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PRISMATIC SUBDIVISION OF A SIMPLICIAL SET IN A TOPOLOGICAL SENSE

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Abstract

We study prismatic sets which are very closely related to simplicial sets. The realization of a prismatic set leads us to the prismatic subdivision of a simplicial set which is a special case of prismatic sets. We show the topological relation between the prismatic subdivision of a simplicial set S and S itself and we give the nerve of this construction.

1. INTRODUCTION

Prismatic sets were introduced and used by Dupont-Ljungmann [5] and prismatic decomposition appeared in many different places (see e.g. Phillips-Stone [12]). In Akyar [1], an important special case of prismatic sets, namely the prismatic subdivision of a simplicial set S in connection with "Lattice Gauge Theory" in the sense of Phillips-Stone [12] was given. It was discovered independently by Lisica-Mardešić [7] and Grayson [6]. The prismatic subdivision had also been used by McClure-Smith [10] to give a solution of Deligne's conjecture. One of the main constructions in the present paper is to give a canonical homeomorphism between the prismatic subdivision of S and S itself (Lemma 3.1). This construction leads us to have topological properties of the prismatic subdivision. Moreover we also give the Alexander-Whitney diagonal map in terms of the prismatic subdivision (Proposition 2.9). We explain how one can get the corresponding nerve of the prismatic subdivision for the covering of the simplicial set.

The organization of the paper is given as follows:

In section 2, for a simplicial set S, we give the definition of a prismatic set and the prismatic subdivision $P_\nabla S_{\sigma ...}$ by an induction on *p* and also we introduce another prismatic set E. First, we define a map between the geometric realizations $|S_\cdot| \to |E_\nabla$ and shortly give the geometric interpretation of $|P_\nabla$. We end this section by giving a homeomorphism between $|P_\nabla$ and $|$ with its cellular inverse.

In the third section, we replace any topological space by the geometric realization of the simplicial set S and have a canonical homeomorphism $\| |PS_\cdot| \| \to \| |S$. Finally we give the relations among $\| |PS_\cdot| \|, |$ and $\|$. This is also one of the main results in the paper (Corollary 3.3 and Corollary 3.4).

Section 4, we recall a new multi-simplicial set whose realization leads us to the nerve of $|P$ for the covering of $|$ by the stars of vertices from Akyar's thesis [1]. We emphasize the role of the prismatic subdivision in gauge theory. Namely this construction will help us to construct a classifying map in the prismatic sense (see Akyar-Dupont [2]).

2. **PRISMATIC SUBDIVISION**

In this section, we introduce two prismatic sets, namely l and i for a given simplicial set S and define the required homeomorphism using the Alexander-Whitney diagonal map. We only give the definition of a simplicial set but a brief exposition of simplicial constructions can be found in e.g. Mac Lane [8], May [9] and Milnor [11].

Definition 2.1. A simplicial set $S = \{S_q\}$ is a sequence of sets with face operators $d_i : S_q \to S_{q-1}$ and degeneracy operators $s_i : S_q \to S_{q+1}$, i = 0,… ,*q*, satisfying the simplicial identities

$$d_i d_j = \begin{cases} d_{j-1} d_i : & i < j \\ d_j d_{i+1} : & i \geq j \end{cases}$$

$$s_i s_j = \begin{cases} s_j s_{i+1} : & i \leq j \\ s_j s_{i+1} : & i > j \end{cases}$$

and

$$d_i s_j = \begin{cases} s_{j-1} d_i : i < j \\ id : i = j, i = j+1 \\ s_j d_{i-1} : i > j+1 \end{cases}$$

Definition 2.2. Let $S = \{S_q\}$, *q*=0,1,… be a simplicial set and suppose that each S_q is a topological space such that all face and degeneracy operators are continuous. Let
$\Delta^q = \{(t_1, \dots, t_q) \in R^q \mid 1 \geq t_1 \geq \cdots \geq t_q \geq 0\}$ be the standard *q*-simplex given with interior coordinates, the face maps $\varepsilon^i : \Delta^q \to \Delta^{q+1}$ and the degeneracy maps $\eta^i : \Delta^q \to \Delta^{q-1}$, i = 0,… ,*q* defined by $\varepsilon^i(t_0, \dots, t_{q-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{q-1})$ and $\eta^i(t_0, \dots, t_q) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, \dots, t_q)$, respectively. Then *S* is called a simplicial space and associated to this is the so-called fat realization $\|S\|$ given by

$$\|S\| = \bigsqcup \Delta^q \times S_q / \sim$$

with the identification

$$(\varepsilon^i t, x) \sim (t, d_i x), t \in \Delta^{q-1}, x \in S_q \ \ and\ i = 0, \dots, q, \quad q = 1,2 \dots.$$

Furthermore we can give the geometric (thin) realization |S| of S with the common extra dentification

$$(\eta^i t, x) \sim (t, s_i x), t \in \Delta^{q+1}, x \in S_q \ \ and\ i = 0, \dots, q, \quad q = 0,1, \dots .$$

Definition 2.3. Given $p \geq 0$, a (*p* + 1)-multi-simplicial set *S* is a sequence $\{S_{q_0, \dots, q_p}\}$ which is a simplicial set in each $q_i, i = 0, \dots, p$ and such that the face and degeneracy operators

$$d_j^i : P_{p, q_0, \dots, q_p} \to P_{p, q_0, \dots, q_i - 1, \dots, q_p}$$

$$s_j^i : P_{p, q_0, \dots, q_p} \to P_{p, q_0, \dots, q_i + 1, \dots, q_p}$$

commute with d_l^k, s_l^k for $i \neq k, i, j = 0, \dots, p$.

Definition 2.4. A prismatic set P is a sequence $\{P_p\}$ of $(p+1)$-multi-simplicial sets for $p > 0$ together with face operators

$d_k\colon P_{p,q_0,\dots,q_p} \to P_{p,q_0,\dots,\hat{q}_k,\dots,q_p}$

commuting with d_j^i and s_j^i (interpreting $d_j^k = s_j^k = id$ on the right) such that $\{P_p\}$ is a Δ-set, that is, there exist only face operators on the space. If similarly there are given degeneracy operators

$$s_k\colon P_{p,q_0,\dots,q_p} \to P_{p+1,q_0,\dots,q_k,q_k,\dots,q_p}$$

we get an ordinary simplicial set $(\{P_p\}, d_k, s_k)$. A prism is a product of simplices, that is, a set of the form $\Delta^{q_0\cdots q_p} = \Delta^{q_0} \times \dots \times \Delta^{q_p}$.

Definition 2.5. For each p, the thin realization

$$|P_p| = \coprod \Delta^{q_0,\dots,q_p} \times P_{p,q_0,\dots,q_p}/\sim \tag{2.6}$$

is given with equivalence relation "~" generated by the face and degeneracy maps

$\varepsilon_j^i\colon \Delta^{q_0\cdots q_i\cdots q_p} \to \Delta^{q_0\cdots q_i+1\cdots q_p}$
$\eta_j^i\colon \Delta^{q_0\cdots q_i\cdots q_p} \to \Delta^{q_0\cdots q_i-1\cdots q_p}$,

respectively. $\{|P_p|\}$ is a Δ-space hence it gives a fat realization

$$\||P|\| = \coprod \Delta^p \times |P_p|/\sim \tag{2.7}$$

only using face operators $|d_i|\colon \pi_i \times d_i\colon \Delta^{q_0\cdots q_p} \times P_{p,q_0,\dots,q_p} \to \Delta^{q_0\cdots\hat{q}_i\cdots q_p} \times P_{p-1,q_0,\dots,\hat{q}_i,\dots,q_p}$

which act on $\Delta^{q_0\cdots q_p}$ as the projection inducing a structure of a simplicial space on $\{|P_p|\}$. In other words, the projection $\pi_i\colon \Delta^{q_0\cdots q_p} \to \Delta^{q_0\cdots\hat{q}_i\cdots q_p}$ deletes the i-th factor. The further equivalence relation on $\||P|\|$ given in (2.7) is generated by

$$(\varepsilon^i t, s, \sigma) \sim (t, \pi_i, d_i\sigma), \qquad t \in \Delta^{p-1}, \qquad s \in \Delta^{q_0\cdots q_p}, \qquad \sigma \in P_{p,q_0,\dots,q_p}.$$

Now, we give a special case which is called the *prismatic subdivision* of a simplicial set S which is denoted by $P_pS_{q_0,\dots}$ and defined by the explicit construction

$P_pS_{q_0,\dots,q_p} := S_{q_0+\dots+q_p+p}.$

Let $q = q_0 + \cdots + q_p$. The face operators

$d_j^i\colon P_pS_{q_0,\dots,q_i,\dots,q_p} = S_{q+p} \to P_pS_{q_0,\dots,q_i-1,\dots,q_p} = S_{q+p-1}$

are defined by

$d_j^i := d_{q_0+\dots+q_{i-1}+i+j}$,

$j = 0,\dots,q_i$. Similarly, the degeneracy operators

$s_j^i\colon P_pS_{q_0,\dots,q_i,\dots,q_p} = S_{q+p} \to P_pS_{q_0,\dots,q_i+1,\dots,q_p} = S_{q+p+1}$

are defined by

$s_j^i := s_{q_0+\dots+q_{i-1}+i+j}$,

$j = 0,\dots,q_i$. The face maps

$d_i\colon P_pS_{q_0,\dots,q_p} \to P_{p-1}S_{q_0,\dots,\hat{q}_i,\dots,q_p}$

are the operators corresponding to the inclusions

$\Delta^{q_0+\cdots+\hat{q}_i+\cdots+q_p+p-1} \to \Delta^{q+p}$

deleting the q_i+1 basis vectors with indices $q_0+\cdots+q_{i-1}+i, \ldots, q_0+\cdots+q_i+i$. For the sequences of spaces $\{|P_{\cdot}S_{\cdot}|\}$, we obtain the fat realization

$$\| |P_{\cdot}S_{\cdot}| \| = \coprod \Delta^p \times |P_p S_{\cdot}| / \sim$$

where

$$|P_p S_{\cdot}| = \coprod \Delta^{q_0 \cdots q_p} \times S_{q+p} / \sim.$$

Remark 1. In order to see the **geometric interpretation** of the prismatic subdivision

$$P_p S_{q_0, \ldots, q_p} = S_{q_0+\cdots+q_p+p}.$$

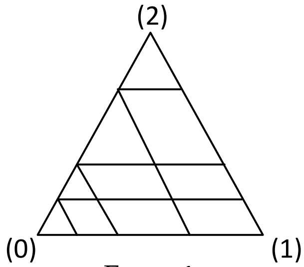
Let $q = q_0 + \cdots + q_p$, in general we can use an induction on *p*. Now let us start with a bisimplicial set $P_1 S_{q_0, q_1}$ as in Akyar [1]. As a motivation, suppose *S* is a simplicial set with face operators $d_i : S_n \to S_{n-1}$ and degeneracy operators $s_i : S_n \to S_{n+1}$, $i = 0, \ldots, n$. We can associate this to a bisimplicial set $P_{\cdot}S_{\cdot}$, where $P_1 S_{q_0, q_1} = S_{q_0+q_1+1}$

and $d_i' = d_i^0, s_i' = s_i^0, i = 0, \ldots, q_0$, $d_j'' = d_{q_0+j+1}$, $s_j'' = s_{q_0+j+1}$, $j = 0, \ldots, q_1$. Now, we give the geometric interpretation of $|P_p S_{\cdot}|^{[1]}$ as follows, here $|P_p S_{\cdot}|^{[1]}$ denotes the 1-skeleton of the realization of $P_p S$. When $p = 1$ we get

$$|P_1 S_{q_0, q_1}|^{[1]} = \coprod_{q_0+q_1=1} \Delta^{q_0 q_1} \times S_{q_0+q_1+1} / \sim$$

$$\equiv \Delta^0 \times \Delta^1 \times S_2 \coprod \Delta^1 \times \Delta^0 \times S_2 / \sim.$$

Let us take $S = \Delta^2$ then the elements $\{0,0,1\}, \{0,0,2\}, \{1.1.2\}, \{0,1,2\}, \{0,1,1\}, \{0,2,2\}$ and $\{1,2,2\}$ are the non-degenerate elements in the prismatic set. See the picture of $|P_p S_{q_0, \ldots, q_p}|$



For *p>1*

Example 2.8. Let S be a simplicial set and $E_p S = S_{\cdot} \times \ldots \times S_{\cdot}$ the $(p+1)$-multi-simplicial set. The face operators $d_i : E_p S \to E_{p-1} S$ project on the *i*-th factor and the degeneracy operators $s_i : E_p S \to E_{p+1} S$ repeat the *i*-th factor. The thin (geometric) realization of $E_p S$ is defined by

$$|E_p S_{\cdot}| = |S_{\cdot} \times \ldots \times S_{\cdot}|$$

$$= |S_\cdot| \times \dots \times |S_\cdot|$$

$$= \coprod_{a_0,\dots,a_p} \Delta^{q_0 \cdots q_p} \times S_{a_0} \times \dots \times S_{a_p} / \sim$$

with the necessary equivalence relations which follow from Definition 2.8. Although the d_i's are cellular, i.e., $d_i\left(E_p S^{(n)}\right) \subset E_{p-1} S^{(n)}$, the s_i's are not cellular, since when we define $\delta_i: \Delta^{q_i} \to \Delta^{q_i} \times \Delta^{q_i}$, we see that s_i does not convert a low cell in $|E_p S_\cdot|$ into a cell in $|E_{p+1} S_\cdot|$. That is why we consider the fat realization of $|E_p S_\cdot|$ instead of the geometric one. So we have $\|\,|E_\cdot S_\cdot|\,\|$ as the fat realization of the simplicial space whose *p*-th term $|S_\cdot| \times \dots \times |S_\cdot|$, (*p*+1)-times, is a contractible space.

Proposition 2.9. *Let S be a simplicial set. One can define the Alexander-Whitney diagonal map*

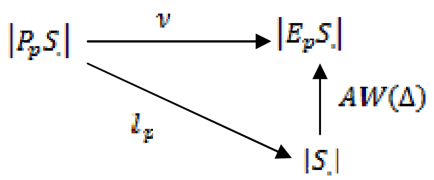
$AW(\Delta): |S_\cdot| \times \dots \times |S_\cdot|$
by $AW(\Delta)\left(l_{a_0,\dots,a_p}(t)(s), y\right) = \nu(s,y)$ *where* $l_{a_0,\dots,a_p}(t): \Delta^{q_0 \cdots q_p} \to \Delta^{q+p}$ *and* $\nu: |P_p S_\cdot| \to |E_p S_\cdot|$.
Proof. The map ν *is defined by using projections on each* $S_{a_i}, i = 0, \dots, p$ *and*
$\nu(s^0, \dots, s^p, y) = (s^0, \dots, s^p, \tilde{d}^{q_1+\dots+q_p+p} y, d_0^{q_0+1} \tilde{d}^{q_2+\dots+q_p+p-1} y, \dots, d_0^{q_0+\dots+q_p+p} y)$
where $\tilde{d}^{n-i}\sigma = d_{i+1} \circ \dots \circ d_n \sigma = d_{i+1\dots n}\sigma, \sigma \in S_n$. *Let* $q = q_0 + \dots + q_p$ *and*
$l_{a_0,\dots,a_p}(t): \Delta^{q_0 \cdots q_p} \to \Delta^{q+p}$ *is defined by*

$$\begin{aligned} l_{a_0,\dots,a_p}(t)(s^0, \dots, s^p) = (&s_1^0(1-t_1)+t_1, \dots, s_{a_0}^0(1-t_1)+t_1, t_1, \\ &s_1^1(t_1-t_2)+t_2, \dots, s_{a_1}^1(t_1-t_2)+t_2, t_2, \\ &, \dots, \\ &s_1^{p-1}(t_{p-1}-t_p)+t_p, \dots, s_{a_{p-1}}^{p-1}(t_{p-1}-t_p)+t_p, t_p, \\ &s_1^p t_p, \dots, s_{a_p}^p t_p) \end{aligned} \tag{2.10}$$

where $\Delta^{q_i} = \left\{\left(s_1^i, \dots, s_{a_i}^i\right) \in \mathbb{R}^{q_i} \mid 1 \geq s_1^i \geq \dots \geq s_{a_i}^i \geq 0\right\}, t = (t_1, \dots, t_p) \in \Delta^p$. *It induces a natural map of realizations* $l_p(t): |PS_\cdot| \to |S_\cdot|$, *that is,*

$$l_p(t): \Delta^{q_0 \cdots q_p} \times P_p S_{a_0,\dots,a_p} \to \Delta^{q+p} \times S_{a+p},$$

where $q = q_0 + \dots + q_p$. *Thus* $l_p(t) = l_{a_0,\dots,a_p}(t) \times id$. *We have a commutative diagram*



By the simplicial construction $P_p S_{a_0,\dots,a_p} = S_{a+p}$ *we know that* $|P_p S_\cdot| \approx |S_\cdot|$. *So the* $AW(\Delta)$ *diagonal map is defined by*
$AW\left(l_{a_0,\dots,a_p}(t)(s), y\right) = \nu(s,y)$.

□

One can see Mac Lane [8] for further information about Alexander-Whitney map .

Proposition 2.11. *1) For* $t \in \overset{\circ}{\Delta}{}^p$ *where* $\overset{\circ}{\Delta}{}^p = \{(t_1, \dots, t_p) \mid 1 > t_1 > \dots > t_p > 0\}$, *the map*
$l_p(t): |P_p S_\cdot| \to |S_\cdot|$
is a *homeomorphism and* $l_p(t)^{-1}$ *is cellular.*

2) Let $C_{*,*}($ be a *bicomplex of* $\,i$ *as a family* $\{(C_{p,n}(P), \partial_H, \partial_V$ *of modules with horizontal* $\,$ *and vertical* *boundary maps such that* $\partial_H \circ \partial_H = 0, \partial_V \circ \partial_V = 0, \partial_H \circ \partial_V + \partial_V \circ \partial_H =$ *For* $t \in ,$ $l_p(t)$ *induces the map of cellular chain complexes* $C_*(S) \to C_{*,*}($ *which is given by*

$$aw(x) = \sum s_{a_0+\cdots+a_{n-1}+p-1} \circ \ldots \circ s_{a_0}(x)_{(a_0,\ldots,a_n)}$$

where $x \in S_n$,

3) *For the i-th face map* $\varepsilon^i : \Delta^{p-1} \to$, *we have* $l_p \circ (\varepsilon^i \times id) = l_{p-1} \circ (id \times |d_i|$ *where* $|d_i| = \pi_i \times$.

Proof. 1) Let us see that $l_p(t)$ is surjective. Consider the case $p = 1$ and show that $\forall u^0 \in \Delta^{q_0+q_1+1}, \exists (s^0, s^1) \in \Delta^{q_0} \times \Delta^{q_1}, t \in$ such that

$$l_1(t)(s^0, s^1) = u^0 = (u_1, \ldots, u_{q_0+q_1+1})$$

By using (2.10) we get

$$s_1^0 = \frac{u_1 \quad u_{q_0+1}}{t}, \ldots, s_{q_0}^0 = \frac{u_{q_0} \quad u_{q_0+1}}{t}, s_1^1 = \frac{u_{q_0+2}}{}, \ldots, s_{q_1}^1 = \frac{u_{q_0+q_1+1}}{}$$

here $s^0 = (s_1^0, \ldots, s_{q_0}^0) \in \Delta^{q_0}$ and $s^1 = (s_1^1, \ldots, s_{q_1}^1) \in \Delta^{q_1}$, since $u^0 = (u_1, \ldots, u_{q_0+q_1+1}) \in \Delta^{q_0+q_1+1}$ satisfies the following

$$1 \geq u_1 \geq \cdots \geq u_{q_0} \geq u_{q_0+1} \geq \cdots \geq u_{q_0+q_1+1} \geq 0$$

Similarly one gets $1 \geq s_1^0 \geq \cdots \geq s_{q_0}^0 \geq 0$ and since $u_{q_0+1} \geq u_{q_0+2}$, we have $1 \geq s_1^1 \geq \cdots \geq s_{q_1}^1 \geq 0$. Thus $\exists s^0 \in \Delta^{q_0}, s^1 \in \Delta^{q_1}, \forall u^0 \in \Delta^{q_0+q_1+1}$. One can show that it is also true for $p > 1$. Now, consider the usual CW structure (see Bredon [4, chapter 4]) on $|P_pS_\cdot|$ and $|S_\cdot|$. The map $l_p(t)^{-1}$ is cellular, since it converts the low dimensional cell in $|S_\cdot|$ into the one in $|P_pS_\cdot|$, that is, $l_p(t)^{-1}(|S_\cdot|^n) \subset |P_pS_\cdot|^n$.

2) It follows from 1) that it induces a chain map of the associated cellular chain complexes. If we let $C_*($ denote the total complex generated by $P_pS_{q_0 \ldots}$ of the double-complex $C_{p,n}($ with horizontal and vertical boundary maps, where $C_{p,*}(P) = \bigoplus_{q_0+\cdots+q_p=n} C_{p,q_0,\ldots,q_p}($. We have a chain map

$$aw(x) = \sum s_{q_0+\cdots+q_{p-1}+p-1} \circ \ldots \circ s_{q_0+q_1+1} \circ s_{q_0}(x)$$

For example for the case p = 2 we consider three differentials in the multi-complex $P_2S_{q_0,q_1,q_2} \cong S_{q_0+q_1+q_2+2}, q_0, q_1, q_2 \geq$ denoted by d', d'', d We need to check

$$d \circ aw(x) = (d' + (-1)^{q_0} d'' + (-1)^{q_0+q_1} d''')(s_{q_0+q_1+1} \circ s_{q_0}(x))$$

in $P_2S_{q_0 q_1}$ where $d' = \sum_{r=0}^{q_0} (-1)^r d_r$, $d'' = \sum_{r=0}^{q_1} (-1)^r d_{q_0+r+1}$, $d''' = \sum_{r=0}^{q_2} (-1)^r d_{q_0+q_1+r+}$
It can be easily shown that it is true for general p.

3) We have a commutative diagram

$$\begin{array}{ccc} \Delta^{p-1} \times |P_pS_\cdot| & \xrightarrow{\varepsilon^i \times id} & \Delta^p \times |P_pS_\cdot| \\ \downarrow{\scriptstyle id \times |d_i|} & & \downarrow{\scriptstyle l_p} \\ \Delta^{p-1} \times |P_{p-1}S_\cdot| & \xrightarrow{l_{p-1}} & |S_\cdot| \end{array}$$

which gives us the following equality

$$l_p \circ \left(\varepsilon^i \times . \right. \qquad \square$$

3. TOPOLOGICAL INTERPRETATION OF THE PRISMATIC SUBDIVISION AND REALIZATIONS

The motivation of this section is as follows: Let X be a topological space which can be considered as a simplicial topological space by saying $X_p = X$ with the identity as face and degeneracy operators. Let $\|X_.\|$ denote the fat realization of X and $|X_.|$ the thin realization of X. We recall the simplicial topological space $E_.X$ where $E_pX = \underbrace{X\times \ldots \times X}_{p+1-times}\ \forall p$ then the diagonal map

$\Delta: X_p \to E_pX \ \ \forall p$

defines a map of simplicial spaces, in particular a map of fat realizations

$\|X_.\| \to \|E_pX_.\|$.

Let us consider the sequence of spaces $|P_pS_.|$ and see that there is a canonical homeomorphism

$L: \||PS_.|\| \to \ \||S_.|\| = \|\Delta^\infty\| \times |S_.|,$

where $\|\Delta^\infty\| = \sqcup_{p\geq 0}\Delta^p/\sim$ given by $\varepsilon^i t \sim t, \forall t \in \Delta^{p-1}, i = 0, \ldots, p, p = 1, \ldots$.

Note. One can notice that X is replaced by $|PS|$ but not by $|S_.|$ on the left-hand-side of the map L.

Lemma 3.1. *Let S be a simplicial set. There exists a homeomorphism* $L: \||PS_.|\| \to \ \||S_.|\|$ *given via* $l_p(t)$.

Proof: We have $\||PS_.|\| \xrightarrow{\|v\|} \||E_.S_.|\|$, where $\||PS_.|\| = \sqcup_{p\geq 0}\Delta^p \times |P_pS_.|/\sim$ and using the inverse of $l_p\,(t)$ we get

$$\sqcup_{p\geq 0}\,\Delta^p \times |S_.|/\sim \xrightarrow{id\times l_p^{-1}(t)} \sqcup_{p\geq 0}\,\Delta^p \times \ |P_pS_.|\ /\sim$$

since $l_p\ (t)$ is a homeomorphism. In particular, we have a commutative diagram for each p and each n

$$\begin{array}{ccccc}
\coprod_{p\geq 0}\Delta^p \times |S_.|/\sim & \xrightarrow{id\times l_p^{-1}(t)} & \||PS_.|\| & \xrightarrow{\|v\|} & \||E_.S_.|\| \\
\uparrow & & \uparrow & & \uparrow \\
\Delta^p \times |S_.|^{(n)}/\sim & \xrightarrow{id\times l_p^{-1}(t)} & \||PS_.|^{(n)}\| & \xrightarrow{\|v\|} & \||E_.S_.|^{(n)}\|
\end{array}$$

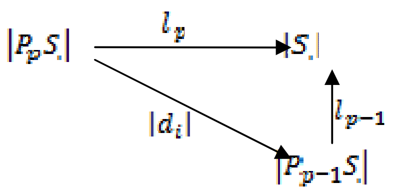
Here $|S_.|^{(n)}$ denotes the $n-th$ skeleton of the realization of the simplicial set. For $n = 0$

the lower row becomes

$$\Delta^p \times S_0 \xrightarrow{id} \Delta^p \times S_0 \xrightarrow{id \times diag} \Delta^p \times S_0 \times \ldots \times S_0$$

and we get
$L_p : \Delta^p \times \left|P_p S_.\right|^{(p)} \xrightarrow{id \times l_p(t)} \Delta^p \times |S_.|^{(p)}$

which follows the existence of L. We note that the maps $l_p : |P_p S_.| \longrightarrow |S_.|$ do not commute with the face operators $|d_i|$ but only up to homotopy. This can be seen by the following diagram



here $l_{p-1} \circ |d_i| \sim l_p$.

The maps $L_p : \Delta^p \times |P_p S_.|/\sim \longrightarrow \Delta^p \times |S_.|/\sim$ given by $L_p(t,x) = \left(t, l_p(t)(x)\right)$ induce a homeomorphism

$$L : \| \; |PS_.| \; \| \longrightarrow \| \; |S_.| \; \| \qquad (3.2)$$

where the right hand side whose the face and the degeneracy operators are given by identity. We can filter both sides of (3.2) by p-skeletons, that is,

$L^{(p)} : \| \; |PS_.|^{(p)} \; \| \longrightarrow \| \; |S_.|^{(p)} \; \|$

and show that $L^{(p)}$ is a homeomorphism by using the fact that $L : \overset{op}{\Delta} \times |PS_.| \rightarrow \overset{op}{\Delta} \times |S_.|$ is a homeomorphism. This can be shown by using an induction on the skeleton. It is a homeomorphism for the zero skeleton and assume that $L^{(p-1)}$ is a homeomorphism and

$$\| |PS_.|^{(p)} \| = \amalg_{p \geq 0} \Delta^p \times |PS_.| / \sim$$
$$= \left(\amalg_{p \geq 0} \Delta^{p-1} \times |PS_.| \amalg_{p \geq 0} \overset{op}{\Delta} \times |PS_.| \right) / \sim .$$

Similarly $\| |S_.|^{(p)} \| = \amalg_{p \geq 0} \Delta^p \times |S_.| / \sim$ and $\amalg_{p \geq 0} \Delta^p \times |S_.| = \amalg_{p \geq 0} \Delta^{p-1} \times |S_.| \amalg_{p \geq 0} \overset{op}{\Delta} \times |S_.|$

We already know that $\overset{op}{\Delta} \times |PS_.| \rightarrow \overset{op}{\Delta} \times |S_.|$ is a homeomorphism and the first part $L^{(p-1)}$ is also a homeomorphism by induction. Thus $L^{(p)}$ is a homeomorphism.

L is well-defined, that is, $L_p(\varepsilon^i t, x) \sim L_{p-1}(t, |d_i| x)$, in other words, $(\varepsilon^i t, x) \sim (t, |d_i| x)$.

$$L_p(\varepsilon^i t, x) = (\varepsilon^i t, l_p(\varepsilon^i t)(x)) \sim \left(t, \left(l_{p-1} \circ |d_i|\right)(t,x)\right) = L_{p-1}(t, |d_i| x),$$

since $l_p \circ (\varepsilon^i \times id) = l_{p-1} \circ (id \times |d_i|)$ for the $i-th$ face map ε^i.

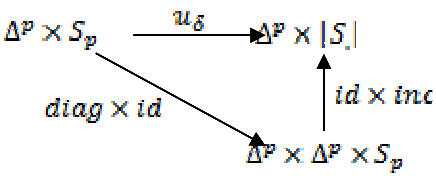
□

Corollary 3.3. $\lambda: \| |PS_.| \| \to |$ *is the composite of and the projection*
$\| |PS_.| \| \longrightarrow \| |S_.| \| \xrightarrow{proj} |S_.|$.
Furthermore, it is a homotopy equivalence.
Proof. The map λ is just induced by $\Delta^p \times |P_p S_.| \to |S_.|$ given by $(t, x) \to l_p(t)(x)$. A homotopy inverse is given by inclusion $|S_.| = \Delta^0 \times |P_0 S_.| \subseteq \| |PS_.| \|$.

□

Remark 2. We have another homotopy equivalence

$u_\delta: \|S_.\| \to \| |S_.| \|$
which is defined by



and it takes (t, x) to (t, t, x), since

$$\|S_.\| \xrightarrow{u_\delta} \| |S_.| \| \xrightarrow{proj} |S_.|$$

is a natural map. We can define the homotopy $u_\delta: \|S_.\| \to \| |S_.| \|$ as follows, that is,

$$u_\delta: \Delta^p \times S_p \to \Delta^{p+1} \times \Delta^p \times S_p$$

is defined by

$$u_\delta(t_1, \dots, t_p, x) = \left([1 - (1 - t_1)\delta, \dots, 1 - (1 - t_p)\delta, 1 - \delta], (t_1, \dots, t_p, x)\right)$$

for $0 < \delta < 1$. Here,

$$u_0(t, x) = (1, \dots, 1, t, x)$$

$$= (\varepsilon_0^{p+1}(0), t, x)$$

$$\sim (0, t, x) \in \Delta^0 \times |S_.|$$

and

$$u_1(t, x) = (t, 0, t, x)$$

$$= (\varepsilon^{p+1} t, t, x)$$

$$\sim (t, t, x)$$

$$= u(t, x).$$

Corollary 3.4. *There is a homotopy equivalence v defined as a composite of* L^{-1} *and* u_δ

$$||S_.|| \xrightarrow{u_\delta} |||S_.||| \xrightarrow{L^{-1}} |||PS_.|||.$$

Proof. It is straight forward since L is a homeomorphism then L^{-1} is continuous. u_δ is a homotopy equivalence then $L^{-1} \circ u_\delta$ is a homeomorphism. □

4. SİMPLİCİAL SETS AND STAR COMPLEX

In this section, we will give an analogy between a nerve for a simplicial complex and a nerve for a simplicial set. Let K be a simplicial complex and K' denote its barycentric subdivision consisting of simplices of the form $[\sigma_p \supseteq \sigma_{p-1} \supseteq \cdots \supseteq \sigma_0]$. This subdivision is the nerve of the simplicial complex K considered as an ordered set and hence a category. This is the nerve of the covering by stars (See Segal [13]).

Definition 4.1. The *star complex* S_q is defined as $S_q := S_{q+1}$ with face and degeneracy operators inherited from those of S_{q+1} as $d_k{:}S_q \to S_{q-1}$ and $s_k{:}S_q \to S_{q+1}$, where $k = 0, \dots, q$.

$i{:}\, S_0 \hookrightarrow |\bar{S}_.|$ and $r{:}\, |\bar{S}_.| \to S_0$ are defined in degree q by $i(y) = (t, s_{q\dots0}y)$ and $r(t.x) = (d_{0\dots q}x)$, where $y \epsilon S_0, x \epsilon S_{q+1}$ and $s_{q\dots0} = s_q \circ \dots \circ s_0, d_{0\dots q} = d_0 \circ \dots \circ d_q$.

For a simplicial set the case is given as follows: For a given simplicial set S we construct another simplicial set $\bar{S}$ so that $S_q = S_{q+1}$ and a retraction $r{:}\, \bar{S} \to S_0$ such that $\{r^{-1}(\sigma) | \sigma \epsilon S_0\}$ corresponds to the covering by stars. If X is a topological space then we have a diagonal map $X \to X \times \dots \times X$, but if we replace X by a simplicial set S, we have seen that we have to replace X_p by $|P_p S_.|$ but not $|S_.|$ because of the diagonal map and the simplicial construction. Here the covering is $\{r^{-1}(\sigma) | \sigma \epsilon S_0\}$ and the nerve of $|P_p S_.|$ covering $r^{-1}(\sigma)(\sigma \epsilon S_0)$ corresponds to $|\bar{P}_p S_.|$ where

$\bar{P}_p S_{q_0,\dots,q_p} := S_{q_0+\dots+q_p+2p+1}$.

Let $q = q_0 + \dots + q_p$. The face and degeneracy operators on $\bar{P}_p S_{q_0,\dots,q_p}$ are inherited from the ones on S_{q+2p+1} as follows:

The face operators $d_j^i{:}S_{q+2p+1} \to S_{q+2p}$ are defined by

$$d_j^i := d_{q_0+\dots+q_{i-1}+j+2i},\ j = 0, \dots, q_i \text{ but } j \neq q_i + 1, i = 0, \dots, p.$$

So $\bar{P}_p S_{q_0,\dots,q_p}$ has only $q + p$ face operators, that is, we are skipping the $p + 1$ face operators $\{d_{q_0+1}, d_{q_0+q_1+3}, \dots, d_{q_0+\dots+q_p+2p+1}\}$.

Similarly the degeneracy operators $s_j^i := S_{q+2p+1} \to S_{q+2p+2}$ are defined by

$$s_j^i := s_{q_0+\dots+q_{i-1}+j+2i}, j = 0, \dots, q_i \text{ but } j \neq q_i + 1, i = 0, \dots, p.$$

The fat realization of $|\bar{P}_p S_.|$ is given by

$$|||\bar{P}_p S_.||| = \coprod_{p \geq 0} \Delta^p \times \Delta^{q_0 \dots q_p} \times \bar{P}_p S_{q_0,\dots,q_p} / \sim$$

with the necessary equivalence relations given as the ones for (2.7).

Remark 3. In the case of a manifold X, the nerve of a covering is the simplicial space

$$NX_{\mathcal{U}_p} = \coprod_{i_0 \dots i_p} (U_{i_0} \cap \dots \cap U_{i_p})$$

Where $\mathcal{U} = \{U_i\}_{i \in I}$ is the covering of X and the disjoint union is taken over all $(p+1)-$ tuples $(i_0, \dots, i_p)$ with $(U_{i_0} \cap \dots \cap U_{i_p}) \neq \emptyset$. In the case of a bundle over a manifold X, the classifying map is a map $\|NX_{\mathcal{U}_.}\| \to BG$. For a simplicial set S, $NX_{\mathcal{U}_p}$ is replaced by $|\bar{P}_p S_.|$ which is homotopy equivalent to the set $|S_.|$ We have the AW map

$$|\bar{P}_p S_.| \to |\bar{S}_.| \times \dots \times |\bar{S}_.|$$

and by the fact that $|\bar{S}_.|$ has the same homotopy type of S_0, we have
$|\bar{P}_p S_.| \to |\bar{S}_.| \times \dots \times |\bar{S}_.| \to S_0 \times \dots \times S_0$.

Proposition 4.2. *The map* $i: S_0 \hookrightarrow |\bar{S}_.|$ *is a deformation retract with retraction* $r: |\bar{S}_.| \to S_0$.

Proof. Let's take the homotopy

$$H_\lambda: |\bar{S}_.| \to |\bar{S}_.|$$

defined by $H_\lambda(t,x) = ((1-\lambda)(t,0) + \lambda(1,\dots,1), s_q x)$ such that $H_1(t,x) \sim i \circ r(t,x)$ and $H_0(t,x) \sim id_{|\bar{S}_q|}$. This can be seen by taking the homotopy as

$$H_\lambda: \coprod \Delta^{q-1} \times S_{q-1} \to \coprod \Delta^q \times S_q \, .$$

Then

$$H_1(t,x) = (\underbrace{1,\dots,1}_{q-times}, s_q x) = (\varepsilon^{q-1} \circ \dots \circ \varepsilon^0(0), s_q x)$$

$$\sim (0, d_{0\dots q-1} s_q x)$$

$$= (0, s_0 d_{0\dots q-1} x)$$

$$= i(0, d_{0\dots q-1} x)$$

$$= i \circ r(t,x) \in S_1.$$

On the other hand

$$H_0(t,x) = (t, 0, s_q x) = (\varepsilon^q t, s_q x)$$

$$\sim (t, d_q s_q x)$$

$$= (t,x)$$

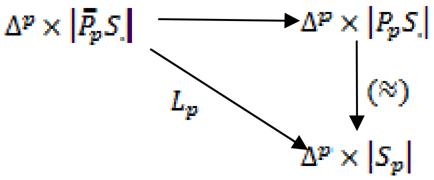
$$= id_{|\bar{S}_q|}.$$

Thus H_λ gives us a deformation retract S_0 of $|\bar{S}_q|$.

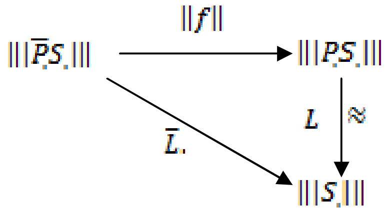
Note. We have an inclusion $\Delta^q \times P_p S_. \subseteq \Delta^{q+1} \times S_{q+2p+1}$. The first part of this inclusion $i: \Delta^q \hookrightarrow \Delta^{q+1}$ is defined by $i(s_1,\dots,s_q) = (s_1,\dots,s_q,0)$. It induces a surjective but not an injective map of realizations. So $f\colon |\bar{P}_p S_.| \to |P_p S_.|$ is not in general a homotopy equivalence. We have an inclusion

$$\Delta^p \times \Delta^{q_0\cdots q_p} \times \bar{P}_p S_{q_0,\dots,q_p} \hookrightarrow \Delta^p \times \Delta^{q_0+1\dots q_p+1} \times S_{q+2p+1}$$

which defines maps of realizations



It follows



where $\bar{L}_. = L \circ \|f\|$. The map $f\colon \Delta^{q_0\cdots q_p} \times S_{q+2p+1} \to \Delta^{q_0\cdots q_p} \times S_{q+p}$ is given by

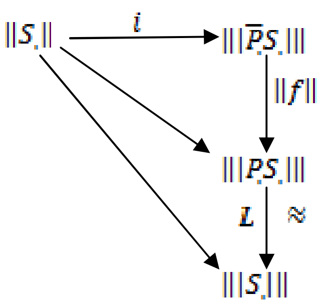
$$f(s^0,\dots,s^p,x) = \left(s^0,\dots,s^p, d_{q_0+1} \circ d_{q_0+q_1+3} \circ \dots \circ d_{q+2p+1} x\right),$$

where $x \in S_{q+2p+1}$.

Proposition 4.3. *Let $i\colon \|S\| \hookrightarrow \||\bar{P}S_.|\|$ be an inclusion defined by $i(t,x) = (t,1,s_{0\dots p}x)$ and $r\colon \||\bar{P}S_.|\| \to \|S_.\|$ be the retraction defined as*
$r(t,s,y) = \left(t, d_{0\dots q_0} \circ d_{q_0+2\dots q_0+q_1+2} \dots \circ d_{q_0+\dots q_{p-1}+2p\dots q+2p} y\right)$.

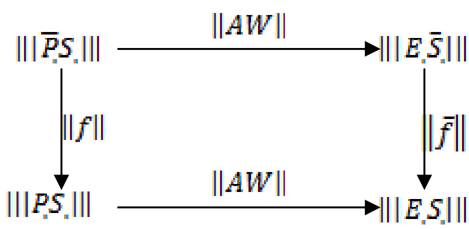
1) i is a deformation retract with the retraction r.

2) There is a diagram of homotopy equivalences



(4.4)

3) There is a commutative diagram



Proof. 1) Let's define the homotopy

$$H_\lambda : \Delta^p \times \Delta^{q_0-1\ldots q_p-1} \times S_{q+p} \to \Delta^p \times \Delta^{q_0\ldots q_p} \times S_{q+2p+1}$$

as

$$H_\lambda(t,s,y)\big(t,(1-\lambda)(s^0,0)+\lambda(1,\ldots,1),\ldots,(1-\lambda)(s^p,0)+\lambda(1,\ldots,1),s_{q+2p}\circ\ldots\circ s_{q_0}y\big),$$

where

$S_{q+p} = \left|\bar{P}_p S_{q_0-1,\ldots,q_p-1}\right|$ and $S_{q+2p+1} = \left|\bar{P}_p S_{q_0,\ldots,q_p}\right|$. Then

$H_1(t,s,y) = i \circ r(t,y) \in S_{2p+1}$.

$H_0(t,s,y) = id_{|||\bar{P}_p S_.|||}$.

This homotopy gives $id_{|\bar{P}_p S_.|} \sim i \circ r$.

2) The first homotopy equivalence i is induced by the inclusion given in 1). We have defined $v\colon ||S_.|| \to |||PS_.|||$ before as $v = L^{-1} \circ u_\delta$ in Corollary 3.4. In the previous note, we have defined

$$\Delta^p \times |\bar{P}S_.| \to \Delta^p \times |PS_.|$$

which induces a homotopy equivalence

$$|||\bar{P}S_.||| \to |||PS_.|||.$$

Furthermore the composition $L \circ ||f|| \circ i$ is a homotopy equivalence and $(L \circ ||f|| \circ i)(t,x) = (t,t,x)$.

3) We can see that the following diagram

$$\begin{array}{ccc}
\Delta^p \times |\bar{P}_p S_.| & \xrightarrow{id \times AW} & \Delta^p \times |\bar{S}_.| \times \ldots \times |\bar{S}_.| \\
\downarrow f & & \downarrow \bar{f} \\
\Delta^p \times |P_p S_.| & \xrightarrow{id \times AW} & \Delta^p \times |S_.| \times \ldots \times |S_.|
\end{array}$$

is commutative since

$$(id \times AW)(t,s^0,\ldots,s^p,y) = (t,s^0,\ldots,s^p,\tilde{d}^{q_1+\cdots+q_p+2p}y, d_0^{q_0+2}\tilde{d}^{q_2+\cdots q_p+2p-1}y,\ldots)$$

$$\bar{f}(t,s^0,\ldots,s^p,\tilde{d}^{q_1+\cdots+q_p+2p}y, d_0^{q_0+2}\tilde{d}^{q_2+\cdots q_p+2p-1}y,\ldots)$$
$$= (t,s^0,\ldots,s^p,d_{q_0+1}\tilde{d}^{q_1+\cdots+q_p+2p}y, d_{q_1+1}d_0^{q_0+2}\tilde{d}^{q_2+\cdots q_p+2p-1}y,\ldots)$$

On the other hand

$$f(t, s^0, \ldots, s^p, y) = (t, s^0, \ldots, s^p, d_{n+p+1} \ldots d_{n+2p+1} y)$$

$$(id \times AW)(t, s^0, \ldots, s^p, d_{n+p+1} \ldots d_{n+2p+1} y) \\ = \left(t, s^0, \ldots, s^p, \tilde{d}^{q_1 + \cdots + q_p + p} d_{n+p+1} \ldots d_{n+2p+1} y, d_0^{q_0+1} \tilde{d}^{q_2 + \cdots q_p + p} d_{n+p+1} \ldots d_{n+2p+1} y, \ldots\right),$$

where $n = p + q$. One can see that $\tilde{\tilde{f}} \circ (id \times AW) = (id \times AW) \circ f$.

We conclude this section by giving the role of the prismatic subdivision in gauge theory. One can define a bundle over a simplicial set (see Akyar [3]) and by pulling back this bundle we get a bundle over $|||\bar{P}S_{.}|||$. The homotopy equivalence $|||\bar{P}S_{.}||| \approx ||S_{.}||$ and the transition functions are used to define a classifying map on $|||\bar{P}S_{.}|||$ (see Akyar-Dupont [2]).

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