Erciyes Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 38, Sayı 1, 2022

Erciyes University
Journal of Institue Of Science and Technology
Volume 38, Issue 1, 2022

On Spherical Inversions in Three Dimensional Tetrakis Hexahedron Space

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(Alınış / Received: 07.01.2022, Kabul / Accepted: 14.04.2022, Online Yayınlanma / Published Online: 30.04.2022)

Keywords
Tetrakis Hexahedron Space,
Spherical Inversion,
Tetrakis Hexahedron Sphere,
Cross Ratio,
Harmonic Conjugates

Abstract: In this research, we study on general properties and basic concepts of spherical inversions in Tetrakis Hexahedron space. We also investigate cross ratio and harmonic conjugates and inverses of lines, planes and Tetrakis Hexahedron spheres in $\mathbb{R}^3_{TH}$ under an inversion with respect to a Tetrakis Hexahedron sphere.

Üç Boyutlu Tetrakis Hexahedron Uzayında Küresel İnversiyonlar Üzerine

Anahtar Kelimeler
Tetrakis Hexahedron Uzayı,
Küresel İnversiyon,
Tetrakis Hexahedron Küresi,
Çifte Oran,
Harmonik Bölme

Öz: Bu çalışmada, Tetrakis Hexahedron uzayında küresel inversiyonların genel özellikleri ve temel kavramları üzerinde çalıştık. Ayrıca çifte oran ve harmonik eşlenik kavramları ile $\mathbb{R}^3_{TH}$ de bir Tetrakis Hexahedron küresine göre bir inversiyon altında doğruların, düzlemlerin ve Tetrakis Hexahedron kürelerinin küresel tersleri araştırıldı.

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1. Introduction

Inversion is one of the most interesting transformation in the plane, since it reveals difficult questions and many challenging problems. Also in geometry many problems become much manageable when an inversion is applied. As it has been stated in [1] Apollonious of Perga was probably the first to reveal this transformation in his last book *Plane Loci*. Later in 1820s Jakob Steiner investigated inversion systematically. Inversion would be used to examine some problems and theorems in geometry as Pappus chain theorem, Feuerbach's theorem, Ptolemy's theorem, Steiner porism, the problem of Apollonius, etc. [2]. When an inversion is considered the first thing that comes to mind is an inversion with respect to a circle, but some authors investigated different inversion maps by using other objects, see [3, 4, 5, 6, 7] and some authors defined new inversion maps by using different distance functions, see [4, 8, 9, 10, 11]. Furthermore inversion has been studied in three dimensional Euclidean and non-Euclidean spaces, see [12, 13, 14].
As it has stated in [15] Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions and only because the distance is not uniform in all directions it is a non-Euclidean geometry. The unit ball of a Minkowski geometry is a general symmetric convex set. Throughout the studies on polyhedra and metric geometry it has seen that unit balls of some Minkowski geometries are convex solids, some of these studies are [16, 17, 18, 19]. In [20, 21, 22, 23, 24, 25, 26, 27] some metrics are given which are induced by some of convex polyhedra such that their unit spheres are corresponding convex solids. Since the only difference of a Minkowski geometry and the Euclidean geometry is the distance, it is interesting to study on the problems of the Euclidean geometry that include the distance concept in different Minkowski geometries. By these motivations in this study first we define the inversion with respect to a sphere in Tetrakis Hexahedron space. Then we investigate general properties and basic concepts of this inversion. Furthermore we give some properties related with spherical inversion in Tetrakis Hexahedron space such as cross-ratio and harmonic conjugates.

2. Material and Method

This section consists of two subsections to give primary definitions of Tetrakis Hexahedron space and spherical inversion in this space.

2.1 Some Basics of Tetrakis Hexahedron Space

Now we give some basic definitions of tetrakis hexahedron space, for more detail see [23]. Geometrical construction of Tetrakis Hexahedron space $\mathbb{R}^3_{TH}$ is similar to the well-known Euclidean space $\mathbb{R}^3$. Set of points and collection of lines are the same, the angles are measured by the same way. The only difference is the definition of the distance. Tetrakis Hexahedron metric in $\mathbb{R}^3$ is defined by using the distance function

$$d_{TH}(P_1, P_2) = max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\} + (\sqrt{3} - 1)mid\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\} \qquad (1)$$

where $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$. Thus the distance is sum of maximum and $(\sqrt{3} - 1)$ times of middle of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. The unit ball in $\mathbb{R}^3_{TH}$ is the set of all points (x, y, z) satisfing the equation

$$max\{|x|, |y|, |z|\} + (\sqrt{3} - 1)mid\{|x|, |y|, |z|\} = 1$$

which is a Tetrakis Hexahedron.

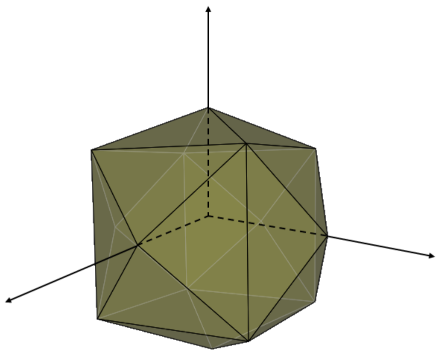

Figure 1. Unit ball in $\mathbb{R}^3_{TH}$

2.2 Preliminaries about Inversions in Tetrakis Hexahedron Space

In this subsection we define an inversion with respect to a sphere in Tetrakis Hexahedron space as an analogue of inversion in $\mathbb{R}^3$. As it has been stated in [4] and [28] an inversion with respect to a circle with radius r is a mapping that transforms points inside out and outside in of the circle such that a point P and its inverse point P' are on a ray emanating from the center O of the circle where the points P and P' satisfies the equation $d(O,P).d(O,P')=r^2$. Since an inversion maps points close to O to points far from O, and maps points far from O to points close to O, this classical definition of inversion excludes O the center of inversion. Thus expanding the Euclidean plane by adjoining one "ideal point", or "point at infinity", we can include O in the domain and range of an inversion.
Now we define the new concept of inversion in $\mathbb{R}^3_{TH}$ as follows:

Definition 2.4 Let $\mathcal{T}$ be a TH-sphere centered at the point O with radius r in $\mathbb{R}^3_{TH}$, and P_∞ be the ideal point adjoined to the Tetrakis Hexahedron space. In $\mathbb{R}^3_{TH}$ the TH-spherical inversion with respect to $\mathcal{T}$ is the transformation

$$I_{\mathcal{T}(O,r)}: \mathbb{R}^3_{TH} \cup \{P_\infty\} \to \mathbb{R}^3_{TH} \cup \{P_\infty\}$$

defined by $I_{\mathcal{T}(O,r)}(O)=P_\infty$, $I_{\mathcal{T}(O,r)}(P_\infty)=O$, $I_{\mathcal{T}(O,r)}(P)=P'$ for $P\neq 0$ and P' lies on the ray $\overrightarrow{OP}$ and

$$d_{TH}(O,P).d_{TH}(O,P')=r^2 \quad (2)$$

$\mathcal{T}$ is called the sphere of the inversion, O is called the center of inversion, the point P' is called the inverse of the point P with respect to the sphere $\mathcal{T}$.

In Euclidean space, an inversion shifts the points outside to the inside of the sphere and vice versa. Now the following theorem states that this property is valid in the Tetrakis Hexahedron space.

Lemma 2.5 Let $\mathcal{T}$ be the TH-sphere with center O and the radius r. If the point P is in the interior of $\mathcal{T}$, the point P' is exterior to $\mathcal{T}$, and viceversa.

Proof. Let us consider the inversion $I_{\mathcal{T}(O,r)}$ with respect to the sphere $\mathcal{T}$ with center O and the radius r and the point P which is in the interior of $\mathcal{T}$. Thus, $d_{TH}(O,P)<r$. Since $P'=I_{\mathcal{T}(O,r)}(P)$ and by Eq. (3), $r^2=d_{TH}(O,P).d_{TH}(O,P')<r.d_{TH}(O,P')$ then $d_{TH}(O,P')>r$. So the point P' is in the exterior of $\mathcal{T}$.

Corollary 2.6 Under a spherical inversion $I_{\mathcal{T}(O,r)}$ in $\mathbb{R}^3_{TH}$, $\mathcal{T}$ itself is left pointwise fixed.

Theorem 2.7 If P and P' is a pair of inverse points with respect to the tetrakis hexahedron spherical inversion $I_{\mathcal{T}(O,r)}$ with center $O=(0,0,0)$ and radius r then

$$P'=\mu P \quad (3)$$

where $\mu=r^2/\left(d_{TH}(O,P)\right)^2$

Proof. Let $P=(x,y,z)$ and $P'=(x',y',z')$ be inverse pair with respect to the tetrakis hexahedron spherical inversion $I_{\mathcal{T}(O,r)}$ with center $O=(0,0,0)$ and radius r. Since the points P and P' are on the ray emanating from O

$$\overrightarrow{OP'}=\mu\overrightarrow{OP} \ ,\mu\in\mathbb{R}^+$$

Thus $(x',y',z')=(\mu x,\mu y,\mu z)$. By the equation (2) we get that $\mu=r^2/\left(d_{TH}(O,P)\right)^2$ and by substituting the resulting value of μ the required result is obtained.

Note that since P and P' is a pair of inverse points with respect to the tetrakis hexahedron spherical inversion $I_{\mathcal{T}(O,r)}$ with center $O=(0,0,0)$ and radius r, the coordinates of P would be obtained by the coordinates of P' by the same way in the Theorem 2.7. Thus $P=\mu P'$ where $\mu=r^2/\left(d_{TH}(O,P')\right)^2$

Corollary 2.8 Let $P=(x,y,z)$ and $P'=(x',y',z')$ is an inverse pair under the tetrakis hexahedron spherical inversion $I_{\mathcal{T}(O,r)}$ with center $O=(x_0,y_0,z_0)$ and radius r then

$$P' - O = \mu(P - O) \tag{4}$$

where $\mu = r^2/(d_{TH}(O,P))^2$.

Proof. It is easy to see that translation preserves distances in $\mathbb{R}^3_{TH}$. Thus by translating $(0,0,0)$ to (x_0, y_0, z_0) in $\mathbb{R}^3_{TH}$ values of x', y', z' would easily be obtained as required.

Theorem 2.9 Let O, P and Q be any three collinear distinct points in $\mathbb{R}^3_{TH}$. If P, P' and Q, Q' are inverse pairs with respect to the tetrakis hexahedron spherical inversion $I_{\mathcal{T}(O,r)}$ then

$$d_{TH}(P', Q') = \frac{r^2 .\ d_{TH}(P,Q)}{d_{TH}(O,P)\ .\ d_{TH}(O,Q)} \tag{5}$$

Proof. Let $I_{\mathcal{T}(O,r)}$ be the spherical inversion with center O and radius r in $\mathbb{R}^3_{TH}$. If P, P' and Q, Q' are inverse pairs with respect to $I_{\mathcal{T}(O,r)}$ then by equation (1), $d_{TH}(O,P).d_{TH}(O,P') = r^2 = d_{TH}(O,Q).d_{TH}(O,Q')$. Since O, P and Q are collinear points and ratios of Euclidean and Tetrakis Hexahedron distances along a line are the same,

$$\begin{aligned} d_{TH}(P', Q') &= |d_{TH}(O,P') - d_{TH}(O,Q')| \\ &= \left|\frac{r^2}{d_{TH}(O,P)} - \frac{r^2}{d_{TH}(O,Q)}\right| \\ &= \frac{r^2 .\ d_{TH}(P,Q)}{d_{TH}(O,P)\ .\ d_{TH}(O,Q)} \end{aligned}$$

is obtained.

Note that converse statement of the theorem above is not true. Also the theorem is not valid for any three non-collinear points in $\mathbb{R}^3_{TH}$. But under some other conditions the equation (5) holds. Now we give the following theorem that shows the equation (5) is satisfied under such conditions.

Theorem 2.10 Let O, P and Q be any three distinct points in $\mathbb{R}^3_{TH}$, P, P' and Q, Q' be inverse pairs with respect to the tetrakis hexahedron spherical inversion $I_{\mathcal{T}(O,r)}$ with center O and radius r, and u and v be direction vectors of the rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, respectively. If $u \in \Delta_i$ and $v \in \Delta_i \setminus \{u\}$ where

$\Delta_1 = \{(1,0,0), (0,1,0), (0,0,1), (-1,0,0), (0,-1,0), (0,0,-1)\}$
$\Delta_2 = \{(1,1,0), (1,0,1), (0,1,1), (1,0,-1), (1,-1,0), (0,1,-1), (0,-1,1), (0,-1,-1), (-1,1,0), (-1,0,1), (-1,0,-1), (-1,-1,0)\}$
$\Delta_3 = \{(1,1,1), (1,1,-1), (1,-1,1), (-1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1), (-1,-1,-1)\}$

and $i = 1,2,3$, then

$$d_{TH}(P', Q') = \frac{r^2\ .\ d_{TH}(P,Q)}{d_{TH}(O,P)\ .\ d_{TH}(O,Q)}$$

Proof. Since all translations are elements of the group of isometries of Tetrakis Hexahedron space it is convenient to consider O the center of inversion as origin. So let $I_{\mathcal{T}(O,r)}$ be the tetrakis hexahedron spherical inversion with center O and radius r in $\mathbb{R}^3_{TH}$. Suppose that $u \in \Delta_1$ and $v \in \Delta_1 \setminus \{u\}$. If $P = (0,0,p)$ and $Q = (q,0,0)$ then the inverses of P and Q with respect to $I_{\mathcal{T}(O,r)}$ are $P' = \left(0,0,\frac{r^2}{p}\right)$ and $Q' = \left(\frac{r^2}{q},0,0\right)$, respectively. Thus we get

$d_{TH}(P', Q') = max\left\{\left|\frac{r^2}{p}\right|, \left|\frac{r^2}{q}\right|\right\} + (\sqrt{3}-1)mid\left\{\left|\frac{r^2}{p}\right|, \left|\frac{r^2}{q}\right|\right\}$. Here there are two subcases;

Case 1: If $|p| \geq |q|$, then $d_{TH}(P', Q') = \frac{r^2(|p| + (\sqrt{3}-1)|q|)}{|p||q|} = \frac{r^2 d_{TH}(P,Q)}{d_{TH}(O,P).d_{TH}(O,Q)}$.
Case 2: If $|p| < |q|$, then $d_{TH}(P', Q') = \frac{r^2(|q| + (\sqrt{3}-1)|p|)}{|p||q|} = \frac{r^2 d_{TH}(P,Q)}{d_{TH}(O,P).d_{TH}(O,Q)}$.

Suppose that $u \in \Delta_2$ and $v \in \Delta_2 \setminus \{u\}$. If $P = (p,p,0)$ and $Q = (q,-q,0)$ then the inverses of P and Q with respect to $I_{\mathcal{T}(O,r)}$ are $P' = \left(\frac{r^2}{p}, \frac{r^2}{p}, 0\right)$ and $Q' = \left(\frac{r^2}{q}, \frac{-r^2}{q}, 0\right)$, respectively. Thus we get

$d_{TH}(P', Q') = max\left\{\left|\frac{r^2}{p} - \frac{r^2}{q}\right|, \left|\frac{r^2}{p} + \frac{r^2}{q}\right|, 0\right\} + (\sqrt{3}-1)mid\left\{\left|\frac{r^2}{p} - \frac{r^2}{q}\right|, \left|\frac{r^2}{p} + \frac{r^2}{q}\right|, 0\right\}$. Here there are two subcases;

Case 1: If $|p-q| \geq |p+q|$, then $d_{TH}(P',Q') = \frac{r^2(|p-q|+(\sqrt{3}-1)|p+q|)}{|p||q|} = \frac{r^2 d_{TH}(P,Q)}{d_{TH}(O,P).d_{TH}(O,Q)}$.
Case 2: If $|p+q| < |p-q|$, then $d_{TH}(P',Q') = \frac{r^2(|p+q|+(\sqrt{3}-1)|p-q|)}{|p||q|} = \frac{r^2 d_{TH}(P,Q)}{d_{TH}(O,P).d_{TH}(O,Q)}$.

Suppose that $u \in \Delta_3$ and $v \in \Delta_3 \setminus \{u\}$. If $P=(p,p,p)$ and $Q=(-q,q,q)$ then the inverses of P and Q with respect to $I_{\mathcal{T}(O,r)}$ are $P' = \left(\frac{r^2}{3p}, \frac{r^2}{3p}, \frac{r^2}{3p}\right)$ and $Q' = \left(\frac{-r^2}{3q}, \frac{r^2}{3q}, \frac{r^2}{3q}\right)$, respectively. Thus we get

$d_{TH}(P',Q') = max\left\{\left|\frac{r^2}{3p} - \frac{r^2}{3q}\right|, \left|\frac{r^2}{3p} + \frac{r^2}{3q}\right|\right\} + (\sqrt{3}-1)mid\left\{\left|\frac{r^2}{3p} - \frac{r^2}{3q}\right|, \left|\frac{r^2}{3p} + \frac{r^2}{3q}\right|\right\}$. Here there are two subcases;

Case 1: If $|p-q| \geq |p+q|$, then $d_{TH}(P',Q') = \frac{r^2(|p-q|+(\sqrt{3}-1)|p+q|)}{3|p||q|} = \frac{r^2 d_{TH}(P,Q)}{d_{TH}(O,P).d_{TH}(O,Q)}$.
Case 2: If $|p+q| < |p-q|$, then $d_{TH}(P',Q') = \frac{r^2(|p+q|+(\sqrt{3}-1)|p-q|)}{3|p||q|} = \frac{r^2 d_{TH}(P,Q)}{d_{TH}(O,P).d_{TH}(O,Q)}$.

For other possible choices of elements in $\Delta_i, i = 1,2,3$, by similar calculations it is easy to see that equality is valid.

3. Results

This section includes two subsections to investigate results and definitions obtained by spherical inversions in tetrakis hexahedron space. We study on inverses of lines, planes and tetrakis hexahedron spheres under an inversion $I_{\mathcal{T}(O,r)}$ as a comparison of inverses of lines and circles in Euclidean plane under a circular inversion. Also we investigate cross-ratio and harmonic conjugates in $\mathbb{R}^3_{TH}$.

3.1. Spherical Inversions of Lines, Planes and Tetrakis Hexahedron Spheres in $\mathbb{R}^3_{TH}$

In Euclidean version inverse of a line is a circle and inverse of a circle is a line, only the lines passing through the inversion center is invariant. In this section, tetrakis hexahedron spherical inversions of lines, planes and tetrakis hexahedron spheres are studied according to their positions in $\mathbb{R}^3_{TH}$.

Theorem 3.11 Let $I_{\mathcal{T}(O,r)}$ be a tetrakis hexahedron spherical inversion with center O and radius r. Any line and any plane containing O is invariant under $I_{\mathcal{T}(O,r)}$.

Proof. Consider the tetrakis hexahedron spherical inversion $I_{\mathcal{T}(O,r)}$ with center O and radius r. By equation (2) it is obvious that a line passing through O is invariant under $I_{\mathcal{T}(O,r)}$. Let $Ax+By+Cz=0$ be a plane containing O. Under $I_{\mathcal{T}(O,r)}$ we get the equation of the plane as;

$$A\frac{r^2 x\prime}{(d_{TH}(O,P\prime))^2} + B\frac{r^2 y\prime}{(d_{TH}(O,P\prime))^2} + C\frac{r^2 z\prime}{(d_{TH}(O,P\prime))^2} = 0.$$

That is $Ax'+By'+Cz'=0$ which completes the proof.

Theorem 3.12 Let $I_{\mathcal{T}(O,r)}$ be a tetrakis hexahedron spherical inversion with center O and radius r. The inverse of a tetrakis hexahedron sphere with center O under $I_{\mathcal{T}(O,r)}$ is a tetrakis hexahedron sphere with center O.

Proof. Since the translation preserves distance in $\mathbb{R}^3_{TH}$ we would take center of inversion $I_{\mathcal{T}(O,r)}$ as $O=(0,0,0)$, thus the tetrakis hexahedron sphere $\mathcal{T}$ with center O and radius r is

$$\mathcal{T} = \{P=(x,y,z): d_{TH}(O,P) = r\}$$

Let $\mathcal{T}_1$ be the tetrakis hexahedron sphere with center O and radius r_1, then

$$\mathcal{T}_1 = \{P=(x,y,z): d_{TH}(O,P) = r_1\}$$

Thus the inverse of $\mathcal{T}_1$ under $I_{\mathcal{T}(O,r)}$ is $\mathcal{T}'_1 = \left\{P' = (x',y',z'): d_{TH}(O,P') = \frac{r^2}{r_1}\right\}$ which is a tetrakis hexahedron sphere.

Theorem 3.13 Let $I_{\mathcal{T}(O,r)}$ be a tetrakis hexahedron spherical inversion with center O and radius r. The inverse of every edges, vertices and faces of $\mathcal{T}$ is itself.

Proof. By Corollary 2.6, $\mathcal{T}$ is pointwise fixed under $I_{\mathcal{T}(O,r)}$. Thus every edges, vertices and faces of $\mathcal{T}$ is invariant under $I_{\mathcal{T}(O,r)}$.

3.2. The Cross Ratio and Harmonic Conjugates in $\mathbb{R}^3_{TH}$

The distance is not invariant under tetrakis hexahedron spherical inversion. Thus, the inversion in tetrakis hexahedron space is not an isometry. However, the fact that the cross-ratio is preserved under inversion reveals the necessity of focusing on the cross-ratio by means of the distance. Therefore, in this section, we investigate the cross ratio and harmonic conjugates in $\mathbb{R}^3_{TH}$ under a spherical inversion.

The following definition will be given in a similar sense of the definition given in [29].

Definition 3.13 For any two points X and Y on a directed line l, the directed tetrakis hexahedron length of the line segment $\overline{XY}$ is denoted by $d_{TH}[X,Y]$. If the line segment $\overline{XY}$ and l have the same direction, then $d_{TH}[X,Y]=d_{TH}(X,Y)$ and if have the opposite direction, then $d_{TH}[X,Y]=-d_{TH}(X,Y)$.

Definition 3.14 Let P,Q,R and S are four distinct points on an oriented line in $\mathbb{R}^3_{TH}$. The tetrakis hexahedron cross-ratio $(PQ,RS)_{TH}$ is defined by

$$(PQ,RS)_{TH}=\frac{d_{TH}[P,R]d_{TH}[Q,S]}{d_{TH}[P,S]d_{TH}[Q,R]} \tag{6}$$

Corollary 3.15 Let P,Q,R and S are four distinct points on an oriented line in $\mathbb{R}^3_{TH}$. The tetrakis hexahedron cross-ratio $(PQ,RS)_{TH}$ is positive if both R and S are between P and Q or if neither R nor S are between P and Q.

Proof. Let both R and S points be between P and Q points. For the directed line PQ the tetrakis hexahedron cross-ratio is

$$\begin{aligned}(PQ,RS)_{TH}&=\frac{d_{TH}[PR]d_{TH}[QS]}{d_{TH}[PS]d_{TH}[QR]}\\&=\frac{d_{TH}(P,R).\left(-d_{TH}(Q,S)\right)}{d_{TH}(P,S).\left(-d_{TH}(Q,R)\right)}=\frac{d_{TH}(P,R).d_{TH}(Q,S)}{d_{TH}(P,S).d_{TH}(Q,R)}\end{aligned}$$

and thus $(PQ,RS)_{TH}$ is positive.

If neither R nor S are between P and Q, then there are six arrays for R and S. Since it is similar to prove for all possible combinations we give the proof for the orientation $R-P-Q-S$. Thus the tetrakis hexahedron cross-ratio is

$$\begin{aligned}(PQ,RS)_{TH}&=\frac{d_{TH}[PR]d_{TH}[QS]}{d_{TH}[PS]d_{TH}[QR]}\\&=\frac{\left(-d_{TH}(P,R)\right).d_{TH}(Q,S)}{d_{TH}(P,S).\left(-d_{TH}(Q,R)\right)}=\frac{d_{TH}(P,R).d_{TH}(Q,S)}{d_{TH}(P,S).d_{TH}(Q,R)}\end{aligned}$$

and thus $(PQ,RS)_{TH}$ is positive.

Corollary 3.16 Let P,Q,R and S are four distinct points on an oriented line in $\mathbb{R}^3_{TH}$. If the pairs $\{P,Q\}$ and $\{R,S\}$ seperate each other, then the tetrakis hexahedron cross-ratio $(PQ,RS)_{TH}$ is negative.

Proof. If the pairs $\{P,Q\}$ and $\{R,S\}$ seperate each other, then there are four arrays for R and S. For the orientation $R-P-S-Q$ the tetrakis hexahedron cross-ratio is

$$\begin{aligned}(PQ,RS)_{TH}&=\frac{d_{TH}[PR]d_{TH}[QS]}{d_{TH}[PS]d_{TH}[QR]}\\&=\frac{\left(-d_{TH}(P,R)\right).\left(-d_{TH}(Q,S)\right)}{d_{TH}(P,S).\left(-d_{TH}(Q,R)\right)}=-\frac{d_{TH}(P,R).d_{TH}(Q,S)}{d_{TH}(P,S).d_{TH}(Q,R)}\end{aligned}$$

and since for other possible arrays, by similar calculations, same results are obtained, thus $(PQ,RS)_{TH}$ is negative.

Theorem 3.17 The tetrakis hexahedron cross-ratio is invariant under tetrakis hexahedron spherical inversion in $\mathbb{R}^3_{TH}$.

Proof. Let $I_{\mathcal{T}(O,r)}$ be a tetrakis hexahedron spherical inversion with center O and radius r, and P, Q, R and S be four points on an oriented line l passing through O. Let P', Q', R' and S' be inverse points of P, Q, R and S respectively under $I_{\mathcal{T}(O,r)}$. Observe that the tetrakis hexahedron spherical inversion preserves the seperation or non-seperation of the pairs $\{P, Q\}$ and $\{R, S\}$ and also it reverses the tetrakis hexahedron - directed distance from the point P to the point Q along a line l to tetrakis hexahedron -directed distance from the point Q' to the point P'. The required result follows from Theorem 2.9;

$$(P'Q', R'S')_{TH} = \frac{d_{TH}(P', R').d_{TH}(Q'S')}{d_{TH}(P'S').d_{TH}(Q'R')}$$

$$= \frac{\frac{r^2.d_{TH}(P,R)}{d_{TH}(O,P).d_{TH}(O,R)}.\frac{r^2.d_{TH}(Q,S)}{d_{TH}(O,Q).d_{TH}(O,S)}}{\frac{r^2.d_{TH}(P,S)}{d_{TH}(O,P).d_{TH}(O,S)}.\frac{r^2.d_{TH}(Q,R)}{d_{TH}(O,Q).d_{TH}(O,R)}}$$

$$= \frac{d_{TH}(P,R).d_{TH}(Q,S)}{d_{TH}(P,S).d_{TH}(Q,R)}$$

$$= (PQ, RS)_{TH}$$

Definition 3.18 Let l be a line in $\mathbb{R}^3_{TH}$. Suppose that P, Q, R and S are four points on l. It is called that P, Q, R and S form a harmonic set if $(PQ, RS)_{TH} = -1$ and it is denoted by $H(PQ, RS)_{TH}$. That is, any pair R and S on l for which

$$\frac{d_{TH}[P,R]d_{TH}[Q,S]}{d_{TH}[P,S]d_{TH}[Q,R]} = -1 \tag{7}$$

is said to divide P and Q harmonically. The points R and S are called tetrakis hexahedron harmonic conjugates with respect to P and Q.

Theorem 3.19 Let T be a tetrakis hexahedron sphere with center O, and line segment $[PQ]$ be a diameter of T in $\mathbb{R}^3_{TH}$. Let R and S be distinct points of the ray $\overrightarrow{OP}$, which divide the segment $[PQ]$ internally and externally. Then R and S are tetrakis hexahedron harmonic conjugates with respect to P and Q if and only if R and S are inverse points with respect to the tetrakis hexahedron spherical inversion $I_{\mathcal{T}(O,r)}$.

Proof. Let R and S are tetrakis hexahedron harmonic conjugates with respect to P and Q. Then

$$(PQ, RS)_{TH} = -1 \Rightarrow \frac{d_{TH}[P, R].d_{TH}[Q, S]}{d_{TH}[P, S].d_{TH}[Q, R]} = -1$$

Since R divides the line segment $[PQ]$ internally and R is on the ray $\overrightarrow{OQ}$,

$$d_{TH}(R, Q) = r - d_{TH}(O, R) \text{ and } d_{TH}(P, R) = r + d_{TH}(O, R).$$

Since S divides the line segment $[PQ]$ externally and S is on the ray $\overrightarrow{OQ}$,

$$d_{TH}(P, S) = r + d_{TH}(O, S) \text{ and } d_{TH}(Q, S) = d_{TH}(O, S) - r.$$

Thus

$$\frac{\left(r + d_{TH}(O, R)\right).\left(d_{TH}(O, S) - r\right)}{\left(r + d_{TH}(O, S)\right).\left(r - d_{TH}(O, R)\right)} = -1$$

$$\Rightarrow \left(r + d_{TH}(O, R)\right).\left(d_{TH}(O, S) - r\right) = \left(r + d_{TH}(O, S)\right).\left(d_{TH}(O, R) - r\right).$$

Simplifying the last equality $d_{TH}(O, R).d_{TH}(O, S) = r^2$ is obtained. So R and S are tetrakis hexahedron spherical inverse points with respect to the tetrakis hexahedron spherical inversion $I_{\mathcal{T}(O,r)}$. For the other condition (S and R are on the ray $\overrightarrow{OP}$) by similar calculations the same conclusion is obtained.

Conversely, if R and S are tetrakis hexahedron spherical inverse points with respect to the tetrakis hexahedron spherical inversion $I_{\mathcal{T}(O,r)}$ the proof is similar.

4. Discussion and Conclusion

Inversion theory is of interest to geometers today, as it used to be, since it suggests challenging problems and when it is applied many problems in geometry became much manageable. Classical inversion is defined with respect to a circle but there are many different definitions of inversion in the literature by using other objects or using different distance functions or expanding dimension. In this study inversion is defined in a three dimensional non-Euclidean geometry and by using obtained results in this space some properties of this inversion is investigated. We hope that this topic would provoke further researches by interested readers or their students.

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