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VARIOUS EXACT SOLUTIONS OF SOME NONLINEAR EQUATIONS BY A DIRECT ALGEBRAIC METHOD

Y. Ugurlu[a] , *I.E. Inan*[b]
[a]*Firat University, Department of Mathematics, 23119 Elazığ, TURKEY*
matematikci_23@yahoo.com.tr
[b]*Firat University, Faculty of Education, 23119 Elazığ, TURKEY*
ieinan@yahoo.com

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Abstract

In this paper, we implemented a direct algebraic method for the exact solutions of the Liouville equation, Dodd-Bullough-Mikhailov equations. By using this method, we find several exact solutions of the Liouville equation, Dodd-Bullough-Mikhailov equations.

Key Words: Direct algebraic method; Liouville equation; Dodd-Bullough-Mikhailov equations.

1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics. Traveling wave solutions of nonlinear equations in mathematical physics play an important role in soliton theory [1, 2]. It is well known that searching for explicit solutions for nonlinear evolution equation, by using different methods, is the goal for many researchers. So many analytic methods have been introduced in literature [3-16]. Some of them are: Bäcklund transformation, Generalized Miura Transformation, Darboux transformation, Cole–Hopf transformation, tanh method, sine–cosine method, Painleve method, homogeneous balance method, similarity reduction method and so on.

Two classes of nonlinear partial equations,

$$u_{xt} + f(u) = 0 \tag{1}$$

and

$$u_{tt} - u_{xx} + f(u) = 0 \tag{2}$$

play an important role in many scientific applications such as solid state physics, nonlinear optics, plasma physics, fluid dynamics, mathematical biology, dislocations in crystals, kink dynamics, and chemical kinetics and quantum field theory. The function $f(u)$ takes many forms such as

$$f(u) = \begin{cases} sinu \\ sinhu \\ e^{u} \\ e^{u} + e^{-2u} \\ e^{-u} + e^{-2u} \end{cases} \tag{3}$$

that characterize the Sine–Gordon equation, *sinh*-Gordon equation, Liouville equation, Dodd–Bullough–Mikhailov equation (DBM), and the Tzitzeica–Dodd–Bullough (TDB) equation respectively.

The first two equations gained its importance when it gave kink and antikink solutions with the collisional behaviors of solitons. A kink is a solution with boundary values 0 and 2π at the left infinity and at the right infinity respectively [17, 18]. The last two equations, Dodd–Bullough–Mikhailov equation and the Tzitzeica–Dodd–Bullough equation, appear in problems varying from fluid flow to quantum field theory. In addition, these two equations are integrable, when boundary conditions are periodic, giving plenty of quasi periodic solutions. Other equations for other forms of $f(u)$ appear as well, such as the Klein–Gordon equation and the$\emptyset^4$ equation if we substitute $f(u) = mu$ and $f(u) = u + u^3$ in (2) respectively [19].

There are several forms of Liouville equation. It is well known that these equations can be converted to the well-known sinh–Gordon equation and other extended combined sinh–cosh–Gordon equations. Wazwaz obtained travelling wave solutions, explicit and implicit for these forms by using variable separated ODE method [20]. Borhanifar and Moghanlu [21]; obtained exact solutions for DBM equation by using $\frac{G'}{G}$- expansion method, Fan and Hon [22]; found some new explicit solutions for DBM equation by using extended tanh method, He and Wu [23]; obtained periodic solution or compact-like solution for DBM equation by using Exp-function method.

In this study, we implemented a direct algebraic method [24]; for finding the exact solutions of Liouville equation, Dodd-Bullough-Mikhailov equations [19].The study is organized as follows. In Section 2, the key idea of proposed method its applications are described. We conclude this paper in Section 3.

2. An Analysis of the Method and applications

Before starting to give a detail of the method, we will give a simple description of a direct algebraic method [24]. For doing this, one can consider in a two variables general form of nonlinear PDE

$$Q(u, u_t, u_x, u_{xx}, \dots) = 0, \quad (4)$$

and transform Eq. (4) with

$$u(x,t) = u(\xi), \quad \xi = kx + wt,$$

where, k and λ are arbitrary constants, respectively. After the transformation, we get a nonlinear ODE for $u(\xi)$

$$Q'(u', u'', u''', \dots) = 0. \tag{5}$$

Then, the solution of the equation (5) we are looking for is expressed in the form as a

$$u(x,t) = u(\xi) = \sum_{i=0}^{M} a_i F^i, \tag{6}$$

M is a positive integer that can be determined by balancing the highest order derivative and with the highest nonlinear terms in equation, and $k, w, a_0, a_1, \dots, a_M$ are parameters to be determined. Substituting solution (6) into Eq. (5) yields a set of algebraic equations for $F'F^i$ or F^i, then, all coefficients of $F'F^i$ or F^i have to vanish. After this separated algebraic equation, we can found $k, w, a_0, a_1, \dots, a_M$ constants.

In this work, we aim to solve the shallow water wave equationby using the direct algebraic method which is introduced by Zhang [24].

$$F'^2 = q_3 F^3 + q_2 F^2 , \quad (7)$$

where $F' = \dfrac{dF}{d\xi}$ and q_2, q_3 are constants. The author has given several cases to get the solutions of Eq. (7).

Example 1. Consider the Liouvilleequation [19]

$$u_{xt} + e^u = 0 . \quad (8)$$

We make transformation $u = \ln v$. Let us consider the traveling wavesolutions $v(x,t) = v(\xi)$, $\xi = kx + wt$, then Eq. (8) becomes

$$kwvv'' - kw(v')^2 + v^3 = 0 .(9)$$

When balancing vv'' with v^3 then gives *M=1*. Therefore, we may choose

$$v = a_0 + a_1 F . \quad (10)$$

Substituting (10) into Eq. (9) yields a set of algebraic equations for $a_0, a_1, q_2, q_{3,} k, w$. These systems are finding as

$$\begin{aligned} &a_0^3 = 0 , \\ &3a_0^2 a_1 + a_0 a_1 k q_2 w = 0 , \\ &3a_0 a_1^2 + \frac{3}{2} a_0 a_1 k q_3 w = 0 \\ &\text{-} a_1^3 + \frac{1}{2} a_1^2 k q_3 w = 0 , \end{aligned} \quad ,(11)$$

From the solutions of the system, we can found

$$a_0 = 0 \ , \ a_1 = a_1 \ , \ w = -\frac{2a_1}{kq_3}, \ q_2 = q_2, \ k \neq 0 \ , \ q_3 \neq 0 \ , \quad (12)$$

with the aid of Mathematica. Substituting (10) and (12) into (7) we have obtained the following solution of equation (8). These solutions are

Solution 1:

$$u(x,t) = Ln\left(\frac{a_1}{c_1 e^{\sqrt{q_2}(kx - \frac{2a_1}{kq_3} t)} + c_2 e^{-\sqrt{q_2}(kx - \frac{2a_1}{kq_3} t)} - \dfrac{q_3}{2q_2}} \right), \quad (13)$$

where $q_2 > 0$, $16 c_1 c_2 q_2^2 = q_3^2$.

Solution 2:

$$u(x,t)=Ln\left(\frac{a_1}{c_1Cos\left(\sqrt{-q_2}\left(kx-\frac{2a_1}{kq_3}t\right)\right)+c_2Sin\left(\sqrt{-q_2}\left(kx-\frac{2a_1}{kq_3}t\right)\right)-\frac{q_3}{2q_2}}\right), \tag{14}$$

where $q_2<0,\ 4q_2^2\left(c_1^2+c_2^2\right)=q_3^2$.

Solution 3:

$$u(x,t)=Ln\left(\frac{a_1}{\frac{q_3}{4}\left(kx-\frac{2a_1}{kq_3}t\right)^2+c_1\left(kx-\frac{2a_1}{kq_3}t\right)+c_2}\right), \tag{15}$$

where $c_1^2=q_3c_2$ when $q_2=0$.

Example 2. Consider the Dodd-Bullough-Mikhailov equation [19]

$$u_{xt}+\alpha e^u+\beta e^{-2u}=0. \tag{16}$$

We make transformation $u=\ln v$. Let us consider the traveling wave solutions $v(x,t)=v(\xi)$, $\xi=kx+wt$, then Eq. (16) becomes

$$kwvv''-kw(v')^2+\alpha v^3+\beta=0. \tag{17}$$

When balancing vv'' with v^3 then gives *M=1*. Therefore, we may choose

$$v=a_0+a_1F. \tag{18}$$

Substituting (18) into Eq. (17) yields a set of algebraic equations for $a_0,a_1,q_2,q_{3,}k,w$. These systems are finding as

$$\begin{gathered}a_0^3\alpha+\beta=0,\\ 3a_0^2a_1\alpha+a_0a_1kq_2w=0,\\ 3a_0a_1^2\alpha+\frac{3}{2}a_0a_1kq_3w=0,\\ a_1^3\alpha+\frac{1}{2}a_1^2kq_3w=0,\end{gathered} \tag{19}$$

From the solutions of the system, we can found

$$a_0=-\sqrt[3]{\frac{\beta}{\alpha}}\ ,\ a_1=-\frac{3}{2}\sqrt[3]{\frac{\beta}{\alpha}}\frac{q_3}{q_2},\quad w=\frac{3\sqrt[3]{\alpha^2\beta}}{kq_2}\ ,\ \alpha\neq 0,\ k\neq 0\ ,\ q_2\neq 0,\ \ q_3\neq 0\,, \tag{20}$$

with the aid of Mathematica. Substituting (18) and (20) into (7) we have obtained the following solutions of equation (16). These solutionsare

Solution 1:

$$u(x,t)=Ln\left(-\sqrt[3]{\frac{\beta}{\alpha}}-\frac{\frac{3}{2}\sqrt[3]{\frac{\beta}{\alpha}}\frac{q_3}{q_2}}{c_1e^{\sqrt{q_2}\,\xi}+c_2e^{-\sqrt{q_2}\,\xi}-\frac{q_3}{2q_2}}\right), \tag{21}$$

where $q_2>0$, $16c_1c_2q_2^2=q_3^2$, $\xi=kx+\frac{3\sqrt[3]{\alpha^2\beta}}{kq_2}t$.

Solution 2:

$$u(x,t)=Ln\left(-\sqrt[3]{\frac{\beta}{\alpha}}-\frac{\frac{3}{2}\sqrt[3]{\frac{\beta}{\alpha}}\frac{q_3}{q_2}}{c_1Cos\left(\sqrt{-q_2}\xi\right)+c_2Sin\left(\sqrt{-q_2}\xi\right)-\frac{q_3}{2q_2}}\right), \tag{22}$$

where $q_2<0$, $4q_2^2\left(c_1^2+c_2^2\right)=q_3^2$, $\xi=kx+\frac{3\sqrt[3]{\alpha^2\beta}}{kq_2}t$.

Solution 3:

In the case of $q_2\neq 0$, there are no solutions.

Example 3. Consider the a different Dodd-Bullough-Mikhailov [19]

$$u_{tt}-u_{xx}+e^{u}+e^{-2u}=0\,, \tag{23}$$

We make transformation $u=\ln v$. Let us consider the traveling wave solutions $v(x,t)=v(\xi)$, $\xi=kx+wt$, then Eq. (23) becomes

$$\left(w^2-k^2\right)vv''-\left(w^2-k^2\right)\left(v'\right)^2+v^3+1=0 \qquad ,(24)$$

The solutions of the systems, we are looking for is stated in the form

$$v(x,t)=v(\xi)=\sum_{i=0}^{M_1}a_iF^i\ ,.$$

When balancing vv'' with v^3 then gives *M=1*. Thus, we can we may choose

$$v = a_0 + a_1 F, \tag{25}$$

Substituting (25) into Eq. (24) yields a set of algebraic equations for $a_0, a_1, a_2, q_2, q_{3,} k, w$. These systems are finding as

$$\begin{aligned} &1 + a_0^3 = 0, \\ &3a_0^2 a_1 - a_0 a_1 k^2 q_2 + a_0 a_1 q_2 w^2 = 0, \\ &3a_0 a_1^2 - \frac{3}{2} a_0 a_1 k^2 q_3 + \frac{3}{2} a_0 a_1 q_3 w^2 = 0, \\ &a_1^3 - \frac{1}{2} a_1^2 k^2 q_3 + \frac{1}{2} a_1^2 q_3 w^2 = 0. \end{aligned} \tag{26}$$

From the solutions of the system, we can found

Case 1:

$$a_0 = -1, \; a_1 = -\frac{3q_3}{2q_2}, \quad k = \mp\sqrt{\frac{-3 + q_2 w^2}{q_2}}, \; q_2 \neq 0, \tag{27}$$

with the aid of Mathematica. Substituting (25) and (27) into (7) we have obtained the solutions of the system of equation (23). These solutions are

Solution 1:

$$u(x,t) = Ln\left(-1 - \frac{\frac{3q_3}{2q_2}}{c_1 e^{\sqrt{q_2}\xi} + c_2 e^{-\sqrt{q_2}\xi} - \frac{q_3}{2q_2}} \right), \tag{28}$$

where $q_2 > 0$, $16c_1 c_2 q_2^2 = q_3^2$, $\xi = \mp\sqrt{\frac{-3 + q_2 w^2}{q_2}}\, x + wt$.

Solution 2:

$$u(x,t) = Ln\left(-1 - \frac{\frac{3q_3}{2q_2}}{c_1 Cos\left(\sqrt{-q_2}\xi\right) + c_2 Sin\left(\sqrt{-q_2}\xi\right) - \frac{q_3}{2q_2}} \right), \tag{29}$$

where $q_2 < 0$, $4q_2^2\left(c_1^2 + c_2^2\right) = q_3^2$, $\xi = \mp\sqrt{\frac{-3 + q_2 w^2}{q_2}}\, x + wt$.

Solution 3:

In the case of $q_2 \neq 0$, there are no solutions.

Case 2:

$$a_0 = \frac{-i+\sqrt{3}}{i+\sqrt{3}},\ a_1 = \frac{3\left(q_3 - i\sqrt{3}\,q_3\right)}{4q_2},\ \ k = \mp\sqrt{\frac{3}{q_2} - \frac{6q_3}{q_2\left(q_3 - i\sqrt{3}\,q_3\right)} + w^2}$$

$$q_2 \neq 0,\ \ q_3 \neq 0,\ \ i^2 = -1, \tag{30}$$

therefore solutions of the Eq.(23)

Solution 1:

$$u(x,t) = Ln\left(\frac{-i+\sqrt{3}}{i+\sqrt{3}} + \frac{\dfrac{3\left(q_3 - i\sqrt{3}\,q_3\right)}{4q_2}}{c_1 e^{\sqrt{q_2}\xi} + c_2 e^{-\sqrt{q_2}\xi} - \dfrac{q_3}{2q_2}}\right), \tag{31}$$

where $q_2 > 0$, $16c_1c_2q_2^2 = q_3^2$, $\xi = \mp\sqrt{\frac{3}{q_2} - \frac{6q_3}{q_2\left(q_3 - i\sqrt{3}\,q_3\right)} + w^2}\ x + wt$.

Solution 2:

$$u(x,t) = Ln\left(\frac{-i+\sqrt{3}}{i+\sqrt{3}} + \frac{\dfrac{3\left(q_3 - i\sqrt{3}\,q_3\right)}{4q_2}}{c_1 Cos\left(\sqrt{-q_2}\xi\right) + c_2 Sin\left(\sqrt{-q_2}\xi\right) - \dfrac{q_3}{2q_2}}\right), \tag{32}$$

where $q_2 < 0$, $4q_2^2\left(c_1^2 + c_2^2\right) = q_3^2$, $\xi = \mp\sqrt{\frac{3}{q_2} - \frac{6q_3}{q_2\left(q_3 - i\sqrt{3}\,q_3\right)} + w^2}\ x + wt$.

Solution 3:

In the case of $q_2 \neq 0$, there are no solutions.

3. Conclusions

In this paper, we present the improved a direct algebraic method by using ansatz (7) and, with aid of Mathematica, implement it in a computer algebraic system. An implementation of the method is given by applying it to Liouville equation, Dodd-Bullough-Mikhailov equations with physics interests. More importantly, for the Liouville equation, Dodd-Bullough-Mikhailov equations, we also obtain some different solutions at same time. The method can be used to many other nonlinear equations or coupled ones. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation on a computer.

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