International Journal of Engineering and Applied Sciences (IJEAS)
Vol.2, Issue 3(2010)26-39

SEMI-ANALYTICAL SOLUTION FOR THE STATIC ANALYSIS OF FUNCTIONALLY GRADED BEAMS RESTING ON TWO PARAMETER ELASTIC FOUNDATION

***S.S.Malihi* [1], *A.Behravan Rad* [1], *F.Nazari* [2]**
[1]*Mech.Eng.Dep. Azad University Karaj*
[2]*Mech.Eng.Dep. Buali Sina University*

Accepted Date:08 March 2010

Abstract

Two-dimensional elasticity solution is presented for static analysis of functionally graded beams with various end conditions and resting on elastic foundation, using the semi-analytical approach, which makes use of the state space method and differential quadrature method. The beams are assumed to be transversely isotropic, with Young's modulus varying exponentially along the thickness, while Poisson's ratio remaining constant. The state space method (SSM) is adopted to obtain analytically the thickness variation of the elastic field and, approximate solution in the longitudinal direction can be obtained using the one dimensional differential quadrature method (DQM). The convergence and accuracy of the present approach is then validated by comparing the numerical results with the exact solutions for the case of simply support functionally graded beam. The influence of material gradient index, coefficient of elastic foundation and the ratio of thickness to length on the behavior of functionally graded beams are finally investigated.

Keywords: FGM beams, Semi-analytical, State space, Elastic foundation, Differential quadrature

Nomenclatures

L, h	beam dimension in x and z directions
E_0, E_h	Young's modulus at the bottom and upper surfaces respectively
n	number of half wave in x direction
I	the second moment of the cross-sectional area
k_w , k_p	Winkler and Pasternak coefficient of elastic foundation respectively
N	number of sampling points
U, W	displacement in x- and z- direction respectively
γ_{zx}	shear strains
σ_i (i=x, z)	normal stress
ε_i (i=x, z)	normal strain
τ_{xz}	shear stresses
δ	state variables

1. Introduction

Analysis of deformation and stress fields in functionally graded materials (FGM) is of fundamental importance in experimental determination of the FGM properties and exact solutions are useful in developing a numerical model. Functionally graded materials possess smooth spatial variations of thermo-mechanical properties which can be made such that the volume fractions of two or more materials are varied continuously along a certain dimension. FGMs are anisotropic in nature. Exact analysis of their elastic responses should be based on the theory of anisotropic elasticity. Suresh and Mortensen [1] provide an excellent introduction to the fundamentals of FGMs. As the use of FGMs increases, for example, in aerospace, automotive and biomedical applications, new methodologies have to be developed to characterize FGMs, and also to design and analyze structural components made of these materials. Reddy [2], in 2000, presented a theoretical formulation and finite element models based on third-order shear deformation theory for the analysis of through-thickness functionally graded plates. The Navier solution for simply supported plates based on the linear third-order theory and the non-linear static and dynamic finite element results based on the first-order theory were presented by Reddy. Sankar [3] established a functionally graded Euler–Bernoulli beam model to treat a static problem of a simply supported beam. Employing the finite element method, Reddy et al. [4] studied thermo-elastic effect and wave propagation in FG beams. Zhu and Sankar [5] solved the two-dimensional elasticity equations for a FGM beam subjected to transverse loads by means of combined Fourier series-Galerkin method, in which the variation of the Young's modulus through the thickness was given by a polynomial in the thickness-coordinate and Poisson's ratio was assumed to be constant. An exact analysis based on state space formulation is presented by Bian [6] to study functionally graded beams integrated with surface piezoelectric actuators and sensors. The free vibration and bending analysis of such structures has been extensively covered by many investigators. Ding et al. [7] derived an elasticity solution for a fixed–fixed plane isotropic beam subjected to uniform load with the aid of Airy stress function. An elasticity solution for a fixed–simply supported plane isotropic beam subjected to uniform load was also presented in [7]. A variety of numerical methods have been proposed to solve problems encountered in engineering and science [10-18]. Among them, the differential quadrature method (DQM), has been widely and successfully applied in many areas. The applications of DQM to the static and dynamic analyses of beams and plates proved that it is a rather efficient numerical technique for analyzing various problems. Chen et al. [8] presented elasticity solution for bending and thermal deformations of FG beams with various end conditions, using the state space method coupled with differential quadrature method. Free and forced vibration of a thermally prestressed, laminated functionally graded beam of variable thickness were investigated by Xiang and Yang [9]. Exact solutions for bending and free vibration of functionally graded beams resting on a Winkler-Pasternak elastic foundation were presented by Ying et al. [10] based on the tow-dimensional theory of elasticity. As the aforementioned works show, the exact solution for FG beams subjected to mechanical load with non-simply support boundary conditions has not been yet considered and the present work attempts to do this. In this paper, the conventional state space method is successfully combined with the differential quadrature method (DQM) and thus a semi-analytical elasticity method is developed and then elasticity solution of FG beam with arbitrary edges under pressure is presented. FG beam is rested on Winkler-Pasternak elastic foundation. Material property of FGM beam is assumed to be graded in the thickness direction according to a simple exponent-law distribution in terms of the volume fractions of the constituents.

2. Basic Equations

Consider functionally graded beam with length L, and thickness h, as shown in Fig.1.

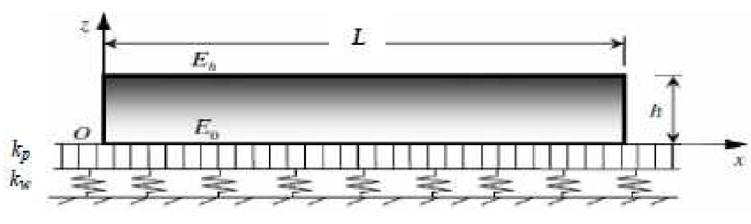


Fig.1 .Functionally graded beam on tow parameter elastic foundation

The beam is assumed in a state of plane stress and rested on two parameter elastic foundation with the foundation module of k_w and k_p. The end boundary conditions are various and the top surface of beam is subjected to uniform pressure, q_0.
The constitutive relations of FG beam in term of displacements can be written as:

$$\sigma_x = c_{11}\frac{\partial u}{\partial x} + c_{13}\frac{\partial w}{\partial z} \quad \sigma_z = c_{13}\frac{\partial u}{\partial x} + c_{11}\frac{\partial w}{\partial z}$$
$$\tau_{xz} = c_{55}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \tag{1}$$

Where σ_i and τ_{xz} are the normal and shear stress components, respectively, and u and w the displacement components.
And the elastic constants for isotropic materials are defined as:
$c_{11} = c_{33} = E/(1-\nu^2),\ \ c_{13} = \nu c_{11},\ \ c_{55} = E/2(1+\nu)$
In the absence of body forces, equilibrium equations can be written as:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{2}$$

The FGM is assumed transversely isotropic with constant Poisson's ratio, ν and all elastic constants and mass density are assumed to vary exponentially through the beam thickness, that is:

$$G = G_0 e^{kz} \tag{3}$$

Where $k = h^{-1}\ln\frac{G_h}{G_0}$ and the subscript ‘0’ and ‘h’ denote the values at the bottom and top surface of the beam. By using Eqs. (1) - (3), following state space equation are obtained;

$$\frac{\partial \sigma_z}{\partial z} = -k\sigma_z - \frac{\partial \tau_{xz}}{\partial x}$$
$$\frac{\partial U}{\partial z} = -\frac{\partial W}{\partial x} + \frac{2(1+\nu)}{E_0}\tau_{xz}$$
$$\frac{\partial W}{\partial z} = \frac{1-\nu^2}{E_0}\sigma_z - \nu\frac{\partial U}{\partial x} \tag{4}$$
$$\frac{\partial \tau_{xz}}{\partial z} = -\nu\frac{\partial \sigma_z}{\partial x} - E_0\frac{\partial^2 U}{\partial x^2} - k\tau_{xz}$$

Induced variables for the beam in term of state variables are as follow

$$\sigma_x = \nu\sigma_z + E_0\frac{\partial U}{\partial x} \tag{5}$$

Eqs. (4) can be written in matrix form, as follow :

$$\frac{d}{dz}\begin{Bmatrix}\sigma_z\\ U\\ W\\ \tau_{xz}\end{Bmatrix}=\begin{bmatrix}-k & 0 & 0 & -\frac{\partial}{\partial x}\\ 0 & 0 & -\frac{\partial}{\partial x} & \frac{2(1+\nu)}{E_0}\\ \frac{1-\nu^2}{E_0} & -\nu\frac{\partial}{\partial x} & 0 & 0\\ -\nu\frac{\partial}{\partial x} & -E_0\frac{\partial^2}{\partial x^2} & 0 & -k\end{bmatrix}\begin{Bmatrix}\sigma_z\\ U\\ W\\ \tau_{xz}\end{Bmatrix} \tag{6}$$

The displacements and stresses boundary conditions are;

$$\sigma_z = k_w w - k_p\frac{\partial^2 w}{\partial x^2},\ \ \tau_{xz}=0 \quad \text{at} \quad z=0 \tag{7a}$$

$$\sigma_z = q_0\ ,\ \tau_{xz}=0 \quad \text{at} \quad z=h \tag{7b}$$

The two ends of the beam (x = 0 and x = L) are subjected to any combinations of the following boundary conditions,

Simply supported (S): $\sigma_x = 0$; w = 0;
Clamped (C): u = 0; w = 0;
Free (F): $\sigma_x = 0$; $\tau_{xz} = 0$.

The dimensionless moduli are $K_w=\frac{k_w L^4}{E_0 I}$, $K_p=\frac{k_p L^2}{E_0 I}$ that refer to Winkler and Pasternak module, respectively.

3. Analytical solution

In order to satisfy the simply supported boundary conditions, displacements and stresses components are given as the following

$$\sigma_x=\bar{\sigma}_x\sin\left(\frac{n\pi x}{L}\right)e^{kz}\qquad \sigma_z=\bar{\sigma}_z\sin\left(\frac{n\pi x}{L}\right)e^{kz}\qquad \tau_{xz}=\bar{\tau}_{xz}\cos\left(\frac{n\pi x}{L}\right)e^{kz}$$
$$u=\bar{U}\cos\left(\frac{n\pi x}{L}\right)\qquad w=\bar{W}\sin\left(\frac{n\pi x}{L}\right) \tag{8}$$

Where quantities with a over bar are termed as the state variables and undetermined function of z coordinate and 'n' is the half-wave number.
Substituting relations (3) and (8) into the Eqs. (1) and (2) leads to the following state-space equations

$$\frac{d\delta}{dz}=A\delta \tag{9}$$

Where A is constant coefficients (see Appendix).
Also the induced variables in term of state variables can be obtained as

$$\bar{\sigma}_x = \nu \bar{\sigma}_z - E_0 \frac{n\pi}{L} \bar{U} \tag{10}$$

General solution to Eq. (9) is

$$\delta = \exp(zA)\delta_0 \quad \text{at} \quad 0 \le z \le h \tag{11}$$

Eq. (11) at z = h yields

$$\delta = \exp(hA)\delta_0 \tag{12}$$

Imposing surface traction at the low and top surface of the beam (Eq. (7a)) to Eq. (12), following equation can be obtained

$$\begin{Bmatrix} \sigma_z \\ U \\ W \\ \tau_{xz} \end{Bmatrix}_h = \begin{bmatrix} a_{12} & \beta a_{11} + a_{13} & a_{14} \\ a_{22} & \beta a_{21} + a_{23} & a_{24} \\ a_{32} & \beta a_{31} + a_{33} & a_{34} \\ a_{42} & \beta a_{41} + a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} U \\ W \\ \tau_{xz} \end{Bmatrix}_0 \tag{13}$$

Where; β = k_w + $\left(\frac{n\pi}{L}\right)^2$ k_p and s_{ij} are the element of '$\exp(hA)$' matrix.

The first and fourth equations of matrix Eq. (13) yields

$$\begin{Bmatrix} \sigma_z \\ \tau_{xz} \end{Bmatrix}_h = \begin{bmatrix} a_{12} & \beta a_{11} + a_{13} \\ a_{42} & \beta a_{41} + a_{43} \end{bmatrix} \begin{Bmatrix} U \\ W \end{Bmatrix}_0 \tag{14}$$

By solving Eq. (14), displacement at the lower surface can be obtained. Once the displacement components at the lower surface of beam are obtained, the state vectors at any coordinate z can be derived from Eq. (11). Finally, inserting the obtained state variables into the induced variable, Eq. (10), axial normal stress can be obtained.

4. Semi-analytical solution

There isn't any exact solution for beams with non-simply support boundary conditions. Differential qudrature method is used to solve partial differential equations for non-simply support boundary conditions. A Semi-analytical procedure with the aids of DQ technique was developed by Chen et al. [11]. In this method, the r_{th}-order partial derivative of a continuous function f(x,z) with respect to x at a given point x_i can be approximated as a linear sum of weighted function values at all of the discrete points in the domain of x, i.e.

$$\left.\frac{\partial^n f(x_i, z)}{\partial x^n}\right|_{x=x_i} = \sum_{r=1}^{N} g_{ir}^{(n)} f(x_r, z) \quad (n = 1,2,...,N-1\,,\; i = 1,2,...,N) \tag{15}$$

Where $g_{ij}^{(n)}$ are the x_i-dependent weight coefficients [12].

Applying Eq. (15) to Eqs. (4) – (5), following state equations at an arbitrary sampling point x_j in the FGM beam are then obtained

$$
\begin{aligned}
\frac{d\sigma_{zi}}{dz} &= -k\sigma_z - \sum_{r=1}^{N} g_{ir}^{(1)}\tau_{xzr} \\
\frac{dU_i}{dz} &= -\sum_{r=1}^{N} g_{ir}^{(1)}W_r + \frac{2(1+\nu)}{E_0}\tau_{xzi} \\
\frac{dW_i}{dz} &= \frac{1-\nu^2}{E_0}\sigma_{zi} - \nu\sum_{r=1}^{N} g_{ir}^{(1)}U_r \\
\frac{d\tau_{xzi}}{dz} &= -\nu\sum_{r=1}^{N} g_{ir}^{(1)}\sigma_{zr} - E_0\sum_{r=1}^{N} g_{ir}^{(2)}U_r - k\,\tau_{xzi}
\end{aligned}
\tag{16}
$$

Where quantities with subscript 'i' means the function value at grid point.
Similarly the induced variable, Eq. (5), is:

$$
\sigma_{xi} = \nu\sigma_{zi} + E_0\sum_{r=1}^{N} g_{ir}^{(1)}U_{ri} \tag{17}
$$

Assembly of Eq. (15) at all sampling points leads to the following global state equation in matrix form:

$$
\frac{d}{dz}\delta = A\delta \tag{18}
$$

Where $\delta = [\sigma_z , u , w , \tau_{xz}]^T$ and A is defined in Appendix and other sub-vectors in Eq. (18) are defined in the same manner. After applying the boundary conditions, Eq. (18) becomes:

$$
\frac{d}{dz}\delta_b = A_b\delta_b \tag{19}
$$

Where, the subscript, b denotes that the state equation contains the boundary conditions and the matrix A_b according to each boundary condition type is given in Appendix. Applying the same procedure, used in Eq. (9) to Eq. (19), stresses and displacements due to static loading are obtained.

5. Numerical results and discussion

In this section, convergence of DQ method and Effects of edge boundary conditions, the foundation parameters, aspect ratio and gradient index on mechanical behavior FGM beams and finally mechanical parameter in two directions are investigated.

5.1 Convergence and accuracy

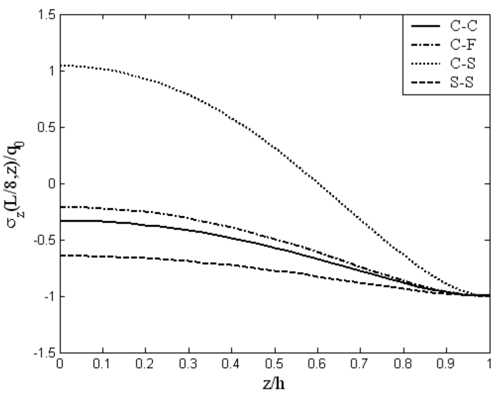
Numerical results are obtained for FGM beam with simply supported edges condition and resting on two parameter (Winkler-Pasternak) elastic foundation and under uniform pressure on the top surface and for validating the convergence and accuracy of the present method, compared with analytical solution (Eq.11). The results of comparison are shown in Table.1. From this table, it is observed that by increasing the number of discrete grid points, the computed results converge rapidly without any discrepancies with that reported by Ying et al. [10]. Also from the table 1, it can be seen that the numerical solution of DQM using only a few discrete grid points is equivalent to the analytical solution.

Table 1. Convergence for the FGM beam, SS, $K_w = 0$, $K_p = 0$

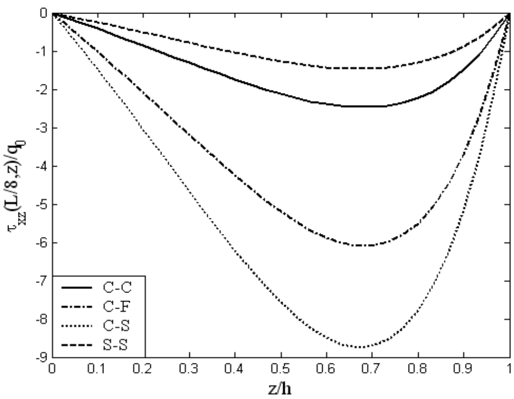
		N = 5	N = 7	N = 9	Analytical(Ref[10])
L/h = 10	σ_z	-0.357	-0.356	-0.356	-0.356
	W	-0.0041	-0.0041	-0.0041	-0.0041
	τ_{xz}	-4.38	-4.42	-4.45	-4.45
	σ_x	-20.57	-20.63	-20.68	-20.68
L/h = 20	σ_z	-0.357	-0.356	-0.356	-0.356
	W	-0.0648	-0.0648	-0.0648	-0.0648
	τ_{xz}	-8.84	-8.89	-8.91	-8.91
	σ_x	-81.79	-81.87	-81.91	-81.91

5.2 Edge boundary conditions

Effect of edges boundary conditions are depicted in Figs.2. As the figures show distribution of transverse normal and shear stresses in CC and CF condition lay between the related distribution for the SS and CS conditions, but curves of axial normal stress for the SS and CC conditions lays between the curves of CF and CS conditions. Also the effect of CS condition in rate of variations transverse normal and shear stresses is greater than the other conditions.

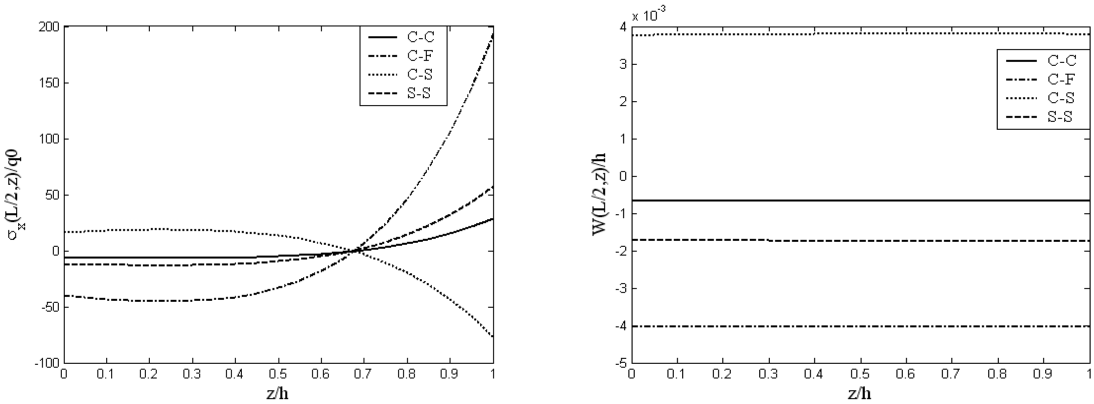


a. Transverse normal stress



b. Transverse shear stress

Influence of edges conditions in axial stress near the outer surface is much more than the lower position in thickness direction, but for transverse normal stress near the lower surface, it is much more than the outer surface. Distribution of transverse displacement across the thickness in contrast with the other quantities is constant.

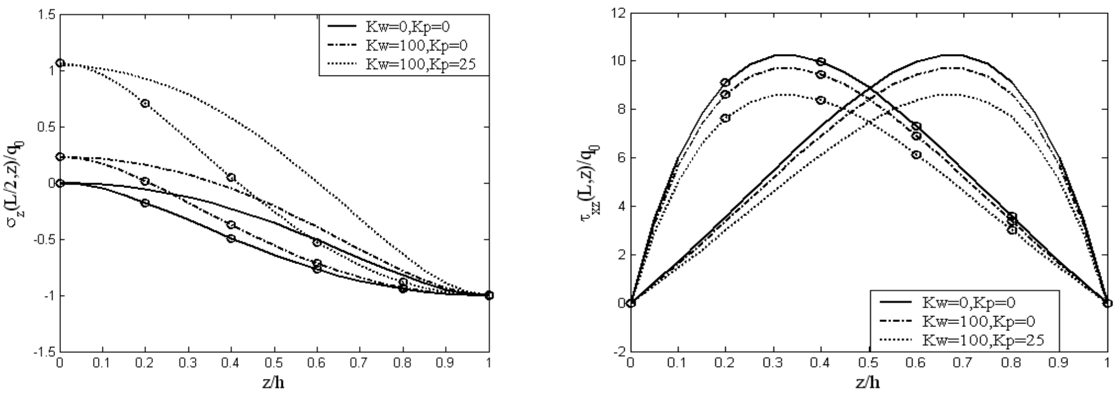


c. Axial normal stress d. Transverse displacement

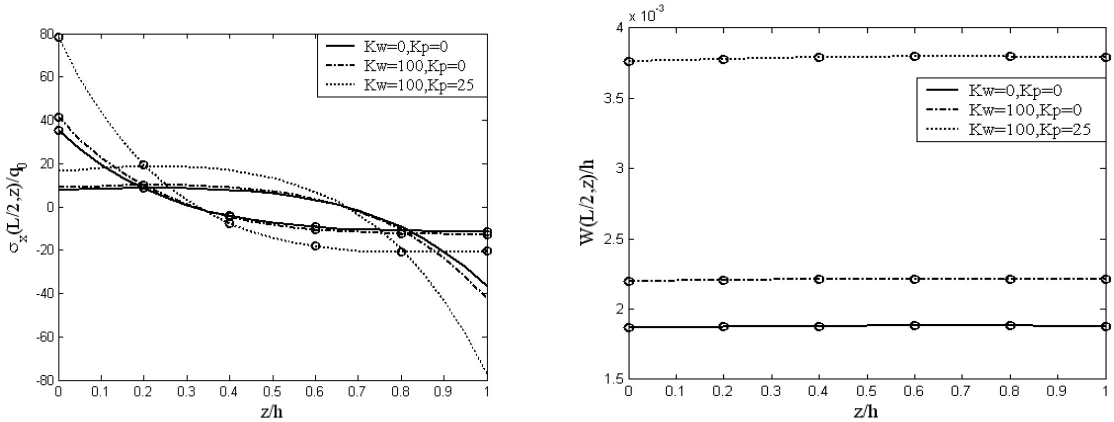
Fig.2. Effect of edge boundary conditions on the behavior of the FGM beam, L/h = 10, K_w=100, K_p=25

5.3 Elastic foundation parameters

First of all, for the convenience of citing, we designate curves without circle (or curves with circle) to the beam with the softer (or harder) surface resting on the elastic foundation. Influence of the elastic foundation on the static behavior of FGM beam is depicted in Figs.3. According to the figures, changing the action surface on elastic foundation can change the values of transverse normal stress at any point of the thickness, but changing the action surface only shifts amounts of the transverse shear and axial normal stresses to other point of the thickness with not changing in amounts. Also changing the action surface don't affect in changing of the transverse displacement. It is observed that the upper and lower surface condition of beam in transverse normal and shear stress figures are satisfied.



a. Transverse normal stress b. Transverse shear stress

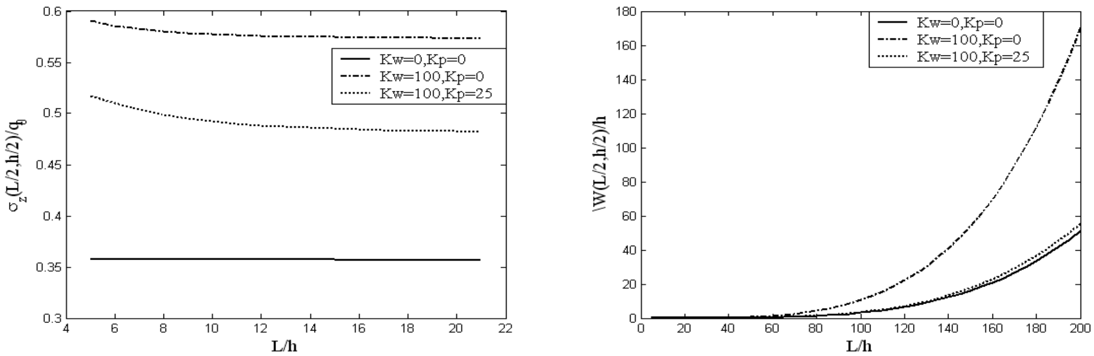


c. Axial normal stress d. Transverse displacement

Fig.3. Variation of stresses and displacement across the thickness, CS, L/h =10

5.4 Aspect ratio

The different aspect ratio, $\frac{L}{h}$, for transverse normal stress and transverse displacement of a CC beam are plotted in Fig. 4.



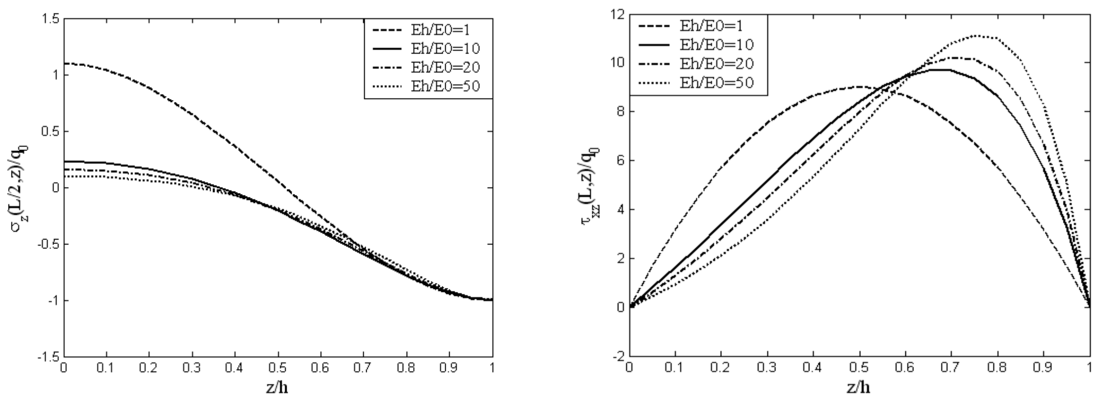
a. Transverse normal stress b. Transverse displacement

Fig.4. Effect of aspect ratio on transverse normal stress and displacement for FGM beam, CF

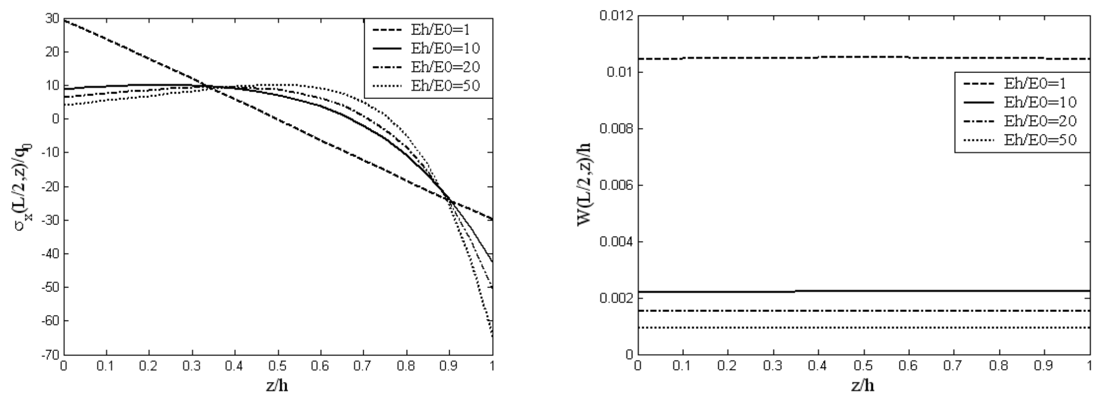
According to the figure and as expected, by increasing the length to thickness ratio transverse normal stress decrease slightly to a constant value, and consequently increasing the aspect ratio causes the FGM beam behave as the thin beam.

5.5 Gradient index

The effects of gradient index on the stresses and displacement of a thick FGM beam with the soft surface subjected to elastic foundation is presented in Figs.5.



a. Transverse normal stress b. Transverse shear stress



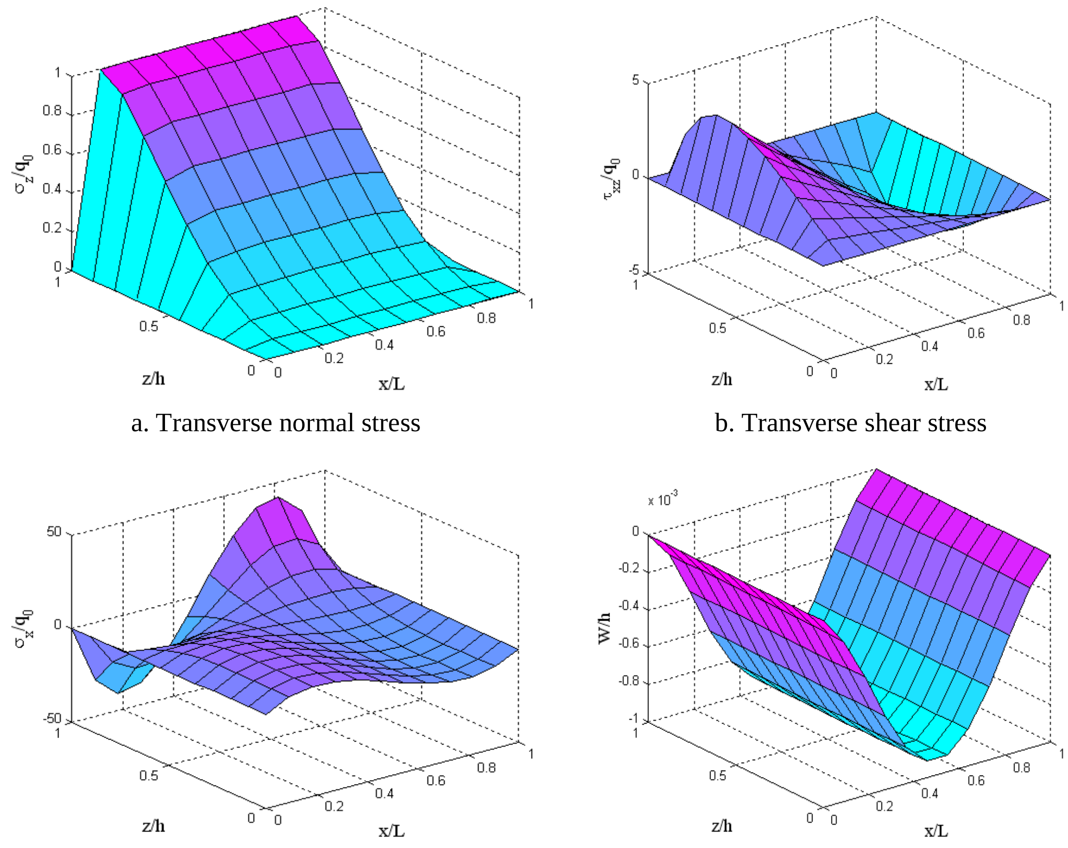
c. Axial normal stress d. Transverse displacement

Fig.5. Effect of gradient index on mechanical parameters for FGM beam, CS, L/h =10, K_w=100, K_p=25

It is observed that as $\frac{E_h}{E_0}$ increases the transverse normal stress decreases gradually and transverse shear stress decreases up to near to the z = h/2 and then increases together with shifting the maximum value near to the hard surface. In contrast with FGM, in Fig.5a it is observed that the distribution of axial normal stress in isotropic material is linear. As the figure depicts by increasing, $\frac{E_h}{E_0}$ axial normal stress decreases near to the z = 0.35h distance and then increases especially rapidly near to the hard surface together with shifting the neutral axis toward the hard surface. As the Fig.5b shows the transverse shear stress curve for isotropic beam is symmetric with respect to the neutral axis and the maximum value moves toward to the vicinity of harder surface of the beam. This point is consistent with physics that, with increasing, the bending rigidity of the upper half of the beam becomes larger than that of the lower half and, hence, the upper half undergoes bigger stress than the lower half.

5.6 Mechanical parameters in two directions

Figs.7 and 8 present the distribution of stresses and displacement for CC end boundary condition, with soft surface subjected to elastic foundation along x and z direction. As the figure shows edge boundary condition are satisfied. Due to the inhomogeneous specification along the thickness direction, axial normal and shear transverse stress with respect to mid-span of the beam has opposite sign.



a. Transverse normal stress

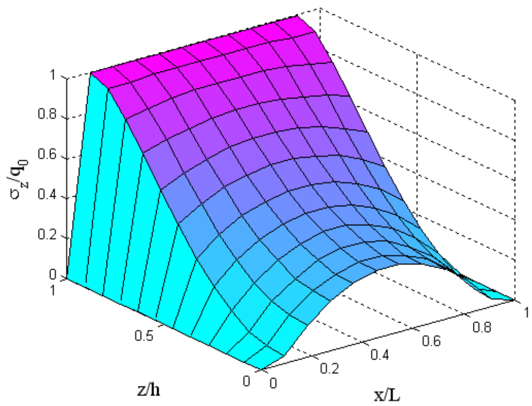
b. Transverse shear stress

c. Axial normal stress

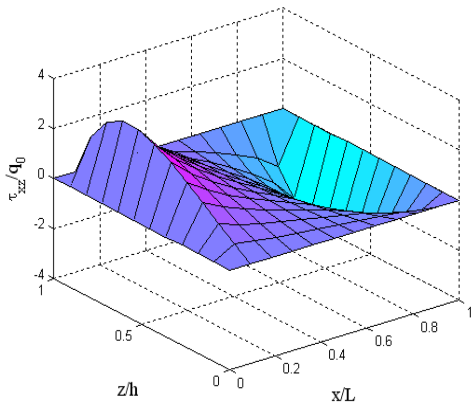
d. Transverse displacement

Fig.7. Mechanical parameters in two directions for FGM beam, CC, $K_w = 0$, $K_p = 0$

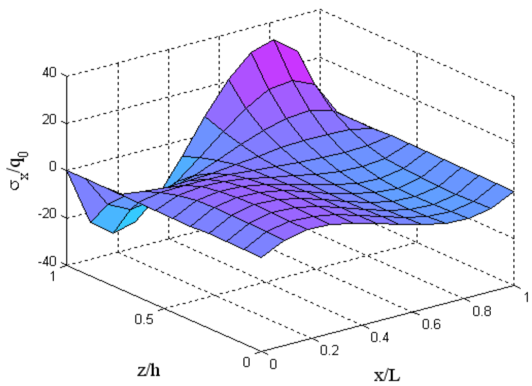
Also the comparing between the beam rests on tow parameter elastic foundation and the beam without foundation is investigated. According to figures, stresses and displacement for the beam rests on elastic foundation is less than the beam without any foundation because of properties of elastic foundation.



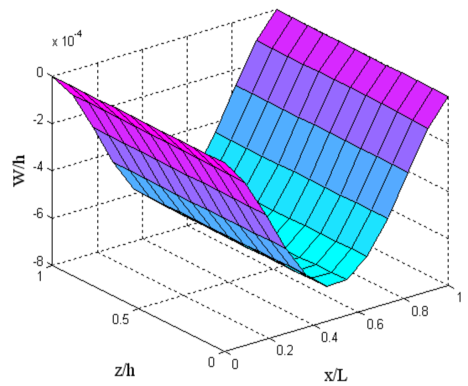
a. Transverse normal stress



b. Transverse shear stress



c. Axial normal stress



d. Transverse displacement

Fig.8. Mechanical parameters in two directions for FGM beam, CC, $K_w = 100$, $K_p = 25$

6. Conclusion

Tow-dimensional elastic deformation of functionally graded beam rested on elastic foundation with various kinds of edges boundary conditions and Young's modulus varying according to exponentially through the thickness has been analyzed. The analysis was carried out by using DQM and state-space approach. The numerical results have revealed that the variations of material properties in the thickness direction affect the response of FG beam.
From this investigation, the following conclusions can be made:

Using only a few discrete grid points in the numerical solution of DQM is equivalent to the analytical solution.

The neutral axis surface of the FG beam is not at mid-surface but depends on the through-thickness variation of Young's modulus.

Maximum stresses at any point in thickness direction of FGM beam in comparison with the isotropic beam are reduced.

Distribution of transverse shear stress in FG beam in contrast with the isotropic beam is not symmetric with respect to the neutral axis and shifts toward the vicinity of hard surface.

Axial normal stress distribution in FGM beam in contrast with isotropic beam is nonlinear and has maximum value at its hard surface.

Results given in the paper can serve as benchmarks for future analyses of FGM beams on elastic foundations.

7. References

Journals:

[1] Suresh S, Mortensen A. Fundamentals of functionally graded materials. London, UK: IOM Communications Limited, 1998.
[2] Reddy JN. Analysis of functionally graded plates. Int J Numer Meth Eng 2000;47:663–84.
[3] Sankar BV. An elasticity solution for functionally graded. Composites Science and Technology 2001; 61: 689-696.
[4] Reddy JN. A new beam finite element for the analysis of functionally graded materials. Int J of Mechanical Sciences 2003; 45: 519-539.
[5] Zhu H, Sankar BW. A combined Fourier series-Galerkin method for the analysis of functionally graded beams. J of Applied Mechanics-Transaction of the ASME 2004; 71 : 421-424.
[6] Bian ZG,Lim CW, Chen WQ. On functionally graded beams with integrated surface piezoelectric layers. Composite Structure 2006; 72: 339-351.
[7] Ding, H.J., Huang, D.J., Wang, H.M. Analytical solution for fixed-end beam subjected to uniform load. Journal of Zhejiang University (SCIENCE) 2005; 6A (8): 779-783.
[8] Chen WQ, Lu CF, Xu RQ. Semi-analytical elasticity solution bi-directional functionally graded beams. Int J of Solids and Structures. 2008; 45: 258-275.
[9] Xiang HJ, Yang J. Free and forced vibration of laminated FGM Timoshenko beam of variable thickness under heat conduction. Composites: Part B 2008; 39: 292-303.
[10] Ying J, Lu CF, Chen WQ. Two-dimensional elasticity solution for functionally graded beams resting on elastic foundations. Composite Structures.2008; 84: 209-219
[11] Chen WQ, Lu CF, Bian ZG. Elasticity solution for free vibration of laminated beams. Composite Structures. 2003; 62: 75-82
[12] Chang Shu, Differential Quadrature and its application in engineering, Springer-Verlag London Berlin Heidelberg, 2000, pp:1-5
[13] Şimşek, M. and Kocatürk, T. (2009), "Free and forced vibration of a functionally graded beam subjected to a concentrated moving harmonic load", Composite Structures 90(4), 465-473.
[14] Kadoli, R., Akhtar, K., Ganesan, N. (2008), "Static analysis of functionally graded beams using higher order shear deformation theory", Applied Mathematical Modeling 32(12), 2509-2525.
[15] Şimşek, M. (2009), "Static analysis of a functionally graded beam under a uniformly distributed load by Ritz Method", International Journal of Engineering and Applied Sciences 1(3), 1-11.
[16] Şimşek, M. (2010), "Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories", Nuclear Engineering and Design 240(4), 697-705.
[17] Li, X.F. (2008), "A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler-Bernoulli beams", Journal of Sound and Vibration 318(4-5), 1210-1229.
[18] Aydogdu, M. and Taskin, V. (2007), "Free vibration analysis of functionally graded beams with simply supported edges" Materials & Design 28(5), 1651-1656.

Appendix:

$$A = \begin{bmatrix} -k & 0 & 0 & p_n \\ 0 & 0 & -p_n & \dfrac{2(1+\nu)}{E_0} \\ \dfrac{1-\nu^2}{E_0} & \nu p_n & 0 & 0 \\ -\nu p_n & E_0 p_n^2 & 0 & -k \end{bmatrix}$$ where $p_n = n\pi/L$

S-S:

$$A = \begin{bmatrix} -kI_{N-2} & 0 & 0 & -g_{ssij} \\ 0 & 0 & -g_{ssji} & \dfrac{2(1+\nu)}{E_0} I_N \\ \dfrac{1-\nu^2}{E_0} I_{N-2} & -\nu g_{ssij} & 0 & 0 \\ -\nu g_{ssij} & E_0(f_{ss1} + f_{ssN} - g_{ssij}^{(2)}) & 0 & -kI_N \end{bmatrix}$$

Where:

$g_{ssij} = g_{ij}(i = 2,...,N-1\ ,\ j = 1,...,N)$, $\qquad g_{ssji} = g_{ij}(i = 1,...,N\ ,\ j = 2,...,N-1)$

$f_{ss1} = g_{i1}g_{1j}(i, j = 1,...,N)$, $\quad f_{ssN} = g_{iN}g_{Nj}(i, j = 1,...,N)$, $\quad g_{ssij}^{(2)} = g_{ij}^{(2)}(i, j = 1,...,N)$

C-C:

$$A = \begin{bmatrix} -kI_{N-2} & 0 & 0 & -g_{ccij} \\ 0 & 0 & -g_{ccij} & \dfrac{2(1+\nu)}{E_0} I_{N-2} \\ \dfrac{1-\nu^2}{E_0} I_{N-2} & -\nu g_{ccij} & 0 & 0 \\ -\nu g_{ccij} & -E_0 g_{ccij}^{(2)} & 0 & -kI_{N-2} \end{bmatrix}$$

Where:

$g_{ccij} = g_{ij}(i, j = 2,...,N-1)$, $\qquad g_{ccij}^{(2)} = g_{ij}^{(2)}(i, j = 2,...,N-1)$

C-F:

$$A=\begin{bmatrix} -kI_{N-1} & 0 & 0 & -g_{cfij} \\ 0 & 0 & -g_{cfji} & \dfrac{2(1+\nu)}{E_0}I_{N-1} \\ \dfrac{1-\nu^2}{E_0}I_{N-1} & -\nu g_{cfji} & 0 & 0 \\ -\nu g_{cfij} & E_0(f_{cfN}-g_{cfij}^{(2)}) & 0 & -kI_{N-1} \end{bmatrix}$$

Where:

$g_{cfij} = g_{ij}(i, j = 1,..., N-1)$, $\quad g_{cfji} = g_{ij}(i, j = 2,..., N)$

$f_{cfN} = g_{iN}g_{Nj}(i = 1,..., N-1,\ j = 2,..., N)$, $\quad g_{cfij}^{(2)} = g_{ij}^{(2)}(i = 1,..., N-1\ ,\ j = 2,..., N)$

C-S:

$$A=\begin{bmatrix} -kI_{N-2} & 0 & 0 & -g_{csij} \\ 0 & 0 & -g_{csji} & \dfrac{2(1+\nu)}{E_0}I_{N-1} \\ \dfrac{1-\nu^2}{E_0}I_{N-2} & -\nu g_{csij} & 0 & 0 \\ -\nu g_{csji} & E_0(f_{csN}-g_{csij}^{(2)}) & 0 & -kI_{N-1} \end{bmatrix}$$

Where:

$g_{csij} = g_{ij}(i = 2,..., N-1\ ,\ j = 2,..., N)$, $\quad g_{csji} = g_{ij}(i = 2,..., N\ ,\ j = 2,..., N-1)$

$f_{csN} = g_{iN}g_{Nj}(i, j = 2,.., N)$, $\quad g_{csij}^{(2)} = g_{ij}^{(2)}(i, j = 2,..., N)$