for only certain eigenmodes. For a more in-depth study on the accuracy of the DQ(BB) and DQ(UB), an extensive comparison for M = 5 to 15, N = 2M + 1 to 31 and all the eigenvalues with absolute relative error being less than 10% is conducted and tabulated in Table 2. We do not consider all the eigenvalues because if the absolute relative error is larger than 10%, it suggests that the method has diverged. Thus, it is not very meaningful to compare accuracy of diverged results. Table 2 shows that the DQ(BB) (implemented on Chebyshev grids) produces a total of 2018 real eigenvalues (for non zero eigenmodes) for a simply-supported beam. These 2018 eigenvalues are compared with the corresponding DQ(UB)'s results. Out of these 2018 eigenvalues, 1712 eigenvalues have at least DQ(BB)'s or DQ(UB)'s (or both) results being less than 10% error. It is found that the DQ(BB) is more accurate than the DQ(UB) for 1090 out of the 1712 eigenvalues. From Table 2, its observed that the DQ(BB) is generally more accurate than the DQ(UB) except for clamped beams. One possible explanation for the higher accuracy of the DQ(BB) is that the DQ(BB) uses a higher-order approximation (near the boundary) than the DQ(UB). The higher order approximation in turn helps the DQ(BB) to produce more accurate results.

Table 1. Comparison study on the stability of the FDDQM. For M = 5 to 15 and N = 2M + 1 to 31**.**

| Method | Grid | No of real eigenvalues | | | | | |
|---|---|---|---|---|---|---|---|
| | | SS | SC | SF | CC | CF | FF |
| DQ(BB) | Uniform | 1490 | 1476 | 1473 | 1482 | 1522 | 1455 |
| DQ(BB) | Chebyshev | 2018 | 2220 | 1887 | 2430 | 2170 | 1799 |
| DQ(UB) | Uniform | 2054 | 1998 | 2095 | 1962 | 2098 | 2111 |
| DQ(UB) | Chebyshev | 2442 | 2442 | 2373 | 2442 | 2434 | 2302 |

Table 2. Comparison study on the accuracy of the DQ(BB) and DQ(UB). For M = 5 to 15 and N = 2M + 1 to 31 on Chebyshev grids.

| | No of eigenvalues | | | | | |
|---|---|---|---|---|---|---|
| | SS | SC | SF | CC | CF | FF |
| Error(DQ(BB))<Error(DQ(UB)) | 1090 (64%) | 1072 (60%) | 1042 (66%) | 884 (50%) | 1135 (62%) | 1029 (69%) |
| No of eigenvalues used in comparsion | 1712 | 1786 | 1583 | 1762 | 1835 | 1496 |
| No of real eigenvalues for DQ(BB) | 2018 | 2220 | 1831 | 2430 | 2170 | 1674 |

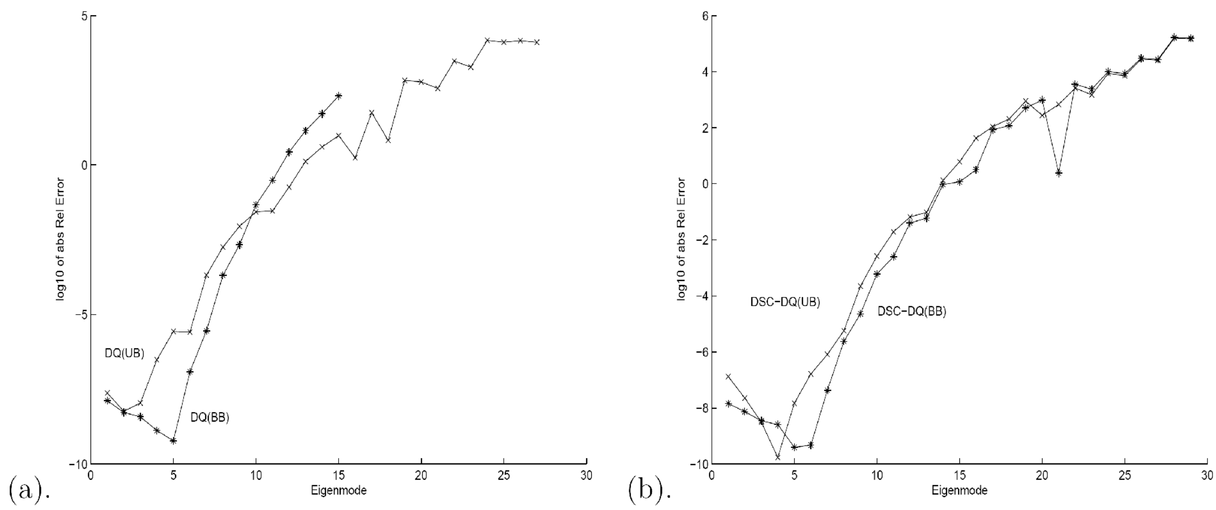


Fig. 3. Comparison of accuracy for an SS beam (N = 31, M = 31, Chebyshev grids). (a) DQ(BB) vs DQ(UB); (b) LaDQM(BB) vs LaDQM(UB).

### 3.3. Comparison study on different types of grids for LaDQM

For the LaDQM, Chebyshev grids produce all real eigenvalues for non-zeros eigenmodes while uniform grids produce some complex eigenvalues. Hence Chebyshev grids are clearly more stable than uniform grids. Figs 4a and 4b show the log10 plot of absolute relative errors for the LaDQM (BB) and LaDQM (UB) for simply-supported beam's fundamental (lowest) eigenfrequency. Both Figs 4a and 4b illustrate that Chebyshev (SymExt) grids are generally more accurate than uniform grids. Furthermore, as N increases, error of Chebyshev (SymExt) grids reduces more rapidly than that of uniform grids. Figs 4c and 4d further illustrate that Chebyshev(SymExt) grids are generally more accurate over a wide range of M (M = 5 to 15). Thus, we can conclude that Chebyshev (SymExt) grids are more accurate than uniform grids. This is the same conclusion as found in the DQ (BB) and DQ(UB). Similar to the FDDQM, it is observed from Figs 4c and 4d that the numerical results for uniform grids diverge when M is large. Furthermore, both the DQ (BB) and LaDQM(BB) diverge at M = 7. On the other hand, both the DQ (UB) and LaDQM (UB) diverge at M = 11. Thus, M should keep less than 7 when an uniform grid is used.

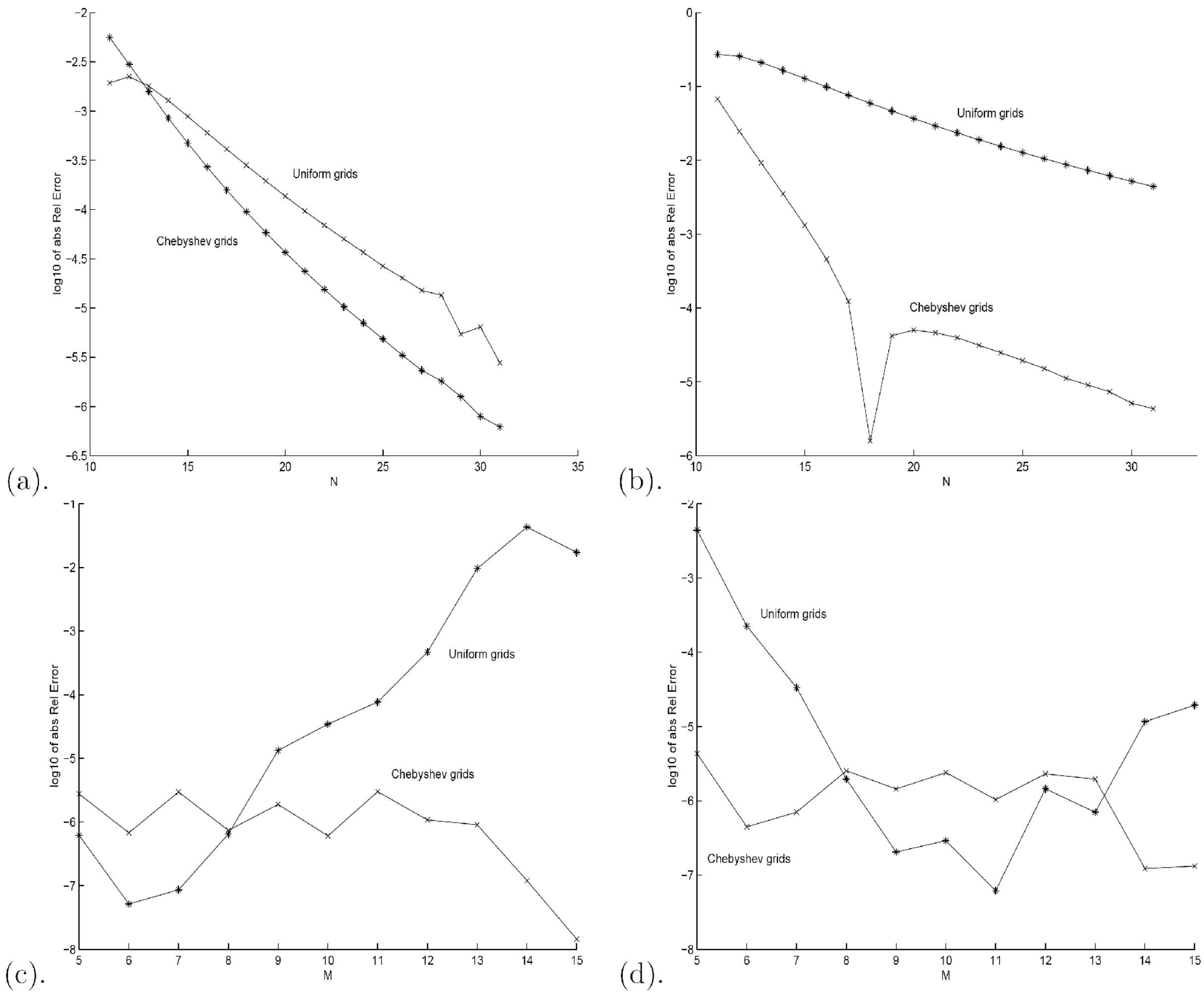


Fig. 4. Comparison of accuracy for 1 by Chebyshev and uniform grids for an SS beam. (a) M = 5, LaDQM(BB) method; (b) M = 5, LaDQM(UB) method; (c) N = 31, LaDQM(BB) method; (d) N = 31, LaDQM(UB) method.

### *3.4. Comparison study on LaDQM(BB) and LaDQM(UB)*

Fig 3b shows log10 plot of absolute relative errors for all the eigenfrequencies of simply-supported beam. It is observed that, for Chebyshev(SymExt) grids, the LaDQM(UB) and LaDQM(BB) have comparable accuracy. An extensive comparison tabulated in TABLE 3 further confirms the finding. For example, for clamped beam, the LaDQM(UB) is more accurate than the LaDQM(BB) for only 1015 (54%) out of 1791 eigenvalues. However, for a free-free supported beam, the LaDQM(UB) is more accurate than the LaDQM(BB) for 847 (47%) out of 1800 eigenvalues. Since 54% and 47% are statistic average over 121 different combinations of N, M and for all the eigenmodes, the LaDQM(BB) and LaDQM(UB) have comparable accuracy. This result is different from the FDDQM. The DQ(BB) has been found being more accurate than the DQ(UB), while the LaDQM(UB) and LaDQM(BB) have comparable accuracy.

### *3.5. Comparison study on Chebyshev(SymExt) and Chebyshev(LTExt) Grids*

With reference to TABLEs 4-5, LaDQM(BB) and LaDQM(UB) implemented on Chebyshev(LTExt) grids are more accurate than on Chebyshev(SymExt) grids. For example, for the LaDQM(BB), Chebyshev(LTExt) grids are more accurate than Chebyshev(SymExt) grids for 1876 (92%) out of 2048 eigenvalues. One possible explanation is that Chebyshev(LTExt) grids are more close to the nature of the orinial Chebyshev grids than Chebyshev(SymExt) grids. Further comparison between the LaDQM(BB) and LaDQM(UB) on Chebyshev(LTExt) grids shows that both methods have comparable accuracy (see TABLE 6). The LaDQM(UB) is more accurate for SC and CC beams while the LaDQM(BB) is more accurate for SS, SF, CF and FF beams. It is interesting to note that the LaDQM(UB) is more accurate for beams with strong boundary conditions (clamped edges) while LaDQM(BB) is more accurate for beams with weak boundary conditions (simply-supported and free edges).

### *3.6. Stability study on the LaDQM and FDDQM*

The LaDQM (Chebyshev grids with SymExt and Chebyshev grids with LTExt) produces all real eigenvalues for non-zeros eigenmodes while FDDQM (Chebyshev grids) produces some complex eigenvalues. Hence the LaDQM is clearly more stable than the FDDQM.

### *3.7. Accuracy study on the LaDQM and DQM*

TABLEs 7-8 show that the LaDQM is generally more accurate than the FDDQM (DQ(BB) and DQ(UB)) for 72% to 94% of the eigenvalues. For example, the LaDQM(BB) is more accurate than the DQ(BB) for 94% of the eigenvalues of a free-free supported beam. A more in-depth comparison study on accuracy is conducted for the LaDQM(BB) and DQ(UB). The ErrorRatio(BB) (see Eq. (77) for the definition) for the LaDQM(BB) and DQ(BB) implemented on Chebyshev(LTExt) grids is computed and its distribution is plotted in FIG 5, where the Error Ratio(BB) is divided into 38 bins which take the values of $-\inf$ to $10^{-4}$, $10^{-4}$ to $10^{-3.5}$, . . . , $10^{3.5}$ to $10^{4}$, $10^{4}$ to inf. It is observed from FIG 5 that most eigenvalues fall into the bin which corresponds to ErrorRatio = 100.25 to 100.5. Thus we can conclude that the LaDQM(BB) is 1.8 to 3.2 times more accurate than the DQ(BB).

Table 3. Comparison study on the accuarcy of the LaDQM(BB) and LaDQM(UB) implemented on Chebyshev(SymExt) grids. For M = 5 to 15 and N = 2M + 1 to 31.

| | No of eigenvalues | | | | | |
|---|---|---|---|---|---|---|
| | SS | SC | SF | CC | CF | FF |
| Error(LaDQM(UB))<Error(LaDQM(BB)) | 986 (54%) | 996 (55%) | 957 (53%) | 1015 (57%) | 1051 (55%) | 847 (47%) |
| No of eigenvalues used in comparsion | 1836 | 1796 | 1791 | 1780 | 1914 | 1800 |
| Total No of real eigenvalues | 2684 | 2684 | 2684 | 2684 | 2805 | 2684 |

Table 4. Comparison study on the accuracy of the LaDQM(UB) implemented on Chebyshev(SymExt) and Chebyshev(TLExt) grids. For M = 5 to 15 and N = 2M + 1 to 31.

| | No of eigenvalues | | | | | |
|---|---|---|---|---|---|---|
| | SS | SC | SF | CC | CF | FF |
| Error(TLExt)<Error(SymExt) | 1538 (78%) | 1576 (81%) | 1514 (76%) | 1670 (87%) | 1604 (76%) | 1579 (77%) |
| No of eigenvalues used in comparsion | 1977 | 1941 | 1993 | 1911 | 2104 | 2057 |
| Total No of real eigenvalues | 2684 | 2684 | 2684 | 2684 | 2805 | 2684 |

Table 5. Comparison study on th accuracy of the LaDQM(BB) implemented on Chebyshev(SymExt) and Chebyshev(TLExt) grids. For M = 5 to 15 and N = 2M + 1 to 31.

| | No of eigenvalues | | | | | |
|---|---|---|---|---|---|---|
| | SS | SC | SF | CC | CF | FF |
| Error(TLExt)<Error(SymExt) | 1738 (88%) | 1711 (88%) | 1803 (91%) | 1742 (91%) | 1930 (92%) | 1876 (92%) |
| No of eigenvalues used in comparsion | 1979 | 1944 | 1984 | 1921 | 2091 | 2048 |
| Total No of real eigenvalues | 2684 | 2684 | 2684 | 2684 | 2805 | 2684 |

Table 6. Comparison study on the accuracy of the LaDQM(BB) and LaDQM(UB) implemented on Chebyshev(TLExt) grids. For M = 5 to 15 and N = 2M + 1 to 31.

| | No of eigenvalues | | | | | |
|---|---|---|---|---|---|---|
| | SS | SC | SF | CC | CF | FF |
| Error(LaDQM(UB))<Error(LaDQM(BB)) | 954 (49%) | 1003 (52%) | 638 (33%) | 977 (52%) | 677 (33%) | 647 (32%) |
| No of eigenvalues used in comparsion | 1953 | 1928 | 1957 | 1893 | 2053 | 1991 |
| Total No of real eigenvalues | 2684 | 2684 | 2684 | 2684 | 2805 | 2684 |

Table 7. Comparison study on the accuracy of the LaDQM and DQ(BB) implemented on Chebyshev(TLExt) grids. For M = 5 to 15 and N = 2M + 1 to 31.

| | No of eigenvalues | | | | | |
|---|---|---|---|---|---|---|
| | SS | SC | SF | CC | CF | FF |
| Error(DQ(BB))>Error(LaDQM(BB)) | 1544 (85%) | 1666 (85%) | 1548 (93%) | 1623 (82%) | 1819 (93%) | 1492 (94%) |
| Error(DQ(BB))>Error(LaDQM(UB)) | 1308 (72%) | 1500 (77%) | 1298 (79%) | 1524 (77%) | 1553 (79%) | 1277 (80%) |
| No of eigenvalues used in comparsion | 1812 | 1952 | 1663 | 1968 | 1956 | 1588 |
| Total No of real eigenvalues | 2018 | 2220 | 1831 | 2430 | 2170 | 1674 |

Table 8. Comparsion study on the accuarcy of the LaDQM and DQ(UB) implemented on Chebyshev(TLExt) grids. For M = 5 to 15 and N = 2M + 1 to 31.

| | No of eigenvalues | | | | | |
|---|---|---|---|---|---|---|
| | SS | SC | SF | CC | CF | FF |
| Error(DQ(UB))>Error(LaDQM(BB)) | 1765 (87%) | 1733 (87%) | 1798 (88%) | 1653 (84%) | 1903 (90%) | 1817 (90%) |
| Error(DQ(UB))>Error(LaDQM(UB)) | 1739 (86%) | 1696 (85%) | 1778 (88%) | 1621 (83%) | 1874 (88%) | 1809 (90%) |
| No of eigenvalues used in comparsion | 1812 | 1952 | 1663 | 1968 | 1956 | 1588 |
| Total No of real eigenvalues | 2442 | 2442 | 2313 | 2442 | 2434 | 2184 |

Table 9. Comparison study on the first non-zero eigenfrequency parameter $\sqrt{\Omega}$ for the FF beam (N = 23). The exact solutions are 9.869604401089358 (SS), 15.418205716980061 (SC), 15.418205716980061 (SF), 22.373285448061324 (CC), 3.5160152685001513 (CF) and 22.373285448061324 (FF).

| BC | M | DQ(BB) with Chebyshev Grid | DQ(UB) with Chebyshev Grid | LaDQM(BB) with uniform Grid | LaDQM(BB) with Chebyshev (TLExt) Grid | LaDQM(UB) with Chebyshev (TLExt) Grid |
|---|---|---|---|---|---|---|
| SS | 8 | 9.869604401086 | 9.869604385355 | 9.869604400364 | 9.869604401119 | 9.869604401076 |
| | 9 | 9.869604401085 | 9.869604400380 | 9.869604393452 | 9.869604401060 | 9.869604401000 |
| | 10 | 9.869604401084 | 9.869604401286 | 9.869604420845 | 9.869604401091 | 9.869604401098 |
| | 11 | 9.869604401087 | 9.869604401119 | 9.869604434995 | 9.869604401095 | 9.869604401092 |
| SC | 8 | 15.418205716982 | 15.418205668485 | 15.418205717532 | 15.418205716821 | 15.418205716926 |
| | 9 | 15.418205716993 | 15.418205713870 | 15.418205709111 | 15.418205716982 | 15.418205716989 |
| | 10 | 15.418205716992 | 15.418205717886 | 15.418205718079 | 15.418205717000 | 15.418205716969 |
| | 11 | 15.418205716979 | 15.418205717098 | 15.418205524910 | 15.418205717056 | 15.418205716983 |
| SF | 8 | 15.418205716809 | 15.418211693395 | 15.418205716082 | 15.418205716922 | 15.418205717832 |
| | 9 | 15.418205716047 | 15.418205800665 | 15.418205717509 | 15.418205716919 | 15.418205716888 |
| | 10 | 15.418205716973 | 15.418205662754 | 15.418205732451 | 15.418205716844 | 15.418205716960 |
| | 11 | 15.418205716935 | 15.418205725202 | 15.418205811727 | 15.418205717048 | 15.418205716939 |
| CC | 8 | 22.373285448071 | 22.373285445696 | 22.373285447981 | 22.373285448057 | 22.373285447941 |
| | 9 | 22.373285448065 | 22.373285448561 | 22.373285447880 | 22.373285448059 | 22.373285448015 |
| | 10 | 22.373285448059 | 22.373285448251 | 22.373285446795 | 22.373285448061 | 22.373285448066 |
| | 11 | 22.373285448064 | 22.373285448062 | 22.373285449527 | 22.373285448061 | 22.373285448069 |
| CF | 8 | 3.516015267751 | 3.516015293763 | 3.516015268342 | 3.516015268743 | 3.516015268738 |
| | 9 | 3.516015264369 | 3.516015269586 | 3.516015269775 | 3.516015268349 | 3.516015268529 |
| | 10 | 3.516015267364 | 3.516015268360 | 3.516015261484 | 3.516015267751 | 3.516015268387 |
| | 11 | 3.516015275348 | 3.516015270149 | 3.516015274731 | 3.516015268260 | 3.516015268715 |
| FF | 8 | 22.373285448377 | 22.373328504491 | 22.373285447900 | 22.373285447972 | 22.373285455111 |
| | 9 | 22.373285447751 | 22.373286514890 | 22.373285447681 | 22.373285448107 | 22.373285447226 |
| | 10 | 22.373285448453 | 22.373284797001 | 22.373285437449 | 22.373285448161 | 22.373285447611 |
| | 11 | 22.373285448348 | 22.373285514269 | 22.373285422534 | 22.373285448010 | 22.373285448083 |

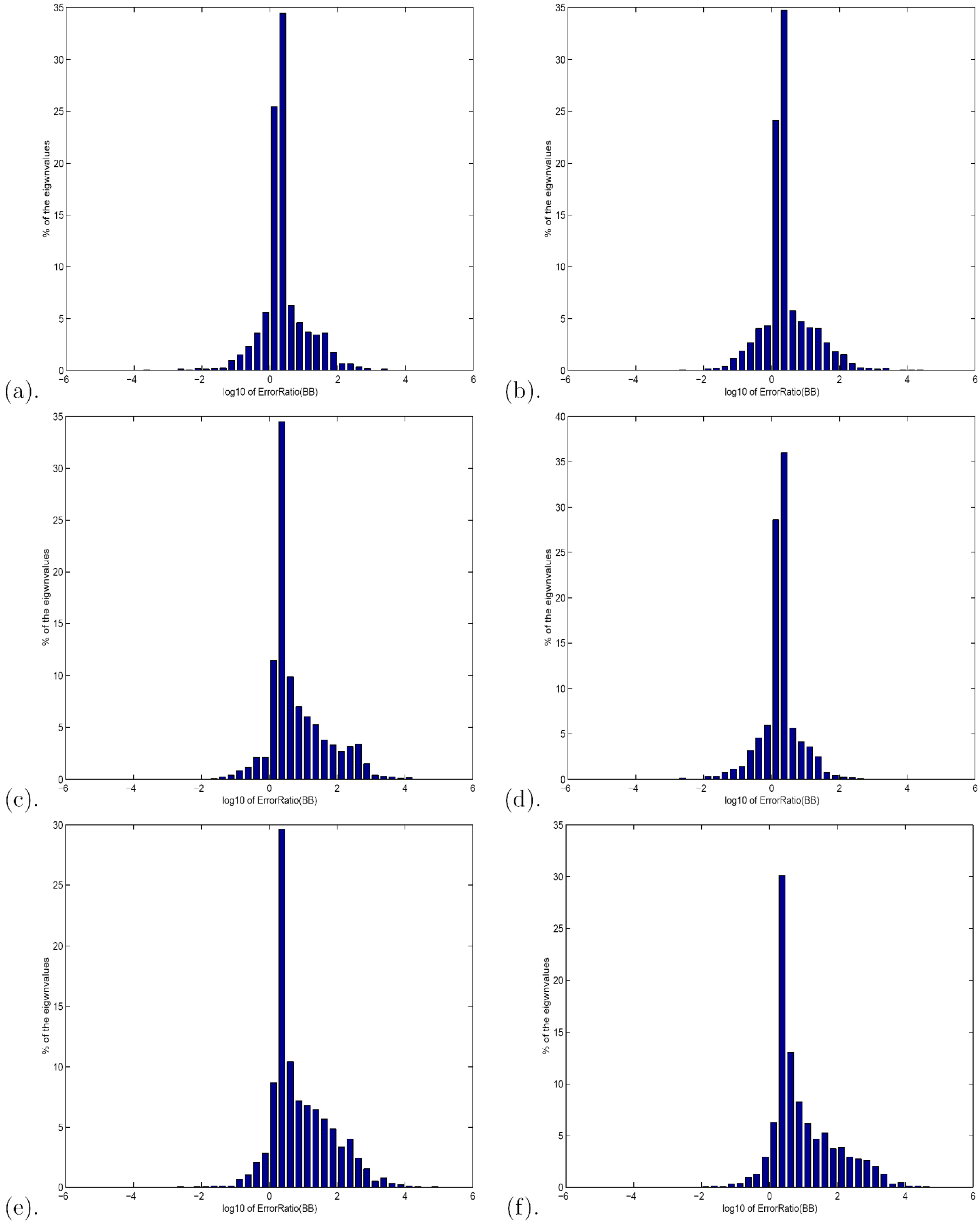


Fig. 5. Comparison of accuracy for the LaDQM(BB) and DQ(BB) (Chebyshev(LTExt) grids). (a) SS beam; (b) SC beam; (c) SF beam; (d) CC beam; (e) CF beam; (f) FF beam.

## 4. Conclusion

Differential quadrature methods (DQMs) become increasingly popular, but are apt to the stability problem, i.e., admitting spurious complex eigenvalues. This paper addresses this issue by presenting a comprehensive comparison among a variety of different local DQM formulations generated by varying the treatment of boundary conditions, the length of stencils and the distribution of grid meshes. The DQM that utilizes fictitious grids and banded matrices, termed local adaptive differential quadrature method (LaDQM) is carefully compared with the DQM that employs banded matrices without using fictitious grids, called finite difference differential quadrature methods (FDDQMs). These local DQMs include the classic global DQMs as special cases given by extending their banded matrices to full matrices.

To qualitatively analyze the performance of the LaDQM and FDDQM, both relative errors and number of real eigenvalues are compared. Uniformly banded (UB) and box banded (BB) matrices are formed for both methods. Two types of grids, uniform and Chebyshev-Gauss-Lobatto grids, are implemented. For the LaDQM, two types of fictitious grids, symmetrically extended (SymExt) and linearly transformed extension (LTExt) are performed. Numerical experiments show that both the FDDQM and LaDQM implemented on Chebyshev grids are more stable and accurate than on uniform grids. Furthermore, with Chebyshev(LTExt) grids, the LaDQM(BB) is generally from 1.8 to 3.2 times more accurate than the DQ(BB). Despite of the similarity between the LaDQM and FDDQM, the LaDQM implemented on Chebyshev (TLExt) grids is found to be more stable and accurate than LaDQM(BB) implemented on uniform grids (equivalence to FDDQM). We believe that the present findings shed light on numerical accuracy and stability of all DQMs.

**Acknowledgement** : This work was supported in part by the National University of Singapore, and the University of Western Sydney. The work of Wei was supported by NSF grant DMS-0616704 and NIH grant CA-127189

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