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On Bi-f-Harmonic Legendre Curves in Sasakian Space Forms

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Abstract: In this study, we consider bi-f-harmonic Legendre curves in Sasakian space forms. We investigate necessary and sufficient conditions for a Legendre curve to be bi-f-harmonic in various cases.

Keywords: Bi-f-harmonic curves, Legendre curves, Sasakian space forms.

1. Introduction

Let (N, g) and $(\bar{N}, \bar{g})$ be two Riemannian manifolds and $\psi : (N, g) \to (\bar{N}, \bar{g})$ be a smooth map. Then, let give the following definitions.

Definition 1.1 *Harmonic maps between two Riemannian manifolds are critical points of the energy functional*

$$E(\psi) = \frac{1}{2} \int_N |d\psi|^2 dv_g$$

for smooth maps $\psi : (N, g) \to (\bar{N}, \bar{g})$*. Namely,* ψ *is called as harmonic if*

$$\tau(\psi) = -d^* d\psi = trace \nabla d\psi = 0.$$

Here $\tau(\psi)$, which is the tension field of ψ, is the Euler-Lagrange equation of the energy functional $E(\psi), d$ is the exterior differentiation, d^* is the codifferentiation, ∇ is the connection induced from the Levi-Civita connection $\nabla^{\bar{N}}$ of $\bar{N}$ and the pull-back connection $\nabla^{\bar{N}}$ [1, 3, 8].

Definition 1.2 ψ *is called as biharmonic if it is critical point, for all variations, of the bienergy functional*

$$E_2(\psi) = \frac{1}{2} \int_N |\tau(\psi)|^2 dv_g.$$

It means that ψ *is a biharmonic map if bitension field* $\tau_2(\psi)$ *equals to*

$$\tau_2(\psi) = trace(\nabla^{\psi}\nabla^{\psi} - \nabla^{\psi}_{\nabla})\tau(\psi) - trace(R^{\bar{N}}(d\psi, \tau(\psi))d\psi) = 0, \tag{1}$$

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where $R^{\bar{N}}$ *is the curvature tensor field of* $\bar{N}$ *[3, 12].*

It is easy to see that any harmonic map is a biharmonic map. On the other hand, a biharmonic map is called as proper biharmonic if it is not harmonic. Now, let us remind the definition of a bi-f-harmonic map.

Definition 1.3 *ψ is called as bi-f-harmonic if it is critical point of the bi-f-energy functional*

$$E_{f,2}(\psi) = \frac{1}{2}\int_N |\tau_f(\psi)|^2 dv_g,$$

where $\tau_f(\psi) = f\tau(\psi) + d\psi(gradf)$ *is the* f*-tension field. The Euler-Lagrange equation for the bi-*f*-harmonic map is given by*

$$\tau_{f,2}(\psi) = trace\big(\nabla^{\psi} f(\nabla^{\psi}\tau_f(\psi)) - f\nabla^{\psi}_{\nabla^N}\tau_f(\psi) + fR^{\bar{N}}(\tau_f(\psi), d\psi)d\psi\big) = 0, \tag{2}$$

here $\tau_{f,2}(\psi)$ *is the bi-*f*-tension field of the map* ψ *and* f *is a smooth positive function on the domain [12].*

Note that overall throughout this paper, we will use SSF instead of Sasakian space form for the sake of simplicity.

The authors of [14] summarized the relationship between biharmonic and bi-f-harmonic maps; by extending bienergy functional to bi-f-energy functional defining a new type of harmonic map called as bi-f-harmonic map.

Bi-f-harmonic maps were introduced by Ouakkas et al. in 2010 [9] and Perktaş et al. obtained bi-f-harmonicity conditions of curves in Riemannian manifolds and derived bi-f-harmonic equations for curves in various spaces such as Euclidean and hyperbolic space in 2019 [12]. Biharmonic Legendre curves were handled in SSF by Fetcu in 2008 [4] and were introduced by Özgür and Güvenç in generalized SSF and S-space forms in 2014 [10, 11]. Subsequently, f-biharmonic Legendre curves were examined by Özgür and Güvenç in SSF in 2017 and were studied by Güvenç in S-space forms in 2019 [6, 7].

Inspired by these papers, in this study, we examined bi-f-harmonic Legendre curves in Sasakian space form. Firstly, in Section 2, we remind definition and properties of a Sasakian space form. Then, in Section 3, we give our main theorems and corollaries.

2. Sasakian Space Forms

Let (N, g) be a framed metric manifold with $dim(N) = (2n + s)$ and a framed metric structure $(\varphi, \xi_\alpha, \eta^\alpha, g)$, where $\alpha \in \{1, ..., s\}$; φ is a $(1,1)$ tensor field defining a $\varphi-$structure of rank $2n$;

$\xi_1, ..., \xi_s$ are vector fields; $\eta^1, ..., \eta^s$ are $1-$forms and g is a Riemannian metric on N.
For all $K, L \in TN$ and $\alpha, \beta \in \{1, ..., s\}$, following formulas are satisfied;

$$\varphi^2 K = -K + \sum_{\alpha=1}^{s} \eta^\alpha(K)\xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta_{\alpha\beta}, \quad \varphi(\xi_\alpha) = 0, \quad \eta^\alpha \circ \varphi = 0, \tag{3}$$

$$g(\varphi K, \varphi L) = g(K, L) - \sum_{\alpha=1}^{s} \eta^\alpha(K)\eta^\alpha(L), \tag{4}$$

$$d\eta^\alpha(K, L) = g(K, \varphi L) = -d\eta^\alpha(L, K), \quad \eta^\alpha(K) = g(K, \xi). \tag{5}$$

If Nijenhuis tensor of φ equals to $-2d\eta^\alpha \otimes \xi_\alpha$ for all $\alpha \in \{1, ..., s\}$, then $(\varphi, \xi_\alpha, \eta^\alpha, g)$ is called $S-$structure and if $s = 1$, a framed metric structure becomes an almost contact metric structure; an $S-$structure becomes a Sasakian structure, then we have [2, 11, 13]:

$$(\nabla_K \varphi)L = \sum_{\alpha=1}^{s} (g(\varphi K, \varphi L)\xi_\alpha + \eta^\alpha(L)\varphi^2 K), \tag{6}$$

$$\nabla \xi_\alpha = -\varphi, \quad \alpha \in \{1, ..., s\}. \tag{7}$$

A plane section in T_pN is a φ-section if there exists a vector $K \in T_pN$ being orthogonal to $\xi_1, ..., \xi_s$ such that $K, \varphi K$ span the section. The sectional curvature of a φ-section is called φ-sectional curvature such that a S-manifold of constant φ-section curvature c is called as S-space form. Finally, if $s = 1$, a S-space form becomes a Sasakian space form [2, 6, 7]. For a SSF, from equations (6) and (7), it is easy to see that

$$(\nabla_K \varphi)L = g(K, L)\xi - \eta(L)K, \tag{8}$$

$$\nabla_K \xi = -\varphi K \tag{9}$$

and the curvature tensor R of a SSF is given by

$$\begin{aligned} R(K, L)M &= \frac{c+3}{4}\big(g(L, M)K - g(K, M)L\big) \\ &+ \frac{c-1}{4}\Big(g(K, \varphi M)\varphi L - g(L, \varphi M)\varphi K + 2g(K, \varphi L)\varphi M + \eta(K)\eta(M)L \\ &- \eta(L)\eta(M)K + g(K, M)\eta(L)\xi - g(L, M)\eta(K)\xi\Big) \end{aligned} \tag{10}$$

for all $K, L, M \in TN$ [2].

Here let's remind the definition of a Legendre curve in a SSF.

Definition 2.1 *A Legendre curve of a SSF* $(N^{2n+1},\varphi,\xi,\eta,g)$ *is a one dimensional integral submanifold of* N *and* $\beta : I \to (N^{2n+1},\varphi,\xi,\eta,g)$ *is a Legendre curve if* $\eta(T) = 0$*, where* T *is the tangent vector field of* β *[6, 7].*

3. Bi-f-harmonic Legendre Curves in Sasakian Space Forms

Let $\beta : I \longrightarrow N$ be an arc-length parametrized curve in a m-dimensional Riemannian manifold (N,g) and $u_1,u_2,,u_r$ are vector fields along β such that

$$\begin{aligned} u_1 = \beta' &= T, \\ \nabla_{u_1}u_1 &= k_1u_2, \\ \nabla_{u_1}u_2 &= -k_1u_1 + k_2u_3, \\ &\vdots \\ \nabla_{u_1}u_r &= -k_{r-1}u_{r-1}. \end{aligned} \tag{11}$$

Then, β is called a Frenet curve of osculating order r, here $k_1,\dots,k_{r-1}$ are positive functions on I and $1 \le r \le m$. With the help of Definition 1.3, β is called a bi-f-harmonic curve if and only if following condition is hold [12],

$$\begin{aligned} \tau_{f,2}(\beta) &= (ff'')'u_1 + (3ff'' + 2(f')^2)\nabla_{u_1}u_1 + 4ff'\nabla^2_{u_1}u_1 + f^2\nabla^3_{u_1}u_1 + f^2R(\nabla_{u_1}u_1,u_1)u_1 \\ &= 0. \end{aligned} \tag{12}$$

Now, let $(N^{2n+1},\varphi,\xi,\eta,g)$ be a Sasakian space form and $\beta : I \to N$ be a Legendre curve. Then, with the help of equation (11) and derivative of $\eta(T) = \eta(u_1) = 0$, following equality

$$\eta(u_2) = 0 \tag{13}$$

is obtained [7]. By using equations (10), (11) and (13), we get the following equalities

$$\begin{aligned} \nabla_{u_1}u_1 &= k_1u_2, \\ \nabla_{u_1}\nabla_{u_1}u_1 &= \nabla^2_{u_1}u_1 = -k_1^2u_1 + k_1'u_2 + k_1k_2u_3, \\ \nabla_{u_1}\nabla_{u_1}\nabla_{u_1}u_1 &= \nabla^3_{u_1}u_1 = -3k_1k_1'u_1 + \big(-k_1^3 + k_1'' - k_1k_2^2\big)u_2 \\ &\quad +\big(2k_1'k_2 + k_1k_2'\big)u_3 + k_1k_2k_3u_4, \\ R(\nabla_{u_1}u_1,u_1)u_1 &= k_1\Big(\frac{c+3}{4}\Big)u_2 + 3k_1\Big(\frac{c-1}{4}\Big)g(u_2,\varphi u_1)\varphi u_1. \end{aligned}$$

Then, by substutiting these equalities into the bi-f-harmonicity condition, namely into the equation (12), we obtain bi-f-harmonicity condition of a Legendre curve in a Sasakian space form as follows,

$$\begin{aligned}
\tau_{f,2}(\beta) &= [(ff'')' - 4k_1^2 ff' - 3k_1k_1' f^2]u_1 \\
&+ \left[(3ff'' + 2(f')^2)k_1 + 4ff'k_1' + \left(-k_1^3 + k_1'' - k_1k_2^2 + k_1(\frac{c+3}{4})\right)f^2\right]u_2 \\
&+ [4ff'k_1k_2 + f^2(2k_1'k_2 + k_1k_2')]u_3 \\
&+ [k_1k_2k_3f^2]u_4 \\
&+ 3f^2k_1(\frac{c-1}{4})g(u_2, \varphi u_1)\varphi u_1 \\
&= 0. \qquad (14)
\end{aligned}$$

It should be noted that if function f is a constant, then bi-f-harmonicity condition turns into a biharmonicity condition. For this reason, the function f will be considered different from a constant throughout the paper.

Now, we give interpretations of bi-f-harmonicity condition given in equation (14).

Remark 3.1 *[12] The property of a curve being bi-f-harmonic in a n-dimensional space $(n > 3)$ does not depend on all its curvatures, but only on k_1, k_2 and k_3.*

Let $k = min\{r, 4\}$. From equation (14), β is a bi-f-harmonic curve if and only if $\tau_{f,2}(\beta) = 0$, namely,

(i) $c = 1$ or $\varphi u_1 \perp u_2$ or $\varphi u_1 \in sp\{u_2, ..., u_k\}$,

(ii) $g(\tau_{f,2}(\beta), u_i) = 0$ for all $i = 1, ..., k$.

Thus, we can give the following main theorem.

Theorem 3.2 *Let β be a non-geodesic Legendre curve of osculating order r in a Sasakian space form $(N^{2n+1}, \varphi, \xi, \eta, g)$ and $k = min\{r, 4\}$. Then, β is a bi-f-harmonic curve if and only if*

(i) $c = 1$ or $\varphi u_1 \perp u_2$ or $\varphi u_1 \in sp\{u_2, ..., u_k\}$,

(ii) the first k of the following differential equations are satisfied

$$\begin{cases}
(ff'')' - 4k_1^2 ff' - 3k_1k_1'f^2 = 0, \\
k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4} + 3(\frac{c-1}{4})g(u_2, \varphi u_1)^2, \\
4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' + 3(\frac{c-1}{4})g(u_2, \varphi u_1)g(u_3, \varphi u_1) = 0, \\
k_2k_3 + 3(\frac{c-1}{4})g(u_2, \varphi u_1)g(u_4, \varphi u_1) = 0.
\end{cases} \qquad (15)$$

From here on, we investigate results of Theorem 3.2 in eight cases.

Case I: If $c = 1,$ then equation (15) reduces to

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + 1, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0, \\ k_2k_3 = 0. \end{cases}$$

Hence, we have Theorem 3.3.

Theorem 3.3 *Let β be a non-geodesic Legendre curve in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$ and $c = 1$. Then, β is a bi-f-harmonic curve iff following differential equations are satisfied*

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + 1, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0, \\ k_2k_3 = 0. \end{cases} \tag{16}$$

Also, we get the following corollary from Theorem 3.2.

Corollary 3.4 *Let β be a non-geodesic Legendre curve in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$ and $c = 1$. Then, β is a bi-f-harmonic curve iff either*

(i) β is of osculating order $r = 2$ and f, k_1 satisfy the following differential equations

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 = 0, \\ k_1^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + 1 \end{cases}$$

or

(ii) β is of osculating order $r = 3$ and f, k_1, k_2 satisfy the following differential equations

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + 1, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0. \end{cases}$$

Proof It is known that if k_2 equals to zero, then β is called as of osculating order 2. Here, if we substitute zero, for k_2 in equation (16), third and fourth equations are vanished, then we obtain the differential equations given in (i). On the other hand, if k_3 equals to zero, then β is called as of osculating order 3 and similarly, substutiting zero for k_3 in equation (16), fourth equation is vanished, so we obtain the differential equations given in (ii).

Case II: If $c=1$ and $(f.f^{''})^{'}=0$, then equation (15) reduces to

$$\begin{cases} 4k_1^2 ff^{'} + 3k_1k_1^{'}f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + 1, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0, \\ k_2k_3 = 0. \end{cases} \tag{17}$$

Hence, we have Theorem 3.5.

Theorem 3.5 *Let β be a Legendre curve with non-constant geodesic curvature in a SSF $(N^{2n+1},\varphi,\xi,\eta,g)$, $c=1$, $(f.f^{''})^{'}=0$ and $n\geq 2$. Then, β is a bi-f-harmonic curve iff either*

(i) β is of osculating order $r=2$ with $f=c_1k_1^{-\frac{3}{4}}$, where c_1 is a positive integration constant and k_1 satisfy the following second order non-linear ordinary differential equation

$$16k_1^4 - 16k_1^2 - 33(k_1^{'})^2 + 20k_1k_1^{''} = 0$$

or

(ii) β is of osculating order $r=3$ with $f=c_1k_1^{-\frac{3}{4}}$, $k_2=c_2k_1$, where c_1,c_2 are positive integration constants and k_1 satisfy the following second order non-linear ordinary differential equation

$$16(1+c_2^2)k_1^4 + 20k_1k_1^{''} - 33(k_1^{'})^2 - 16k_1^2 = 0.$$

Proof By using the first equation of (17), we get

$$\frac{f^{'}}{f} = -\frac{3}{4}\frac{k_1^{'}}{k_1}, \quad \frac{f^{''}}{f} = \frac{21}{16}(\frac{k_1^{'}}{k_1})^2 - \frac{3}{4}\frac{k_1^{''}}{k_1}. \tag{18}$$

Thus from equation (18), we obtain $f=c_1k_1^{-\frac{3}{4}}$, where c_1 is an integration constant. Then, we know that if $k_2=0$, β is called as of osculating order $r=2$ and if $k_2=0$, third and fourth equations of (17) are vanished. Finally, by substutiting equation (18) to the second equation of (17), we obtain a second order non-linear ordinary differential equation $16k_1^4 - 16k_1^2 - 33(k_1^{'})^2 + 20k_1k_1^{''} = 0$.

On the other hand, we know that if $k_3 = 0$, β is called as of osculating order $r = 3$ and if $k_3 = 0$, fourth equation of (17) is vanished. Then, by substutiting equation (18) to the third equation of (17), we obtain that $k_2 = c_2 k_1$ for a positive integration constant c_2. Finally, by using these results in the second equation of (17), we get second order non-linear ordinary differential equation $16(1+c_2^2)k_1^4 + 20k_1 k_1'' - 33(k_1')^2 - 16k_1^2 = 0$. So, the proof is complete. □

Case III: If $c \neq 1$ and $\varphi u_1 \perp u_2$, then equation (15) reduces to

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1 k_1' f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4}, \\ 4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' = 0, \\ k_2 k_3 = 0. \end{cases}$$

Then, before giving Theorem 3.7, we need the following proposition.

Proposition 3.6 *[5] Let β be a Legendre curve of osculating order 3 in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$ and $\varphi u_1 \perp u_2$. Then, $\{u_1, u_2, u_3, \varphi u_1, \nabla_{u_1}\varphi u_1, \xi\}$ is linearly independent at any point of β. Consequently, $n \geq 3$.*

Now, we can give Theorem 3.7.

Theorem 3.7 *Let β be a non-geodesic Legendre curve in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ and $\varphi u_1 \perp u_2$. Then, β is a bi-f-harmonic curve iff following differential equations are satisfied*

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1 k_1' f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4}, \\ 4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' = 0, \\ k_2 k_3 = 0. \end{cases}$$

Now, we can introduce the Corollary 3.8 of Theorem 3.7.

Corollary 3.8 *Let β be a non-geodesic Legendre curve in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ and $\varphi u_1 \perp u_2$. Then, β is a bi-f-harmonic curve iff either*

(i) β is of osculating order $r = 2$ and f, k_1 satisfy the following differential equations

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1 k_1' f^2 = 0, \\ k_1^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4} \end{cases}$$

or

(ii) β *is of osculating order* $r = 3$, $n \geq 3$ *and* f, k_1, k_2 *satisfy the following differential equations*

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1 k_1^{'} f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + \frac{c+3}{4}, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0. \end{cases}$$

Proof The proof is similar to the proof of Corollary 3.4. □

Now, let investigate the Case IV.

Case IV: If $c \neq 1$, $\varphi u_1 \perp u_2$ and $(ff^{''})^{'} = 0$, then equation (15) reduces to

$$\begin{cases} 4k_1^2 ff^{'} + 3k_1 k_1^{'} f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + \frac{c+3}{4}, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0, \\ k_2 k_3 = 0. \end{cases}$$

Now, with the help of Proposition 3.6, we can give the Theorem 3.9.

Theorem 3.9 *Let* β *be a Legendre curve with non-constant geodesic curvature in a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ *and* $\varphi u_1 \perp u_2$. *Then,* β *is a bi-*f*-harmonic curve iff either*

(i) β *is of osculating order* $r = 2$ *with* $f = c_1 k_1^{-\frac{3}{4}}$, $\{u_1, u_2, \varphi u_1, \nabla_{u_1}\varphi u_1, \xi\}$ *is linearly independent,* $n \geq 2$ *and* k_1 *satisfy the following second order non-linear ordinary differential equation*

$$16k_1^4 - 4(c+3)k_1^2 - 33(k_1^{'})^2 + 20k_1 k_1^{''} = 0$$

or

(ii) β *is of osculating order* $r = 3$ *with* $f = c_1 k_1^{-\frac{3}{4}}$, $k_2 = c_2 k_1$, $\{u_1, u_2, u_3, \varphi u_1, \nabla_{u_1}\varphi u_1, \xi\}$ *is linearly independent,* $n \geq 3$ *and* k_1 *satisfy the following second order non-linear ordinary differential equation*

$$16(1 + c_2^2)k_1^4 + 20k_1 k_1^{''} - 33(k_1^{'})^2 - 4(c+3)k_1^2 = 0.$$

Proof It is proved as similar to the proof of Theorem 3.5. □

Case V: Let $c \neq 1$ and $\varphi u_1 \parallel u_2$.

In this case, since $\varphi u_1 \parallel u_2$, we can write $\varphi u_1 = \mp u_2$. Hence, $g(u_2, \varphi u_1) = \mp 1, g(u_3, \varphi u_1) = g(u_3, \mp u_2) = 0$ and similarly, $g(u_4, \varphi u_1) = g(u_4, \mp u_2) = 0$. Then, equation (15) reduces to

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1 k_1^{'} f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + c, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0, \\ k_2 k_3 = 0. \end{cases} \tag{19}$$

Remark 3.10 *In [11], it is proved that in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$ if $c \neq 1$ and $\varphi u_1 \parallel u_2$, then $k_2 = 1$.*

Hence, we give the Theorem 3.11.

Theorem 3.11 *Let β be a non-geodesic Legendre curve in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ and $\varphi u_1 \parallel u_2$. Then, β is a bi-f-harmonic curve iff it is of osculating order $r = 3$ with $f = c_1 k_1^{-\frac{1}{2}}$ and k_1 satisfies the following differential equations*

$$\begin{cases} 18(k_1^{'})^3 - 11k_1 k_1^{'} k_1^{''} + 4k_1^2 k_1^{'''} + 8k_1^4 k_1^{'} = 0, \\ 4k_1^4 - 3(k_1^{'})^2 + 2k_1 k_1^{''} - 4(c-1)k_1^2 = 0. \end{cases}$$

Proof First of all from Remark 3.10, we know that $k_2 = 1$ and by choosing β as a curve of osculating order $r = 3$, we get $k_3 = 0$. Then, when we substitute these informations into the equation (19), we get

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1 k_1^{'} f^2 = 0, \\ k_1^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + c - 1, \\ 2\frac{f^{'}}{f} + \frac{k_1^{'}}{k_1} = 0. \end{cases} \tag{20}$$

Then, with help of third equation of (20), we obtain

$$\frac{f^{'}}{f} = -\frac{1}{2}\frac{k_1^{'}}{k_1}, \quad \frac{f^{''}}{f} = \frac{3}{4}(\frac{k_1^{'}}{k_1})^2 - \frac{1}{2}\frac{k_1^{''}}{k_1}. \tag{21}$$

Finally, if equation (21) is substituted into the first and second equation of (20), then two equations are found for k_1 and the proof is completed. □

Case VI: If $c \neq 1$, $\varphi u_1 \parallel u_2$ and $(ff^{''})^{'} = 0$, then by using Remark 3.10, equation (15) reduces to

$$\begin{cases} 4k_1^2 ff^{'} + 3k_1 k_1^{'} f^2 = 0, \\ k_1^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + c - 1, \\ 2\frac{f^{'}}{f} + \frac{k_1^{'}}{k_1} = 0. \end{cases} \tag{22}$$

In this case, if we take into consideration first and third equations of (22), then it is easy to see that f is a constant. Therefore, we obtain Theorem 3.12.

Theorem 3.12 *There is no bi-f-harmonic Legendre curve in a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$, *where* $c \neq 1$, $\varphi u_1 \parallel u_2$ *and* $(ff^{''})^{'} = 0$.

Considering that f is a constant, then we get Corollary 3.13.

Corollary 3.13 *Let* β *be a Legendre curve in a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$, *where* $c \neq 1$, $\varphi u_1 \parallel u_2$ *and* $(ff^{''})^{'} = 0$. *Then,* β *is a biharmonic curve if and only if it is a helix with* $k_1 = \sqrt{c-1}$ *and* $k_2 = 1$.

Case VII: Let $c \neq 1$ and $g(u_2, \varphi u_1)$ is not equal to $-1, 0$ or 1.

Now, let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a SSF and $\beta : I \longrightarrow N$ be a Legendre curve of osculating order r, where $4 \leq r \leq 2n+1$ and $n \geq 2$. We know that if β is bi-f-harmonic, then $\varphi u_1 \in sp\{u_2, u_3, u_4\}$. Here, let denote the angle between φu_1 and u_2 by $\phi(t)$, namely,

$$g(u_2, \varphi u_1) = cos\phi(t). \tag{23}$$

By differentiating $g(u_2, \varphi u_1)$ along β with the help of (8) and (11), the equality

$$-\phi^{'}(t) sin\phi(t) = k_2 g(u_3, \varphi u_1) \tag{24}$$

is obtained. Also, we can write

$$\varphi u_1 = g(u_2, \varphi u_1)u_2 + g(u_3, \varphi u_1)u_3 + g(u_4, \varphi u_1)u_4. \tag{25}$$

For details, see [7]. By using these results, we obtain Theorem 3.14 and Theorem 3.15.

Theorem 3.14 *Let* β *be a non-geodesic Legendre curve in a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ *and* $g(u_2, \varphi u_1)$ *is not equal to* $-1, 0$ *or* 1. *Then,* β *is a bi-*f*-harmonic curve iff following differential* *equations are satisfied*

$$\begin{cases}(ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 = 0,\\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + \frac{c+3}{4} + 3(\frac{c-1}{4})cos^2\phi(t),\\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} + 3(\frac{c-1}{4})g(u_3, \varphi u_1)cos\phi(t) = 0,\\ k_2k_3 + 3(\frac{c-1}{4})g(u_4, \varphi u_1)cos\phi(t) = 0.\end{cases}$$

Proof It is easy to see that if equation (23) substituted into equation (15), then the proof is completed. □

Case VIII: If $c \neq 1$ and $g(u_2, \varphi u_1)$ is not equal to $-1, 0$ or 1 and $(ff^{''})^{'} = 0$, then equation (15) reduces to

$$4k_1^2 ff^{'} + 3k_1k_1^{'}f^2 = 0, \tag{26}$$

$$k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + \frac{c+3}{4} + 3(\frac{c-1}{4})cos^2\phi(t), \tag{27}$$

$$4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} + 3(\frac{c-1}{4})g(u_3, \varphi u_1)cos\phi(t) = 0, \tag{28}$$

$$k_2k_3 + 3(\frac{c-1}{4})g(u_4, \varphi u_1)cos\phi(t) = 0. \tag{29}$$

Now, let give the interpretation of Case VIII.

First of all, from equation (26), it is easy to see that $\frac{f^{'}}{f} = -\frac{3}{4}\frac{k_1^{'}}{k_1}$ and $\frac{f^{''}}{f} = \frac{3}{4}\left(\frac{k_1^{'}}{k_1}\right)^2 - \frac{1}{2}\frac{k_1^{''}}{k_1}$. Then, by using these equalities in the equations (27) and (28), we get

$$k_1^2 + k_2^2 = \frac{33}{16}(\frac{k_1^{'}}{k_1})^2 - \frac{5}{4}\frac{k_1^{''}}{k_1} + \frac{c+3}{4} + 3(\frac{c-1}{4})cos^2\phi(t), \tag{30}$$

$$-k_2(\frac{k_1^{'}}{k_1}) + k_2^{'} + 3(\frac{c-1}{4})g(u_3, \varphi u_1)cos\phi(t) = 0, \tag{31}$$

respectively. Then, by multiplying equation (31) with $2k_2$ and using equation (24), we get

$$2k_2k_2^{'} - 2k_2^2\frac{k_1^{'}}{k_1} + 3(\frac{c-1}{4})(-2\phi^{'}(t)cos\phi(t)sin\phi(t)) = 0. \tag{32}$$

Let ϕ be a constant. Then, from (24), we get $g(u_3, \varphi u_1) = 0$ and also, from (25), we get $g(u_4, \varphi u_1) = \mp sin\phi$ since $\|\varphi u_1\| = 1$. Finally, from (32), we obtain $k_2 = c_2k_1$, where c_2 is a positive integration constant. Then, by using these informations, equations (29) and (30) reduces to $c_2k_1k_3 = \mp\frac{3(c-1)sin(2\phi(t))}{8}$ and

$$33(k_1^{'})^2 - 20k_1k_1^{''} + k_1^2\big(4(c+3) + 3(c-1)cos^2\phi(t) - 16k_1^2 - 16c_2^2k_1^2\big) = 0.$$

Now, we can state the Theorem 3.15.

Theorem 3.15 *Let β be a Legendre curve with non-constant geodesic curvature of osculating order r in a SSF $(N^{2n+1},\varphi,\xi,\eta,g)$, where $c \neq 1$, $g(u_2,\varphi u_1)$ is not equal to $-1, 0$ or 1, $(ff^{''})^{'} = 0$, $r \geq 4$, $n \geq 2$ and ϕ be a constant. Then, β is a bi-f-harmonic curve iff $f = c_1k_1^{-\frac{3}{4}}, k_2 = c_2k_1$ and k_1, k_3 satisfy following differential equations*

$$33(k_1^{'})^2 - 20k_1k_1^{''} + k_1^2\big(4(c+3) + 3(c-1)cos^2\phi(t) - 16k_1^2 - 16c_2^2k_1^2\big) = 0,$$

$$c_2k_1k_3 = \mp\frac{3(c-1)sin(2\phi(t))}{8},$$

where c_1 and c_2 are positive integration constants.

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

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